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Extended results on restrained domination number and connectivity of a graph

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Abstract

A subset S of V is called a dominating set in G if every vertex in $V - S$ is adjacent to at least one vertex in S. A dominating set S is said to be a restrained dominating set if $\langle V - S\rangle$ contains no isolated vertices. The minimum cardinality of a restrained dominating set of G is called the restrained domination number of G and is denoted by $\gamma_r(G)$. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. In this paper we characterized the graphs with sum of restrained domination number and connectivity is equal to $2n - 6$.

Keywords: Restrained domination number, connectivity.
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1 Introduction

The graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. The degree of any vertex u in G is the number of edges incident with u and is denoted by $d(u)$. The minimum and maximum degree of a graph G is denoted by $\delta(G)$ and $\Delta(G)$ respectively. $H(m_1, m_2, ..., m_n)$ denotes the graph obtained from the graph H by attaching m_i edges to the vertex $v_i \in V(H), 1 \leq i \leq n$. The graph $K_2(r, s)$ is called a bistar and is also denoted by $B(r, s)$. $H(P_{m_1}, P_{m_2}, ..., P_{m_n})$ is the graph obtained from the graph H by attaching an end vertex of P_{m_i} to the vertex v_i in $H, 1 \leq i \leq n$. The graph $G(r)$ is obtained from a graph $G \cup K_1$ where G is a regular graph, by adding r number of edges between the vertex of K_1 and any r vertices of G. For graph theoretic terminology we refer to Chartrand and Lesniak [1] and Haynes et.al [3, 4].

A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set S is said to be a restrained dominating set if the induced subgraph $\langle V - S \rangle$ contains no isolated vertices. The minimum cardinality of a restrained dominating set of G is called the restrained domination number of G and is denoted by $\gamma_r(G)$. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. J. Paulraj Joseph and S. Arumugam [5] proved that $\gamma(G) + \kappa(G) \leq n$ and characterized the corresponding extremal graphs. P. Selvaraju and M.P. Kulandaivel [6] proved that $\gamma_r(G) + \kappa(G) \leq 2n - 1$ and they characterized the corresponding extremal graphs. Also they characterized the extremal graphs with the sum of restrained domination number and connectivity upto $2n - 5$.

In this paper we characterized the graphs with sum of restrained domination number and connectivity equals to $2n - 6$. We use the following theorems to prove our result.

Theorem 1.1. [2] For any connected graph G, $\gamma_r(G) \leq n$. Further, equality holds if and only if G is a star.

Theorem 1.2. [2] $\gamma_r(K_n) = 1,\ n > 2$.

Theorem 1.3. [2] If G is a connected graph of order n and G is not a star, then $\gamma_r(G) \leq n-2$.

Theorem 1.4. $\kappa(G) \leq \delta(G)$.

Theorem 1.5. [6] For any connected graph G, $\gamma_r(G) + \kappa(G) \leq 2n - 1$ and equality holds if and only if G is isomorphic to K_2.

2 Main Results

Theorem 2.1. For any connected graph G, $\gamma_r(G)+\kappa(G) = 2n-6$ if and only if G is isomorphic to any one of the following graphs (i) $K_{1,6}$ (ii) $K_{3,2}$ (iii) K_6 (iv) $B(2,1)$ (v) $K_3(2,0,0)$ (vi) $C_4(2)$ (vii) $C_4(3)$ $(viii)$ P_5 (ix) $C_3(1,1,0)$ (x) $K_5 - Y$ where Y is a matching in K_5 (xi) $K_6 - M$ where M is a perfect matching in K_6.

Proof: Let $\gamma_r(G) + \kappa(G) = 2n - 6$. Then there are five cases to consider (i) $\gamma_r(G) = n$ and $\kappa(G) = n - 6$ (ii) $\gamma_r(G) = n - 2$ and $\kappa(G) = n - 4$ (iii) $\gamma_r(G) = n - 3$ and $\kappa(G) = n - 3$ (iv) $\gamma_r(G) = n - 4$ and $\kappa(G) = n - 2$ (v) $\gamma_r(G) = n - 5$ and $\kappa(G) = n - 1$.

Case 1: $\gamma_r(G) = n$ and $\kappa(G) = n - 6$.

Then G is a star which gives $\kappa(G) = 1 = n - 6$ and hence $n = 7$. Then G is isomorphic to $K_{1,6}$.

Case 2: $\gamma_r(G) = n - 2$ and $\kappa(G) = n - 4$.

Then $n - 4 \leq \delta(G)$. If $\delta(G) = n - 1$ then G is a complete graph which is a contradiction to $\kappa(G) = n - 4$.

If $\delta(G) = n - 2$ then G is isomorphic to $K_n - Y$ where Y is a matching in G. Hence $\gamma_r(G) \leq 2$. Then $n \leq 4$ which is a contradiction to $\kappa(G) = n - 4$. Suppose $\delta(G) = n - 3$. Let $X = \{v_1, v_2, \cdots, v_{n-4}\}$ be a minimum vertex cut of G and let $V - X = \{x_1, x_2, x_3, x_4\}$.

If $\langle V - X \rangle$ contains at least one isolated vertex then $\delta(G) \leq n - 4$ which is a contradiction. Hence $\langle V - X \rangle$ is isomorphic to $K_2 \cup K_2$. Also every vertex of $V - X$ is adjacent to all the vertices of X. Then X is a restrained dominating set of G. Hence $\gamma_r(G) \leq n - 4$ which is a contradiction. Thus $\delta(G) = n - 4$.

Sub Case 2.1: $\langle V - X \rangle = \overline{K_4}$.

Then every vertex of $V - X$ is adjacent to all the vertices in X. Suppose $E(\langle X \rangle) = \phi$. Then $|X| \leq 4$ and hence G is isomorphic to $K_{s,4}, 1 \leq s \leq 4$. But $\gamma_r(G) + \kappa(G) \neq 2n - 6$.

Suppose $E(\langle X \rangle) \neq \phi$. If any one of the vertex in X say v_1 is adjacent to all the vertices in X and hence $\gamma_r(G) = 1$. Then $n = 3$ which is impossible. Hence every vertex in X is not adjacent to at least one vertex in X. Hence $\gamma_r(G) = 2$. Then $n = 4$ which is also impossible.

Sub Case 2.2: $\langle V - X \rangle = P_3 \cup K_1$.

Let x_1 be the isolated vertex in $\langle V - X \rangle$ and let (x_2, x_3, x_4) be a path in $\langle V - X \rangle$. Then x_1 is adjacent to all the vertices in X and x_2 and x_4 are not adjacent to at most one vertex in X and x_3 is not adjacent to at most two vertices in X. If $|X| \geq 3$ then $X \cup \{x_1\}$ is a restrained dominating set of cardinality $n - 3$ which is a contradiction. If $|X| = 2$ then $\{x_3, x_4, v_2\}$ is a restrained dominating set of G or G is isomorphic to C_6. Both the cases we get a contradiction. If $|X| = 1$ then G is isomorphic to P_5 or $B(2, 1)$ or $C_3(1, 1, 0)$ or $C_4(1, 0, 0)$ or the graph G_1 which is obtained from $(K_4 - e) \cup K_1$ by adding an edge between a vertex of K_1 and a vertex of degree three in $K_4 - e$. But $\gamma_r(C_4(1, 0, 0)) = \gamma_r(G_1) = 2 \neq n - 2$. Hence G is isomorphic to P_5 or $B(2, 1)$ or $C_3(1, 1, 0)$.

Sub Case 2.3: $\langle V - X \rangle = K_3 \cup K_1$.

Let x_1 be the isolated vertex in $\langle V - X \rangle$ and let $\langle \{x_2, x_3, x_4\} \rangle$ be the complete graph. Then x_1 is adjacent to all the vertices in X and x_2, x_3, x_4 are not adjacent to at most two vertices in X. If $|X| \geq 3$ then $X \cup \{x_1\}$ is a restrained dominating set of cardinality $n - 3$ which is a contradiction. If $|X| = 2$ then $\{v_1, x_1, x_2\}$ or $\{v_1, x_1, x_3\}$ or $\{v_1, x_1, x_4\}$ is a restrained dominating set of G. Hence $\gamma_r(G) \leq 3$. Then $n \leq 5$ which is a contradiction. If $|X| = 1$ then $\gamma_r(G) \leq 2$ and hence $n \leq 4$ which is a contradiction.

Sub Case 2.4: $\langle V - X \rangle = K_2 \cup K_2$.

Let $x_1 x_2$, $x_3 x_4 \in E(G)$. Since $\delta(G) = n - 4$ each $x_i, 1 \leq i \leq 4$ is non-adjacent to at most one vertex in X. If $|X| \geq 2$ then X is a restrained dominating set of cardinality $n-4$ which is a contradiction. Hence $|X| = 1$. Then G is isomorphic to P_5 or $C_3(P_3, P_1, P_1)$ or the graph G_2 which is obtained from $C_3(2,0,0)$ by joining the pendant vertices by an edge. But $\gamma_r(C_3(P_3, P_1, P_1)) = 2 \neq n-2$ and $\gamma_r(G_2) = 1 \neq n-2$ which is a contradiction. Hence G is isomorphic to P_5.

Sub Case 2.5: $\langle V - X \rangle = K_2 \cup \overline{K_2}$.

Let $x_1 x_2 \in E(G)$ and $x_3 x_4 \in E(\overline{G})$. Then each x_i, $i = 1$ or 2 is non adjacent to at most one vertex in X and each x_j, $j = 3$ or 4 is adjacent to all the vertices in X. For this graph $\gamma_r(G) \leq 3$ and hence $n \leq 5$. Thus $n = 5$. Then $|X| = 1$. Hence G is isomorphic to $B(2,1)$ or $K_3(2,0,0)$.

Case 3: $\gamma_r(G) = n-3$ and $\kappa(G) = n-3$.

Then $n-3 \leq \delta(G)$. If $\delta = n-1$ then G is a complete graph which is a contradiction to $\kappa(G) = n-3$. If $\delta = n-2$ then G is isomorphic to $K_n - Y$ where Y is a matching in K_n. Then $\gamma_r(G) \leq 2$. If $\gamma_r(G) = 1$ then $n = 4$. Hence G is isomorphic to $K_4 - e$. But $\kappa(K_4 - e) = 2 \neq n-3$ which is a contradiction. If $\gamma_r(G) = 2$ then $n = 5$. There is no graph satisfies this condition. Hence $\delta(G) = n-3$. Let $X = \{v_1, v_2, \cdots, v_{n-3}\}$ be a minimum vertex cut of G and let $V - X = \{x_1, x_2, x_3\}$.

Sub Case 3.1: $\langle V - X \rangle = \overline{K_3}$.

Then every vertex of $V - X$ is adjacent to all the vertices in X. Suppose $E(\langle X \rangle) = \phi$. Then $|X| \leq 3$ and hence G is isomorphic to $K_{s,3}$, $s = 2$ or 3. If $s = 3$ then G is isomorphic to $K_{3,3}$. But $\gamma_r(K_{3,3}) = 2 \neq n-3$. Hence G is isomorphic to $K_{3,2}$. Suppose $E(\langle X \rangle) \neq \phi$. If any $v_1 \in X$ is adjacent to all the vertices in X and hence $\gamma_r(G) = 1$. Then $n = 4$ which is a contradiction. Hence every vertex in X is not adjacent to at least one vertex in X. Hence $\gamma_r(G) = 2$. Then $n = 5$. Hence G is isomorphic to $K_{3,2}$.

Sub Case 3.2: $\langle V - X \rangle = K_1 \cup K_2$.

Let $x_1 x_2 \in E(G)$. Since $\delta = n-3$ we have x_3 is adjacent to all the vertices of X. Suppose $d(x_1)$ or $d(x_2)$ is $n-2$. Let $d(x_1) = n-2$. Then $\{x_2, x_3\}$ is a restrained dominating set of G and hence $\gamma_r(G) \leq 2$. If $\gamma_r(G) = 1$ then $n = 4$ which is impossible. If $\gamma_r(G) = 2$ then $n = 5$. Hence G is isomorphic to $C_4(2)$ or $C_4(3)$. Suppose $d(x_i) = n-3, 1 \leq i \leq 2$. Then $\gamma_r(G) = 2$ or 3. If $\gamma_r(G) = 3$ then $n = 6$. Then we get the graphs with $\gamma_r(G) + \kappa(G) \neq 2n-6$. If $\gamma_r(G) = 2$ then $n = 5$. Hence G is isomorphic to C_5 or $C_3(P_3, P_1, P_1)$. For these graphs $\gamma_r(G) + \kappa(G) \neq 2n-6$.

Case 4: $\gamma_r(G) = n-4$ and $\kappa(G) = n-2$.

Then $\delta(G) \geq n-2$. If $\delta(G) = n-1$ then G is a complete graph which is a contradiction. Hence $\delta(G) = n-2$. Then G is isomorphic to $K_n - M$ where M is a matching in K_n. Thus $\gamma_r(G) \leq 2$. If $\gamma_r(G) = 1$ then $n = 5$. Hence G is isomorphic to $K_5 - M$ where M is a matching in K_5. If $\gamma_r(G) = 2$ then $n = 6$ and hence G is isomorphic to $K_6 - M$ where M is a perfect matching in K_6.

Case 5: $\gamma_r(G) = n - 5$ and $\kappa(G) = n - 1$.

Then G is isomorphic to a complete graph. Hence $\gamma_r(G) = 1 = n - 5$. Thus $n = 6$. Hence G is isomorphic to K_6. The converse is obvious. ∎

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