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Eternal m-security in graphs

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Abstract

Eternal 1-secure set of a graph $G = (V, E)$ is defined as a set $S_0 \subseteq V$ that can defend against any sequence of single-vertex attacks by means of single guard shifts along edges of G. That is, for any k and any sequence $v_1, v_2, \ldots, v_k$ of vertices, there exists a sequence of guards $u_1, u_2, \ldots, u_k$ with $u_i \in S_{i-1}$ and either $u_i = v_i$ or $u_iv_i \in E$, such that each set $S_i = (S_{i-1} - \{u_i\}) \cup \{v_i\}$ is dominating. It follows that each S_i can be chosen to be an eternal 1-secure set. The *eternal 1-security number*, denoted by $\sigma_1(G)$, is defined as the minimum cardinality of an eternal 1-secure set. The *Eternal m-security* number $\sigma_m(G)$ is defined as the minimum number of guards to handle an arbitrary sequence of single attacks using multiple-guard shifts. In this paper we characterize the class of trees and split graphs for which $\sigma_m(G) = \gamma(G)$. We also characterize the class of trees, unicyclic graphs and split graphs for which $\sigma_m(G) = \beta(G)$.

Keywords: Eternal security, domination number, independence number.
AMS Subject Classification(2010): 05C69.

1 Introduction

Burger et al. [2, 3], introduced a dynamic form of domination which has been designated *eternal security* by Goddard et al. [6]. The concept calls for a fixed number of guards which are positioned on the vertices of a graph $G = (V, E)$, at most one to a vertex. A guard on a vertex w can respond to an attack at a vertex v by moving along an edge from w to v (assuming v does not already have a guard). Informally, if such a response can be made no matter what vertex is attacked and if the changing position of the guards can continue to respond forever, we say that the guards form an *eternally secure set.*

Two versions of the eternal security problem were considered. In the first version, which they call *1-security*, only one guard moves in response to an attack; in the second, which they call *m-security* all guards can move in response to an attack. The first version was introduced by Burger et al. [2, 3], though being able to withstand two attacks with a single-guard movement was explored in [4, 5, 10–12]. On the other hand, the idea that all guards may move in response to an attack appears to have been considered only in [12].

They defined an *eternal 1-secure* set of a graph $G = (V, E)$ as a set $S_0 \subseteq V$ that can defend against any sequence of single-vertex attacks by means of single-guard shifts along the edges of G. That is, for any k and any sequence $v_1, v_2, \ldots, v_k$ of vertices, there exists a sequence of guards $u_1, u_2, \ldots, u_k$ with $u_i \in S_{i-1}$ and either $u_i = v_i$ or $u_i v_i \in E$, such that each set $S_i = (S_{i-1} - \{u_i\}) \cup \{v_i\}$ is dominating. It follows that each S_i can be chosen to be an eternal 1-secure set. They defined the *eternal 1-security number*, denoted by $\sigma_1(G)$, as the minimum cardinality of an eternal 1-secure set. This parameter was introduced by Burger et al. [3] using the notation γ_∞.

In order to reduce the number of guards needed for eternal security, consider allowing more guards to move. Suppose that in responding to each attack, every guard may shift along an incident edge. The *eternal m-security* number $\sigma_m(G)$ is defined as the minimum number of guards to handle an arbitrary sequence of single attacks using multiple-guard shifts. A suitable placement of the guards is called an *eternal m-secure set*, call such a set a σ_m-set of G. They observed that $\sigma_m(G) \le \sigma_1(G)$, for all graphs G.

A set S is a *dominating set* if $N[S] = V(G)$ or equivalently, every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G, and a dominating set S of minimum cardinality is called a γ-set of G. A set S is a *2-dominating set* if every vertex in $V - S$ is dominated by at least two vertices in S. The minimum cardinality of a 2-dominating set is called the *2-domination number* $\gamma_2(G)$. A set S of vertices is called *independent* if no two vertices in S are adjacent. The *independence number* $\beta(G)$ is the maximum cardinality of a independent set in G.

Wayne Goddard et al. [6] have proved that $\gamma(G)$ and $\beta(G)$ are lower and upper bounds of $\sigma_m(G)$ respectively for any graph G. They have also proved that the *2-domination* number $\gamma_2(G)$ of a graph is also an upper bound for $\sigma_m(G)$. Further they have found the value of $\sigma_m(G)$ when G is a path, cycle, complete graph, and complete bipartite graph. In [13] we have obtained specific values of $\sigma_m(G)$ for certain classes of graphs, namely, grid graphs, binary trees, caterpillars, circulant graphs and generalized Petersen graphs. More results related to these parameters $\sigma_1(G)$ and $\sigma_m(G)$ are found in [1, 8, 9]. Wayne Goddard et al. [6] also have proved that $\sigma_m(G) = \gamma(G)$ when G is a Cayley graph and they have mentioned that $\sigma_m(G) = \gamma(G)$ is probably true for any vertex transitive graph. In this paper we give an example to disprove this statement. Further we characterize trees and split graphs for which $\sigma_m(G) = \gamma(G)$. We also characterize trees, unicylic graphs and split graphs for which $\sigma_m(G) = \beta(G)$.

2 Notations

Let $G = (V, E)$ be a simple and connected graph of order $|V| = n$. For graph theoretic terminology we refer to Harary [7]. For any vertex $v \in V$, the *open neighbourhood* of v is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighbourhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood is $N[S] = N(S) \cup S$. The *external private neighbourhood* $epn(v, S)$ of a vertex $v \in S$ is defined by $epn(v, S) = \{u \in V - S : N(u) \cap S = \{v\}\}$. For any graph G, $\delta(G) = \min\{deg\ v : v \in V(G)\}$ and $\Delta(G) = \max\{deg\ v : v \in V(G)\}$.

A vertex of degree one in a graph is a *pendant vertex.* A vertex of G adjacent to pendant vertices is called a *support.* We call a support vertex adjacent to exactly one pendant vertex a *weak support* and a support vertex adjacent to at least two pendant vertices a *strong support.*

A *unicylic graph* is a graph with exactly one cycle. A connected graph having no cycle is called a *tree.* A *rooted tree* is a tree in which one of the vertices is distinguished from others. The distinguished vertex is called the *root* of the tree.

A graph G is *k-partite*, $k \geq 1$ if it is possible to partition $V(G)$ into k subsets $V_1, V_2, \ldots, V_k$ (called partite set) such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i \neq j$. If G is a 1-*partite* graph of order n, then $G = \overline{K_n}$. For $k = 2$, such graphs are called *bipartite* graphs.

A *split graph* is a graph G, whose vertices can be partitioned into X and Y, where the vertices in X are independent and the vertices in Y form a complete graph. For $v \in Y$, $N_X(v)$ denotes the neighbours of v in X.

A graph G is *vertex-transitive* if and only if for any two vertices u and v of G, there exists an *automorphism* ϕ of G such that $\phi\ (u) = v$.

3 Eternal m-Security on Petersen Graph

Wayne Goddard et al. [6] have mentioned that $\sigma_m(G) = \gamma(G)$ is probably true for every vertex-transitive graph. Here we prove that for the *Petersen graph* G, which is a vertex transitive graph, $\sigma_m(G) > \gamma(G)$.

Theorem 3.1. For the Petersen graph G, $\sigma_m(G) > \gamma(G)$.

Proof: Consider the Petersen graph G. Let u_i, v_i, $1 \leq i \leq 5$ be the vertices on the inner and outer cycles of G respectively. We know that $\gamma(G) = 3$ and any γ-set of G contains either 2 vertices from the inner cycle and 1 vertex from the outer cycle or 2 vertices from the outer cycle and 1 vertex from the inner cycle.

Without loss of generality, let $S = \{v_4, u_1, u_2\}$ be a γ-set of G. Here $u_4 \in V - S$ is a non-private neighbour of S. If there is an attack at u_4, then the guard at either v_4 or u_1 or u_2

responds to it. In each case, there are two possibilities of movements of guards. We list the possibilities in each case separately.

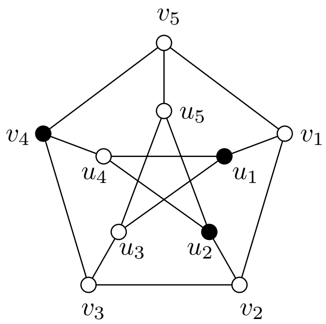

Figure 1: Petersen Graph with $\sigma_m(G) > \gamma(G)$.

Case (i): The guard at v_4 moves to u_4.
(a) $v_4 \to u_4$, $u_1 \to v_1$, $u_2 \to v_2$.
(b) $v_4 \to u_4$, $u_1 \to u_3$, $u_2 \to v_2$.
In (a), u_3 and u_5 are left undefended whereas in (b), v_5 is undefended.
(By an *undefended vertex* we mean a vertex which is neither equipped with a guard nor adjacent to a vertex which is equipped with a guard).
Case (ii): The guard at u_1 moves to u_4.
(a) $u_1 \to u_4$, $v_4 \to v_5$, $u_2 \to v_2$.
(b) $u_1 \to u_4$, $v_4 \to v_3$, $u_2 \to u_5$.
In (a), u_3 is left undefended and in (b), v_1 is undefended.
Case (iii): The guard at u_2 moves to u_4.
(a) $u_2 \to u_4$, $u_1 \to v_1$, $v_4 \to v_3$.
(b) $u_2 \to u_4$, $u_1 \to u_3$, $v_4 \to v_5$.
In (a), u_5 is left undefended and in (b), v_1 is left undefended.

In all the above cases, we see that $\sigma_m(G) > \gamma(G)$. In fact $\sigma_m(G) = 4$. To prove $\sigma_m(G) = 4$. G contains two cycles $C_1 = (u_1, u_2, u_3, u_4, u_5, u_1)$ and $C_2 = (v_1, v_2, v_3, v_4, v_5, v_1)$ and we know that $\sigma_m(C_n) = \lceil \frac{n}{3} \rceil$. We have $\sigma_m(C_1) = \lceil \frac{5}{3} \rceil$ and $\sigma_m(C_2) = \lceil \frac{5}{3} \rceil$. Hence $\sigma_m(G) = \sigma_m(C_1) + \sigma_m(C_2) = \lceil \frac{5}{3} \rceil + \lceil \frac{5}{3} \rceil = 4$. ∎

Remark 3.2. Let G_n be the graph obtained from the Petersen graph by expanding each vertex to a complete graph K_n. Then G_n is a vertex-transitive graph with $\sigma_m(G_n) > \gamma(G_n)$. Therefore there exist infinitely many vertex-transitive graphs for which $\sigma_m(G) > \gamma(G)$.

We give below a list of theorems proved by Wayne Goddard et al. [6] which will be useful for our study.

Theorem 3.3. [6] For any graph G, $\gamma(G) \leq \sigma_m(G) \leq \beta(G)$.

Theorem 3.4. [6]

(a) $\sigma_m(K_n) = 1$

(b) $\sigma_m(K_{r,s}) = 2$ for $r, s \geq 1$, $r + s \geq 3$.

(c) $\sigma_m(P_n) = \lceil \frac{n}{2} \rceil$.

(d) $\sigma_m(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.

Theorem 3.5. [6] For any Cayley graph G, $\sigma_m(G) = \gamma(G)$.

4 Graphs with $\sigma_m(G) = \gamma(G)$

First we prove that in a split graph G, $\sigma_m(G) = \gamma(G)$ or $\sigma_m(G) = \gamma(G) + 1$.

Theorem 4.1. For any split graph G, $\sigma_m(G) = \gamma(G)$ or $\sigma_m(G) = \gamma(G) + 1$.

Proof: Let G be a split graph with bipartition (X, Y) with X is independent and $G[Y]$ is complete and $|X| = m$ and $|Y| = n$.

If $deg(y) = n$ for each $y \in Y$ then $\sigma_m(G) = \gamma(G) = m$. Otherwise, let S be any γ-set of G. Then $S \cap Y \neq \emptyset$ and clearly $|epn(v, S)| \geq 1$ for all $v \in S \cap Y$. Further a member of $S \cap X$ and a member of $S \cap Y$ cannot be adjacent. Suppose $|epn(w, S)| \geq 2$ for some $w \in S \cap Y$ and if there is an attack at $z \in epn(w, S)$, then the guard at w responds to it whereas the members of $epn(w, S) - \{z\}$ are left undefended which implies that $\sigma_m(G) > \gamma(G)$.

Suppose $|epn(v, S)| = 1$ for all $v \in S \cap Y$. In this case, there exist at least two vertices $z_1, z_2 \in S \cap Y$ such that z_1 and z_2 have a common neighbour say z. Clearly $z \notin S$. If there is an attack at z, none of the members in S can respond to the attack, (since $|epn(v, S)| = 1$ for all $v \in S \cap Y$ and z is a non-private neighbour of S) which implies that $\sigma_m(G) > \gamma(G)$. Now, we claim that $\sigma_m(G) = \gamma(G) + 1$. Choose a γ-set S such that $S \subseteq Y$. If $S = Y$ then $S' = S \cup \{z\}$ is a σ_m-set of G for some $z \in X$. For, if there is an attack at some vertex $x \in X$, then the guard at the vertex $y \in Y$ which is adjacent to x responds to it. If z and y are adjacent, then the guard at z moves to y. If z and y are not adjacent, then the guard at z moves to some vertex $w \in Y$ which is adjacent to z and the guard at w moves to y. Hence the guards in S' can respond eternally to any sequence of attacks.

If $S \subset Y$ then $S' = S \cup \{z\}$ is a σ_m-set of G for some $z \in Y - S$. For, if there is an attack at $u \in Y \cap (V - S')$ then the guard at z responds to the attack and if there is an attack at some member of $epn(v, S)$ say v_1,then the guard at v responds to the attack and the guard at u moves to v to protect the rest of the external private neighbours of v. Further, if there is a subsequent attack at another member of $epn(v, S)$ (if any) then the guard at v responds to the attack while the guard at v_1 moves back to its original position (Refer figure 2). Hence the guards in S' can respond eternally to any sequence of attacks. Hence $\sigma_m(G) = \gamma(G) + 1$. ∎

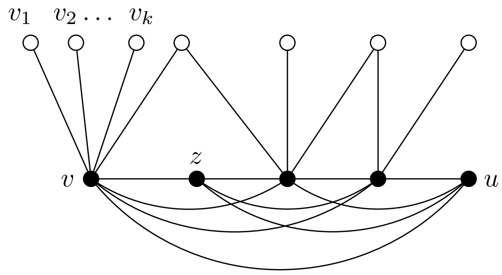

Figure 2: A Graph illustrating the proof of Theorem 4.1.

Now we characterize trees T, for which $\sigma_m(T) = \gamma(T)$.

Theorem 4.2. For any tree T, $\sigma_m(T) = \gamma(T)$ if and only if every vertex of degree at least two is a weak support.

Proof: Suppose $\sigma_m(T) = \gamma(T)$. Let S be any γ-set of T such that S contains all the supports of T. Since $\sigma_m(T) = \gamma(T)$, S is a σ_m-set of T. Let $v \in V(T)$ such that $deg(v) > 1$. If v is a strong support of T, then clearly $v \in S$. Since S is a γ-set, for any $z \in N(v) \cap S$, $epn(z, S) \neq \emptyset$. Hence, if there is an attack at a leaf say x adjacent to v then the guard at v responds to it where as none of the guards at $N(v) \cap S$ can move to v in which case all the leaves other than x adjacent to v are left undefended. Hence $\sigma_m(G) > \gamma(G)$, a contradiction.

Now we claim that v is a weak support. Suppose not. Since $deg(v) > 1$, there exist two vertices v_1 and v_2 which are adjacent to v. Suppose $v \in S$. Then we have the following cases.
Case (i): $|epn(v, S)| \geq 1$.
If both $v_1, v_2 \in epn(v, S)$ and there exists a vertex $w \in S$ at a distance 2 from v_2. Now, suppose there is a sequence of attacks at v_1 and v_2 then the guard at v moves to v_1 and the guard at w moves two steps to respond to the attack at v_2. Further there exists a (v_2, z)- path say Q in T such that $deg(z) = 1$ and the vertex adjacent to z in Q is of degree two and while the guard at w responds to the attack at v_2 and other guards at the vertices of $(S \cap Q) - \{v\}$ move either one or two steps towards v_2 which leaves the vertex z undefended which implies that $\sigma_m(T) > \gamma(T)$, a contradiction.

Suppose $v_1 \in epn(v, S)$ and $v_2 \notin epn(v, S)$ and if there is a sequence of attacks at v_1 and v_2 then the guard at v moves to v_1 and subsequently the guards at the vertices of $S \cap Q$ move one step towards v_2, which leaves z undefended. Hence $\sigma_m(T) > \gamma(T)$, a contradiction.
Case (ii): $epn(v, S) = \emptyset$.
Suppose there is an attack at v_1 and some member y $(\neq v)$ in S with $deg(y) > 1$ has to respond to this attack then as in Case (i),there exists a (v_1, z') - path say Q' in T such that $deg(z') = 1$ and the vertex adjacent to z' in Q' is of degree two and while the guard at y responds to the attack at v_1 and the other guards at the vertices of $S \cap Q'$ move one step towards v_1 which will

leave z' undefended, which implies that $\sigma_m(T) > \gamma(T)$, a contradiction.

Now, if v has to respond to the attack at v_1 and in addition if there is a second attack at v_2 then as in Case (i) the vertex z of Q is left undefended, a contradiction. Similarly if there is an attack at v_2 we get a contradiction. Hence v is a weak support. Converse of the theorem is obvious. ∎

5 Graphs with $\sigma_m(G) = \beta(G)$

In this section, we characterize the class of trees, unicyclic graphs and split graphs for which $\sigma_m(G) = \beta(G)$. For this purpose we first introduce two families of graphs $\mathcal{T}_1$ and $\mathcal{T}_2$ as follows.

Let G be a graph with $\delta(G) = 1$. We prune the vertices of G as follows. Corresponding to each support vertex u, remove u and exactly one pendant vertex adjacent to u. Let G' be the resulting graph. Again corresponding to each support vertex w, remove w and one pendant vertex adjacent to w. Let G'' be the resulting graph. Repeat the above process until no such vertices remain. Let G^* be the final graph.

Now we define a family $\mathcal{T}_1$ of trees as follows. A tree $T \in \mathcal{T}_1$ if either $T^* = \emptyset$ or $T^* \cong K_1$ where T^* is obtained from T as discussed above (Refer Figure 5). We also define a family of unicyclic graphs $\mathcal{T}_2$ as follows. A unicyclic graph $G \in \mathcal{T}_2$ if either $G^* = \emptyset$ or $G^* \cong K_1$ or $G^* \cong C_n$, $n = 3, 4, 5$ or 7 (Refer Figure 5).

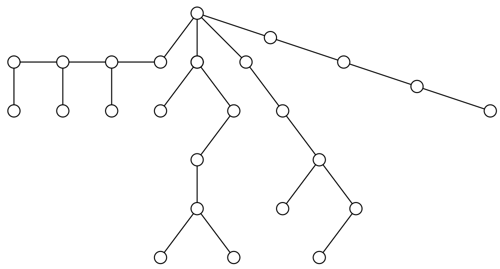

Figure 3: A tree $T \in \mathcal{T}_1$.

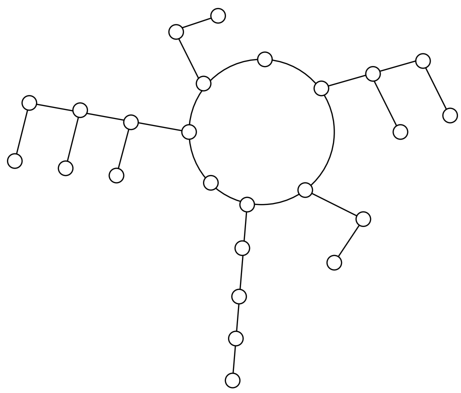

Figure 4: A unicyclic graph $G \in \mathcal{T}_2$.

Remark 5.1. We see that any graph $G \in \mathcal{T}_1 \cup \mathcal{T}_2$ has at most one strong support with exactly two leaves.

Theorem 5.2. For any tree T, $\sigma_m(T) = \beta(T)$ if and only if $T \in \mathcal{T}_1$.

Proof: If $T \in \mathcal{T}_1$, then by placing guards at each of the pendant vertices of the different trees obtained during the pruning and a guard at T^* when $T^* \cong K_1$, we see that these guards can safeguard the corresponding pendant vertices and the neighbouring support vertices.

Let S be the set of all pendant vertices removed during the pruning, then S or $S \cup \{w\}$ is a σ_m-set according as $T^* = \emptyset$ or $T^* \cong K_1$, $V(K_1) = \{w\}$. Hence $\sigma_m(T) = \beta(T)$.

Conversely, suppose $\sigma_m(T) = \beta(T)$. Let T^* be the final tree obtained by pruning T successively as in the definition of T^*. We claim that $T^* = \emptyset$ or $T^* \cong K_1$. Suppose not. Then T^* contains at least two vertices say x_1 and x_2. Let T_1 be the graph obtained from T by deleting the vertices x_1 and x_2. Now $\beta(T_1) = \sigma_m(T_1)$ and $\beta(T) = \beta(T_1) + 2$.

Case (i): x_1 and x_2 have a common neighbour.

Let $v \in N(x_1) \cap N(x_2)$ then there exists a vertex z which is adjacent to v in some T_1 such that removal of v and z would have left x_1 and x_2 isolated. (Refer Figure 5). In this case

$$\begin{aligned} \sigma_m(T) &= \sigma_m(T_1) + 1 \\ &\leq \beta(T_1) + 1 \\ &\leq \beta(T) - 2 + 1 \\ &\leq \beta(T) - 1 \\ \textit{Hence } \sigma_m(T) &< \beta(T). \end{aligned}$$

Case (ii): x_1 and x_2 do not have a common neighbour.

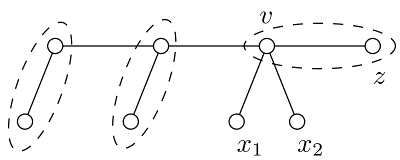

Figure 5: A subtree of T illustrating Case (i) of Theorem 5.2.

Let P be the (v_1, v_2)-path in T, where v_1, v_2 are the neighbours of x_1 and x_2 respectively. Let w_1 and w_2 be the vertices adjacent to v_1 and v_2 respectively in P. Clearly w_1 and w_2 are pruned already (otherwise by case (i) we get a contradiction) (Refer figure 5).

Let S be any σ_m-set of T_1. Further the length of P is an even number say k. Hence S contains $k/2$ vertices of P. Without loss of generality, let $v_1, v_2 \in S$. Now either $S' = S \cup \{x_1\}$ (or $S \cup \{x_2\}$) is a σ_m-set of T. For, if there is an attack at x_2 (or x_1), the guard at v_2 (or v_1)

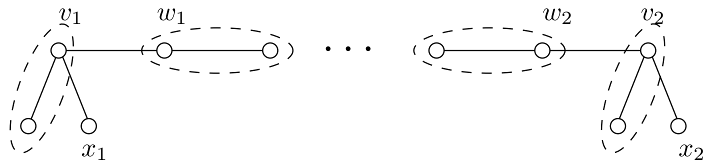

Figure 6: A subtree of T illustrating Case (ii) of Theorem 5.2.

will respond to it and the other guards in S' move in such a way that no vertex is undefended. Hence $\sigma_m(T) = \sigma_m(T_1) + 1$. This implies that $\sigma_m(T) < \beta(T)$. Hence in both the cases we get a contradiction. So either $T^* = \emptyset$ or $T^* \cong K_1$. Hence $T \in \mathcal{T}_1$. ■

We need the following Lemma to characterize unicyclic graphs G for which $\sigma_m(G) = \beta(G)$.

Lemma 5.3. For cycles C_n, $\sigma_m(C_n) = \beta(C_n)$ iff $n = 3, 4, 5, 7$.

Proof: Proof follows from Theorem 3.4(d) and the fact that $\beta(C_n) = \lfloor \frac{n}{2} \rfloor$. ■

Theorem 5.4. For any unicyclic graph G with $\Delta(G) \geq 3$, $\sigma_m(G) = \beta(G)$ if and only if $G \in \mathcal{T}_1 \cup \mathcal{T}_2$.

Proof: Let $C_n = \{v_1, v_2, \ldots, v_n, v_1\}$ be the unique cycle in G. Suppose $G \in \mathcal{T}_1 \cup \mathcal{T}_2$.

Let S be the set of all pendant vertices removed during the pruning of G and let G^* be the resulting graph.

Case (i): If $G^* = \emptyset$, then S is a σ_m-set of G.

Case (ii): If $G^* \cong K_1$, then $S \cup \{x\}$ is a σ_m-set of G where $V(K_1) = \{x\}$.

Case (iii): If $G^* \cong C_n$, $n = 3, 4, 5, 7$ then define

$$S' = \begin{cases} S \cup \{v_1\} & \text{if } n = 3 \\ S \cup \{v_1, v_3\} & \text{if } n = 4, 5 \\ S \cup \{v_1, v_3, v_5\} & \text{if } n = 7. \end{cases}$$

Hence S' is a σ_m-set of G. In all the above cases, we clearly see that the respective σ_m-sets are the β-sets of G. Hence $\sigma_m(G) = \beta(G)$.

Conversely, suppose $\sigma_m(G) = \beta(G)$. We claim that either $G^* = \emptyset$ or $G^* \cong K_1$ or $G^* \cong C_n$, $n = 3, 4, 5, 7$. Suppose not. Then either G^* contains at least two vertices and does not have a cycle or G^* is a cycle C_k with $k \neq 3, 4, 5, 7$. In the former case, as in the proof of Theorem 5.2, we get a contradiction. In the latter case, we see that by Lemma 5.3, $\sigma_m(G^*) < \beta(G^*)$, a contradiction. ■

Next we characterize split graphs for which $\sigma_m(G) = \beta(G)$.

For a split graph G with bipartition (X, Y) where X is independent and $G[Y]$ is complete, we define two families $\mathcal{G}_1$ and $\mathcal{G}_2$ as follows,

A split graph $G \in \mathcal{G}_1$ if each vertex in Y has at most one neighbour in X.

A split graph $G \in \mathcal{G}_2$ if the following conditions hold.

(i) $deg(y) \leq |Y| + 1$ for every $y \in Y$.

(ii) If $Y' = \{y \in Y | deg(y) = |Y| + 1\}$, then $Y' \neq \emptyset$ and for some vertex $v \in Y'$, every vertex in $Y - \{v\}$ has exactly one neighbour in $X - N_X(v)$. If Y' has a support then v is so chosen such that v is a support.

Theorem 5.5. Let G be a split graph with bipartition (X, Y) where X is independent and $G[Y]$ is complete. Then $\sigma_m(G) = \beta(G)$ if and only if either $G \in \mathcal{G}_1$ or $G \in \mathcal{G}_2$.

Proof: Suppose either $G \in \mathcal{G}_1$ or $G \in \mathcal{G}_2$. If $G \in \mathcal{G}_1$ then each vertex in Y has at most one neighbour in X. Hence, clearly $\sigma_m(G) = \beta(G) = |X|$, if $deg(y) = |Y|$ for every $y \in Y$ and $\sigma_m(G) = \beta(G) = |X| + 1$, otherwise.

If $G \in \mathcal{G}_2$, let $Y' = \{y \in Y | deg(y) = |Y| + 1\}$. Now every vertex in $Y - \{v\}$ has exactly one neighbour in $X - N_X(v)$ where $v \in Y'$ is a vertex as mentioned in the definition of $\mathcal{G}_2$. Hence $\sigma_m(G) = 2 + |X| - 2 = |X| = \beta(G)$.

Conversely, suppose $\sigma_m(G) = \beta(G)$. If each vertex in Y has at most one neighbour in X, then $G \in \mathcal{G}_1$. Otherwise, let $Y' = \{y \in Y | deg(y) = |Y| + 1\}$, clearly $Y' \neq \emptyset$. Let $v \in Y'$. If Y' has a support, then v is so chosen such that v is a support. Now we claim that each vertex in $Y - \{v\}$ has exactly one neighbour in $X - N_X(v)$.

In this case, either $\beta(G) = |X| + 1$ or $|X|$ and $\sigma_m(G) = 2 + k$, $k \geq 0$.

If $\beta(G) = |X| + 1$ then $\sigma_m(G) \leq 2 + |X| - 2 = |X|$, which is a contradiction. If $\beta(G) = |X|$, then $\sigma_m(G) = \beta(G)$ which implies that $k = |X| - 2$. This is possible only when, each vertex in $Y - \{v\}$ has exactly one neighbour in $X - N_X(v)$. Hence $G \in \mathcal{G}_2$. ∎

References

[1] M. Anderson, C. Barrientos, R. Brigham, J. Carrington, R. Vitray and J. Yellan, *Maximum demand graphs for eternal security*, J. Combin. Math. Combin. Comput., 61 (2007), 111–128.

[2] A.P. Burger, E.J. Cockayne, W.R. Grundlingh, C.M. Mynhardt, J.H. Van Vuuren and W. Winterbach, *Finite order domination in graphs*, J. Combin. Math. Combin. Comput., 49 (2004), 159–175.

[3] A.P. Burger, E.J. Cockayne, W.R. Grundlingh, C.M. Mynhardt, J.H. Van Vuuren and W. Winterbach, *Infinite order domination in graphs*, J. Combin. Math. Combin. Comput., 52 (2005), 169–180.

[4] E.J. Cockayne, O. Favaron and C.M. Mynhardt, *Secure domination, weak Roman domination and forbidden subgraphs*, Bull. Inst. Combin. Appl., 39 (2003), 87–100.

[5] E.J. Cockayne, P.J.P. Grobler, W.R. Grundlingh, J. Munganga and J.H. Van Vuuren, *Protection of a graph*, Utilitas Math., 67 (2005), 19–32.

[6] W. Goddard, S.M. Hedetniemi and S.T. Hedetniemi, *Eternal security in graphs*, J. Combin. Math. Combin. Comput., 52 (2005), 169–180.

[7] F. Harary, *Graph theory*, Addison-Wesley Publ. Comp., Reading, Massachusettes, 1969.

[8] W.F. Klostermeyer and G. Mac Gillivray, *Eternal security in graphs of fixed independence number*, J. Combin. Math. Combin. Comput., 63 (2007).

[9] Micheal P. Baldado Jr. and Sergio R. Canoy Jr., *Eternal 1-security number of the corona and cartesian product of graphs*, International Mathematical Forum, Vol. 6 (2011), No. 12, 557–564.

[10] C.M. Mynhardt, H.C. Swart and E. Ungener, *Excellent trees and secure domination*, Utilitas Math., 67 (2005), 255–267.

[11] D. Ochmanek, *Time to restructure U.S. defense forces*, Issues in Science and Technology, Winter (1996).

[12] A. Pagourtzis, P. Penna, K. Schlude, K. Steinhofel, D.S. Taylor and P. Widmayer, *Server placements, Roman domination and other dominating set variants*, 2^{nd} IFIP International Conference on Theoretical Computer Science, Montreal (2002), 280–291.

[13] P. Roushini Leely Pushpam and G. Navamani, *Eternal m-security in certain classes of graphs*, J.Combin. Math. Combin. Comput.-to appear