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Neighborhood-prime labeling

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Abstract

In this paper we introduce neighborhood-prime labeling and investigate neighborhood-prime labelings for paths, cycles, helm, closed helm and flower. Also we establish a sufficient condition for a graph to admit neighborhood-prime labeling.

Keywords: Prime labeling, neighborhood of a vertex, neighborhood-prime labeling.
AMS Subject Classification(2010): 05C78.

1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow Gross and Yellen [2] for graph theoretic terminology and notations.

Definition 1.1. Let $G = (V(G), E(G))$ be a graph with n vertices. A bijective function $f : V(G) \rightarrow \{1, 2, 3, \ldots, n\}$ is said to be a prime labeling, if for every pair of adjacent vertices u and v, $gcd\,(f(u), f(v)) = 1$. A graph which admits prime labeling is called a prime graph.

The notion of prime labeling for graphs originated with Roger Entringer and was introduced in a paper by Tout et al. [5] in the early 1980s and since then it is an active field of research for many scholars. For a comprehensive list of results regarding prime graphs the readers may refer to the dynamic survey of graph labeling by Gallian [1]. Motivated by the study of prime labeling, we introduce the notion of neighborhood-prime labeling in this paper and we believe that it will be an interesting area of research in future for many researchers.

Definition 1.2. For a vertex v in G, the neighborhood of v is the set of all vertices in G which are adjacent to v and is denoted by $N(v)$.

Definition 1.3. : Let $G = (V(G), E(G))$ be a graph with n vertices. A bijective function $f : V(G) \rightarrow \{1, 2, 3, \ldots, n\}$ is said to be a neighborhood-prime labeling, if for every vertex $v \in V(G)$ with $deg(v) > 1$, $gcd\,\{f(u) : u \in N(v)\} = 1$. A graph which admits neighborhood-prime labeling is called a **neighborhood-prime graph**.

Remark 1.4. If in a graph G, every vertex is of degree at most 1, then such a graph is neighborhood-prime vacuously.

Definition 1.5. The helm H_n is the graph obtained from the wheel $W_n = C_n + K_1$ by attaching a pendent edge at each vertex of the cycle C_n.

Definition 1.6. A closed helm is a graph obtained from a helm by joining each pendent vertex to form a cycle.

Definition 1.7. A flower is a graph obtained from a helm by joining each pendent vertex to the central vertex of the helm.

The concepts of prime graphs and neighborhood-prime graphs are independent in the sense that a prime graph may or may not be neighborhood-prime and vice versa.

2 Main Results

Theorem 2.1. Let $G = (V(G), E(G))$ be a graph with $n > 2$ vertices. If there exists a vertex $v_0 \in V(G)$ of degree $n-1$, then G is neighborhood-prime.

Proof: Let $f : V(G) \to \{1, 2, 3, \ldots, n\}$ be a bijective function such that $f(v_0) = 1$. Then for every $v \in V(G) - \{v_0\}$ with $deg(v) > 1$, $gcd\{f(u) : u \in N(v)\} = 1$, since the set $\{f(u) : u \in N(v)\}$ contains the number 1.

Also $gcd\{f(u) : u \in N(v_0)\} = 1$, since $\{f(u) : u \in N(v_0)\} = \{2, 3, \ldots, n-1, n\}$. Thus f is a neighborhood-prime labeling. ∎

It is established in [3] that the wheel graph $W_n = C_n + K_1$ is prime graph iff n is even and K_n is prime iff $n \leq 3$. However, by Theorem 2.1 and Remark 1.4 we observe that K_n, W_n, $K_{1,n}$ and the fan $F_n = P_n + K_1$ are neighborhood-prime graphs for all n.

Theorem 2.2. The path P_n is a neighborhood-prime graph for every n.

Proof: P_n is vacuously neighborhood-prime graph if $n \leq 2$ and so we assume that $n > 2$. Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of path P_n.

Define $f : V(P_n) \to \{1, 2, 3, \ldots, n\}$ as follows:

Case 1: n is odd.

$$\begin{aligned} f(v_{2j-1}) &= \frac{n-1}{2} + j, \quad 1 \leq j \leq \frac{n+1}{2}, \\ f(v_{2j}) &= j, \qquad\qquad 1 \leq j \leq \frac{n-1}{2}. \end{aligned}$$

Case 2: n is even.

$$\begin{aligned} f(v_{2j-1}) &= \frac{n}{2} + j, \quad 1 \leq j \leq \frac{n}{2}, \\ f(v_{2j}) &= j, \qquad\quad 1 \leq j \leq \frac{n}{2}. \end{aligned}$$

We claim that f is a neighborhood-prime labeling.

Let v_i be any vertex of P_n whose degree is greater than 1. Then $v_i \neq v_1, v_n$ and moreover $N(v_i) = \{v_{i-1}, v_{i+1}\}$. By definition of f, $f(v_{i-1})$ and $f(v_{i+1})$ are consecutive integers and so the gcd of the labels of vertices in $N(v_i)$ is 1. ■

Example 2.3. Neighborhood-prime labelings of paths P_8 and P_9 are shown in Fgure 1.

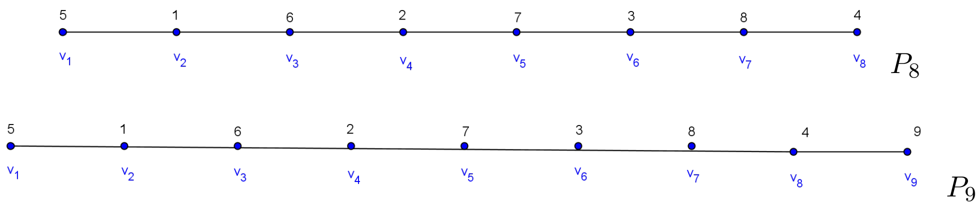

Figure 1: Neighborhood-prime labelings of paths P_8 and P_9.

Next we show that if $n \not\equiv 2 \pmod 4$, then the above labeling is a neighbourhood-prime labeling for the cycle C_n also.

Theorem 2.4. The cycle C_n is neighborhood-prime if $n \not\equiv 2 \pmod 4$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of cycle C_n and $f : V(C_n) \to \{1, 2, 3, \ldots, n\}$ be defined as in the proof of Theorem 2.2. We claim that f is a neighborhood prime labeling. If v_i is any vertex of C_n different from v_1 and v_n then $N(v_i) = \{v_{i-1}, v_{i+1}\}$. Since $f(v_{i-1})$ and $f(v_{i+1})$ are consecutive integers, gcd of the labels of vertices in $N(v_i)$ is 1. Since $N(v_1) = \{v_n, v_2\}$ and $f(v_2) = 1$, the gcd of the labels of vertices in $N(v_1)$ is 1. Finally for $N(v_n) = \{v_{n-1}, v_1\}$, we have to show that the gcd of the labels of vertices in $N(v_n)$ is 1. We consider the following two cases.

Case 1: n is odd.
In this case, $f(v_{n-1}) = \dfrac{n-1}{2}$ and $f(v_1) = \dfrac{n+1}{2}$ which are consecutive integers and hence relatively prime.

Case 2: n is even.
In this case, $f(v_{n-1}) = n$ and $f(v_1) = \dfrac{n}{2} + 1$. But when n is even and $n \not\equiv 2 \pmod 4$, n and $\dfrac{n}{2} + 1$ are relatively prime.

Hence, C_n is neighborhood-prime if $n \not\equiv 2 \pmod 4$. ■

Example 2.5. The following figure shows the neighborhood-prime labelings of the cycles C_8 and C_9.

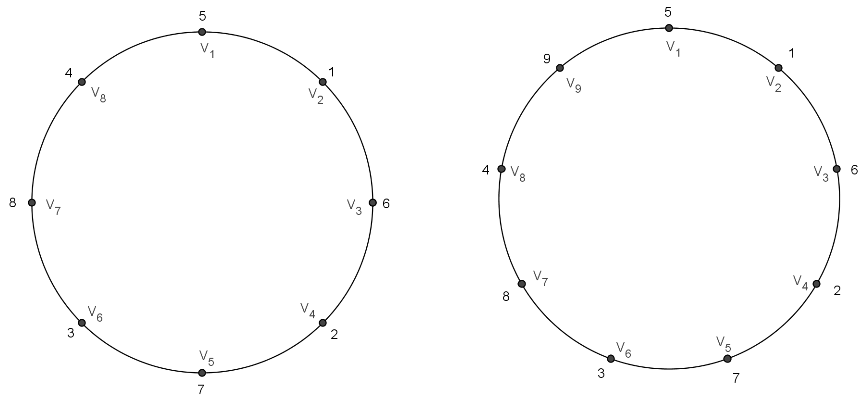

Figure2 : Neighborhood-prime labeling of the cycles C_8 and C_9.

In order to show that C_n is not neighborhood-prime if $n \equiv 2 \pmod 4$, we need to establish the following lemma.

Lemma 2.6. Let n be any positive integer such that $n \equiv 2 \pmod 4$. Suppose $v_1, v_2, \ldots, v_n$ are the consecutive vertices of the cycle C_n which are all labeled with 0 or 1 in such a way that the vertices labeled with 0 and the vertices labeled with 1 are equal in number. Then there exists at least one i, $1 \leq i \leq n$, such that v_{i-1} and v_{i+1} are labeled with 0 where the indices are taken modulo n.

Proof: Note that the conclusion of the lemma states that there exists at least one pair of alternate vertices in C_n labeled with 0. We prove the lemma by induction on k.
The case $k = 1$ (so that $n = 6$) can be verified easily. We prove the lemma by induction.

Assume that the lemma is true for the cycle C_{4k+2}. We have to prove $C_{4(k+1)+2}$. The proof is by contradiction.

Let $u_1, u_2, \ldots, u_{4(k+1)+2}$ be the consecutive vertices of the cycle $C_{4(k+1)+2}$ and suppose there does not exist i $(1 \leq i \leq 4(k+1)+2)$, such that u_{i-1} and u_{i+1} are labeled with 0. Since the vertices labeled with 0 and 1 are equal in number, there exist two consecutive vertices in $C_{4(k+1)+2}$ labeled with 0. Let u_j and u_{j+1} be two consecutive vertices labeled with 0. By our supposition, no two alternate vertices are labeled with 0 and hence $u_{j-2}, u_{j-1}, u_{j+2}, u_{j+3}$ are labeled with 1. Now consider the cycle C with vertices $u_1, u_2, \ldots, u_{j-2}, u_{j+3}, u_{j+4} \ldots, u_{4(k+1)+2}$. Note that C is a cycle of length $4k+2$ in which u_{j-2} and u_{j+3} are labeled with 1. This along with our supposition suggests that C does not contain a pair of alternate vertices labeled with 0. Thus C is a cycle of length $4k+2$ in which no pair of alternate vertices are labeled with 0, which is a contradiction to our assumtion. By the principle of mathematical induction the lemma follows for all k. ■

Theorem 2.7. The cycle C_n is not neighborhood-prime if $n \equiv 2 \pmod 4$.

Proof: Let $v_1, v_2, \ldots, v_{4k+2}$ be the consecutive vertices of the cycle C_{4k+2} and $f : V(C_{4k+2}) \to \{1, 2, 3, \ldots, 4k+2\}$ be a bijective function. If we identify all even and odd integers of the set $\{1, 2, 3, \ldots, 4k+2\}$ by 0 and 1 respectively, then in view of Lemma 2.6 it follows that there exists at least one i such that $f(v_{i-1})$ and $f(v_{i+1})$ are even integers. Hence f is not neighbourhood-prime labeling. ∎

It has been shown in [4] that helm and flower are prime graphs. We now show that a helm, a closed helm and a flower graph are neighborhood-prime graphs.

Theorem 2.8. Helm, closed helm and flower are neighborhood-prime graphs.

Proof: The central vertex of a flower graph is adjacent to every other vertex and so it is neighborhod-prime by Theorem 2.1. Next we define a neighborhood-prime labeling for a closed helm and the readers can verify that the same labeling works in the case of helm.
Let G be a closed helm with $2n+1$ vertices in which u_0 is the central vertex; $u_1, u_2, \ldots, u_n$ are the vertices of inner cycle and $v_1, v_2, \ldots, v_n$ are the vertices of the outer cycle such that each u_i is adjacent to v_i. Define a bijective function $f : V(G) \to \{1, 2, \ldots, 2n+1\}$ as follows:

$$\begin{aligned} f(u_0) &= 1 \\ f(u_i) &= 2i+1, \quad 1 \le i \le n \\ f(v_i) &= 2i, \quad\quad 1 \le i \le n. \end{aligned}$$

In order to show that f is a neighborhood-prime labeling we need to establish that if w is an arbitrary vertex of G, then

$$gcd\{f(p) : p \in N(w)\} = 1. \quad (1)$$

We consider the following four cases.

Case 1: If $w = u_0$, then $\{f(p) : p \in N(w)\} = \{3, 5, \ldots, 2n+1\}$ and so (1) follows.

Case 2: If $w = u_i$ for $1 \le i \le n$, then (1) follows since $u_0 \in N(w)$ and $f(u_0) = 1$.

Case 3: If $w = v_i$ for $1 \le i \le n-1$, then $N(w) \supset \{u_i, v_{i+1}\}$. But $f(u_i) = 2i+1$ and $f(v_{i+1}) = 2i+2$, and so (1) follows.

Case 4: If $w = v_n$, then $N(w) \supset \{u_n, v_1\}$. As $f(u_n) = 2n+1$ and $f(v_1) = 2$, the result follows. ∎

Example 2.9. Neighborhood-prime labelings of a helm and a closed helm are given in Figure 3.

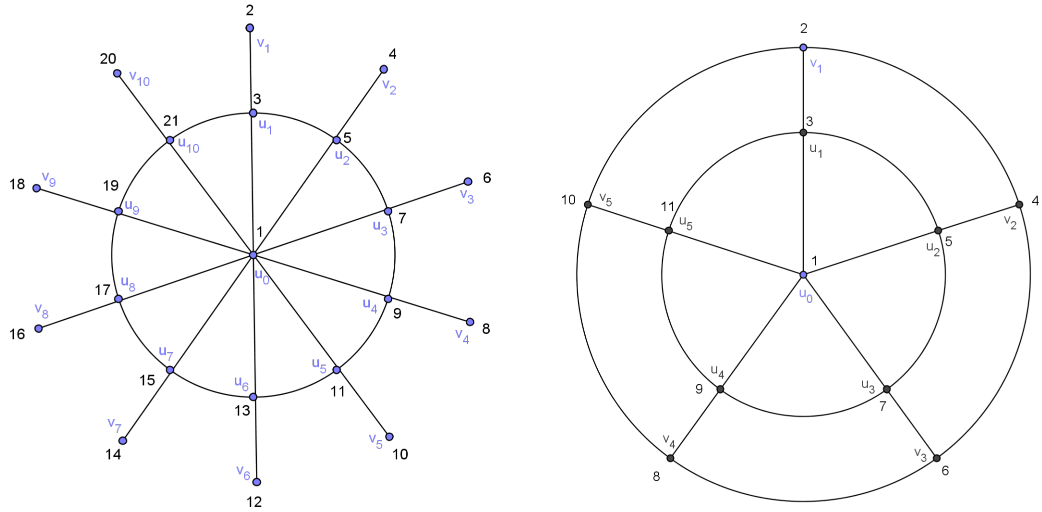

Figure 3: Neighborhood-prime labeling of a helm and a closed helm.

3 Further results on neighborhood-prime graphs

Vaidya and Kanani have shown in [6] that the graph obtained by duplicating an arbitrary vertex in the cycle C_n is a prime graph. A similar result for path has been established by Vaidya and Prajapati [7]. We derive similar results in the context of neighborhood-prime labeling.

Definition 3.1. Duplication of a vertex v of a graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$.

Theorem 3.2. Let $G = (V(G), E(G))$ be a neighborhood-prime graph of n vertices and v be a vertex in G which is not adjacent to any of its pendent vertices. If $G' = (V(G'), E(G'))$ is a graph obtained by duplicating the vertex v, then G' is neighborhood-prime.

Proof: Let v' be the duplicated vertex of v so that $V(G') = V(G) \cup \{v'\}$. If f is a neighborhood-prime labeling for the graph G, then we claim that the function $g : V(G') \to \{1, 2, 3, \ldots, n, n+1\}$ defined by

$$\begin{aligned} g(u) &= f(u), \text{ if } u \neq v' \\ g(v') &= n+1, \end{aligned}$$

is a neighborhood-prime labeling for the graph G'. Let u be any vertex in G' whose degree (in the graph G') is greater than 1. We consider the following three cases.
Case 1: $u = v$ or v'.
If $N_G(u)$ and $N_{G'}(u)$ denote the neighborhood of u in G and G' respectively, then it is clear that $N_G(v) = N_{G'}(v) = N_{G'}(v')$. Hence if $u = v$ or v', then $gcd\{g(w) : w \in N_{G'}(u)\} = gcd\{f(w) :$

$w \in N_G(v)\} = 1$, since f is a neighborhood-prime labeling for the graph G.

Case 2: $u \neq v, v'$ and $u \notin N_G(v)$.
In this case $N_{G'}(u) = N_G(u)$ and so the claim follows.

Case 3: $u \neq v, v'$ and $u \in N_G(v)$.
Here $u \in N_G(v)$ and so the hypothesis of the theorem implies that the degree of u in G is greater than 1. Further, since f is a neighborhood-prime labeling of G, $gcd\{f(w) : w \in N_G(u)\} = 1$. But $N_{G'}(u) = N_G(u) \cup \{v'\}$ and g restricted to $V(G)$ is f. Hence, $gcd\{g(w) : w \in N_{G'}(u)\} = 1$. ∎

Now we show that the graph obtained by the duplication of an arbitrary vertex of cycle C_n or path P_n is neighborhood-prime.

Theorem 3.3. The graph obtained by the duplication of an arbitrary vertex of cycle C_n or path P_n is neighborhood-prime.

Proof: For $n \leq 2$, this is obvious and so we assume that $n > 2$. We first consider the case of cycle C_n. Note that if n is odd then the result follows by Theorem 2.4 and Theorem 3.2. Hence we assume that n is even. Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of C_n. Let C'_n be the graph obtained by duplication of an arbitrary vertex of C_n. Due to symmetry we may assume that C'_n is obtained by duplication of the vertex v_n. Let v'_n be the duplicated vertex of v_n. Define the function $g : \{v_1, v_2, \ldots, v_n, v'_n\} \to \{1, 2, 3, \ldots, n, n+1\}$ as follows:

$$\begin{aligned} g(v_{2i-1}) &= \frac{n}{2} + i, \quad 1 \leq i \leq \frac{n-2}{2} \\ g(v_{2i}) &= i, \qquad\quad 1 \leq i \leq \frac{n}{2} \\ g(v_{n-1}) &= n+1 \\ g(v'_n) &= n. \end{aligned}$$

We claim that g is a neighborhood-prime labeling by considering the following four cases. Let u be any vertex of C'_n.
Case 1: $u \neq v_1, v_n, v'_n, v_{n-2}$.
In this case $u = v_i$ for some i different from $1, n, n-2$ and so $N(u) = \{v_{i-1}, v_{i+1}\}$, where v_{i-1}, v_{i+1} are different from v_{n-1}. Now it follows from the definition of g that $g(v_{i-1})$ and $g(v_{i+1})$ are two consecutive integers and hence gcd of the labels of vertices in $N(u)$ is 1.
Case 2: $u = v_1$.
In this case $N(u)$ contains the vertex v_2. But $g(v_2) = 1$ and hence gcd of the labels of vertices in $N(u)$ is 1.
Case 3: $u = v_{n-2}$.
Here $N(u) = \{v_{n-3}, v_{n-1}\}$. But $g(v_{n-3}) = n-1$, $g(v_{n-1}) = n+1$ and n is even, so the gcd of the labels of vertices in $N(u)$ is 1.

Case 4: $u = v_n, v'_n$.
Here $N(u) = \{v_{n-1}, v_1\}$. But $g(v_{n-1}) = n + 1$ and $g(v_1) = \frac{n}{2} + 1$, and as a result the gcd of the labels of vertices in $N(u)$ is 1.

Now consider the case of path P_n. Let $v_1, v_2, \dots, v_n$ denote the consecutive vertices of path P_n and P'_n be the graph obtained by duplication of an arbitrary vertex of P_n. If this vertex is different from v_2 and v_{n-1} then the result follows by Theorem 2.2 and Theorem 3.2. Since the graphs obtained by duplication of the vertex v_2 and v_{n-1} respectively are isomorphic it is enough to prove the theorem for any one of them.

Let P'_n bethe graph obtained by duplication of the vertex v_2. Let v'_2 be the duplicated vertex of v_2 and f be a neighborhood-prime labeling of path P_n as in the proof of Theorem 2.2. Define the function $g : \{v_1, v_2, \dots, v_n, v'_2\} \to \{1, 2, 3, \dots, n, n + 1\}$ by defining $g(u) = f(u)$ if $u = v_j$ for $1 \le j \le n$ and $g(v'_2) = n + 1$. Using the fact that f is a neighborhood-prime labeling of path P_n and $g(v_2) = f(v_2) = 1$, it can be easily verified that g is a neighborhood-prime labeling of P'_n. ■

Example 3.4. Neighborhood-prime labelings of the graphs C'_6 and P'_{10} which are obtained by duplicating a vertex of C_6 and P_{10} respectively are shown in Figure 4(a) and 4(b).

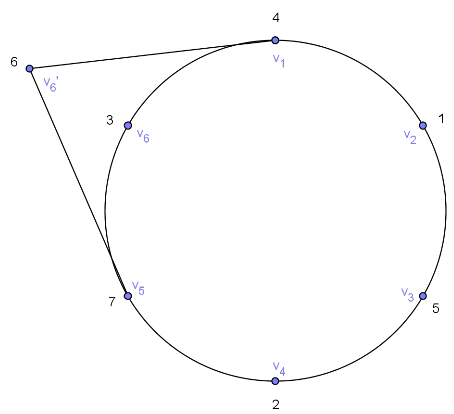

Figure 4(a): Neighborhood-prime labeling of the graph C'_6.

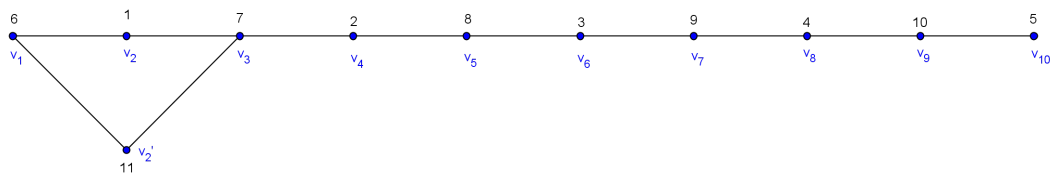

Figure 4(b): Neighborhood-prime labeling of the graph P'_{10}.

It has been shown by Seoud et al in [4] that the graph $K_{2,n}$ is prime for all n where as $K_{3,n}$ is prime for $n \neq 3, 7$. In the same paper they proved that $P_m + \overline{K_n}$ is not prime for $n \geq 3$ where as $P_m + \overline{K_2}$ is prime iff $m = 2$ or m is odd. We establish the following general result in order to show that such graphs are neighborhood- prime.

Theorem 3.5. Let G_1 and G_2 be graphs of order at least 2. Then the graph $G = G_1 + G_2$ is neighborhood-prime.

Proof: Suppose $V(G_1) = \{u_1, u_2, \ldots, u_m\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_n\}$. Define a function $f : V(G) \to \{1, 2, \ldots, m+n\}$ as follows:

$$\begin{aligned} f(u_i) &= i, & 1 \leq i \leq m \\ f(v_i) &= m+i, & 1 \leq i \leq n. \end{aligned}$$

Then f is a neighborhood-prime labeling because if w is an arbitrary vertex of G, then the set $\{f(p) : p \in N(w)\}$ contains at least two consecutive integers. ∎

In view of Theorem 3.5 and Theorem 2.1 (along with Remark 1.4), we can say that $K_{m,n}$, $P_m + \overline{K_n}$ and $P_m + K_n$ are neighborhood-prime graphs for all m and n.

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