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Ascending domination decomposition of subdivision of graphs

K. Lakshmiprabha[1], K. Nagarajan[2]

[1] Research Scholar
Department of Mathematics, Sri S.R.N.M.College
Sattur - 626 203, Tamil Nadu, India.
prabhalakshmi95@gmail.com

[2] Department of Mathematics, Sri S.R.N.M.College
Sattur - 626 203, Tamil Nadu, India.
k_nagarajan_srnmc@yahoo.co.in

Abstract

In this paper, the two major fields of graph theory namely decomposition and domination are connected and new concept called Ascending Domination Decomposition (ADD) of a graph G is introduced. An ADD of a graph G is a collection $\psi = \{G_1, G_2, \ldots, G_n\}$ of subgraphs of G such that, each G_i is connected, every edge of G is in exactly one G_i and $\gamma(G_i) = i$, $1 \leq i \leq n$. In this paper, we prove the subdivision of some standard graphs admit ADD.

Keywords: Domination, decomposition, ascending domination decomposition, subdivision of graphs.
AMS Subject Classification(2010): 05C69, 05C70.

1 Introduction

By a graph, we mean a finite, undirected, non-trivial, connected graph without loops and multiple edges. The order and size of a graph are denoted by p and q respectively. For terms not defined here we refer to Harary [3] .

The theory of domination is one of the fast growing areas in graph theory, which has been investigated by Walikar et. al. [7]. A set $D \subseteq V$ of vertices in a graph G is a dominating set if every vertex v in $V - D$ is adjacent to a vertex in D. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$.

Another important area in graph theory is decomposition of graphs. A decomposition of a graph G is a collection ψ of edge disjoint subgraphs $G_1, G_2, \ldots, G_n$ of G such that every edge of G is in exactly one G_i. If each G_i is isomorphic to a subgraph H of G, then ψ is called a H-decomposition . Several authors studied various types of decompositions by imposing conditions on G_i in the decomposition.

Using these two concepts, we introduced the concept called Ascending Domination Decomposition (ADD) [5] of a graph which is motivated by the concepts of Ascending Subgraph Decomposition (ASD) and Continuous Monotonic Decomposition (CMD) of a graph. The concept of Ascending Subgraph Decomposition was introduced by Alavi et al [1].

Definition 1.1. A decomposition of G into subgraphs G_i (not necessarily connected) such that $|E(G_i)| = i$ and G_i is isomorphic to a proper subgraph of G_{i+1} is called an Ascending subgraph decomposition.

Now we define Ascending Domination Decomposition (ADD) as follows.

Definition 1.2. [5] An ADD of a graph G is a collection $\psi = \{G_1, G_2, \ldots, G_n\}$ of subgraphs of G such that

(i) Each G_i is connected.

(ii) Every edge of G is in exactly one G_i and

(iii) $\gamma(G_i) = i$, $1 \leq i \leq n$.

If a graph G has an ADD, we say that G admits ADD.

Example 1.3. A graph G and its ADD are given in Figure 1(a) and 1(b) respectively.

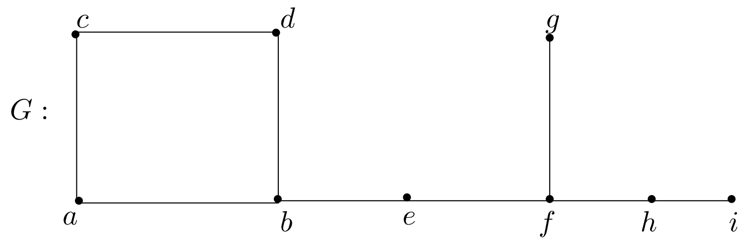

Figure 1(a)

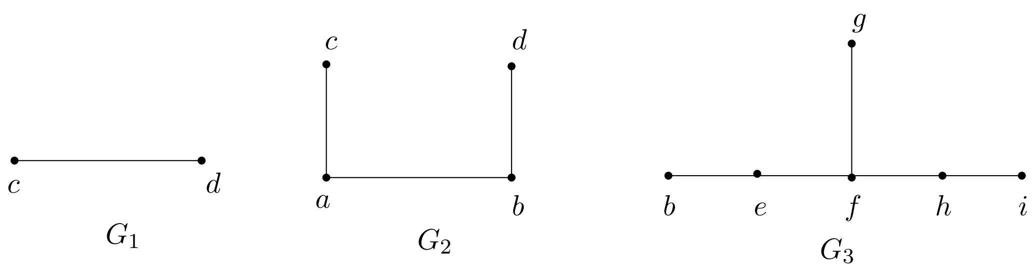

Figure 1(b): ADD $\{G_1, G_2, G_3\}$ of G given in Figure 1(a).

We proved K_p, W_p, $K_{m,k}$, P_p, C_p, path corona P_p^+, cycle corona C_p^+ and star corona $K_{1,p-1}^+$ admit ADD with certain conditions in [5].

Definition 1.4. A subdivision of a graph G is a graph obtained by inserting a new vertex in each edge of G and is denoted by $S(G)$.

Example 1.5. Subdivision of the complete graph on 5 vertices, $S(K_5)$ is given in Figure 2.

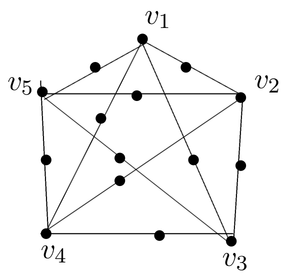

Figure 2: Subdivision graph $S(K_5)$.

Definition 1.6. The corona $G1 \odot G2$ of two graphs G_1 and G_2 is defined as a graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.

The following theorem is used to prove that the subdivision of a path admits ADD into n-parts.

Theorem 1.7. [5] A path P_p has an ADD $\psi = \{G_1, G_2, \ldots, G_n\}$ if and only if $\frac{3n^2-3n+2}{2} \leq q \leq \frac{3n^2+n}{2}$.

First we see that the subdivision of a path admits ADD into n-parts.

Note 1.8. In general, if P_p admits ADD, then $S(P_p)$ need not admit ADD and vice - versa, because we cannot apply the range of q as in Theorem 1.7 to $S(P_p)$.

2 Main results

Theorem 2.1. Let P_p be a (p,q)- path. $S(P_p)$ has an ADD $\psi=\{G_1, G_2, \ldots, G_n\}$ if and only if $\frac{n^2+3n-8}{2} \leq q \leq \frac{n^2+5n-10}{2}$.

Proof: Let $P_p = v_1\, v_2\, v_3 \ldots v_p$ be a path. Then $S(P_p) \cong P_{2p-1}$ has $2p-2$ edges.

Suppose $S(P_p) \cong P_{2p-1}$ admits ADD $\psi=\{G_1, G_2, \ldots, G_n\}$. From Theorem 1.7, P_p admits ADD if and only if $\frac{3n^2-3n+2}{2} \leq q \leq \frac{3n^2+n}{2}$. Now we can find the range of q if and only if $S(P_p)$ admits ADD.

From Note 1.8, we cannot apply the above range for q in P_p to $2q$ in $S(P_p)$. Hence using the range of q in P_p as in Theorem 1.7 to $S(P_p)$, we have the following possibilities for $S(P_p)$ to admit ADD, which is shown in Table 1.

Table 1

No. of decompositions n	No. of edges in P_p	No. of edges in $S(P_p)$
1	1	2
2	2 3	4 6
3	5 6 7	10 12 14
4	10 11 12 13	20 22 24 26
5	16 17 18 19 20	32 34 36 38 40
6	23 24 25 26 27 28	46 48 50 52 54 56
7	31 32 33 34 35 36 37	62 64 66 68 70 72 74

We find the upper and lower bound of q for P_p such that $S(P_p)$ admits ADD, using Newton's forward difference table.

First we find the lower bound of q using Table 1.

Table 2

n	q	Δq	$\Delta^2 q$	$\Delta^3 q$
3	5			
		5		
4	10		1	
		6		0
5	16		1	
		7		0
6	23		1	
		8		
7	31			

$$\begin{aligned} n &= n_0 + xh \\ &= x + 3 \\ x &= n - 3 \\ q &= q_0 + x\ \Delta q_0 + \tfrac{x(x-1)}{2!}\ \Delta^2 q_0 + \ldots \\ &= 5 + (n-3)\ (5) + \tfrac{(n-3)(n-4)}{2!}\ (1) \\ &= \tfrac{n^2+3n-8}{2}. \end{aligned}$$

Next we find the upper bound of q using Table 1.

Table 3

n	q	Δq	$\Delta^2 q$	$\Delta^3 q$
3	7			
		6		
4	13		1	
		7		0
5	20		1	
		8		0
6	28		1	
		9		
7	37			

$$\begin{aligned} n &= n_0 + xh \\ &= x + 3 \\ x &= n - 3 \\ q &= q_0 + x\ \Delta q_0 + \tfrac{x(x-1)}{2!}\ \Delta^2 q_0 + \ldots \\ &= 7 + (n-3)(6) + \tfrac{(n-3)(n-4)}{2!}\ (1) \\ &= \tfrac{n^2+5n-10}{2}. \end{aligned}$$

We see that, if $S(P_p)$ an ADD, then $\frac{n^2+3n-8}{2} \leq q \leq \frac{n^2+5n-10}{2}$.

Conversely, suppose $S(P_p)$ does not admit ADD.
Consider the upper bound for q as $q \leq \frac{n^2+5n-10}{2}$.
First we prove, if we add 1 or 2 edges to the upper bound for q, then it would not admit an ADD.

For, if $q = \frac{n^2+5n-10}{2} + 1$ or $q = \frac{n^2+5n-10}{2} + 2$, then we have extra 1 or 2 edges in any one or two $G_i{}^{'}s$. Then $\gamma(G_i) \neq i$ for one or two $i^{'}s$.
This gives a contradiction to our assumption that $q \leq \frac{n^2+5n-10}{2}$.
Now, we prove, if we reduce 1 or 2 from the lower bound for q, then it would not admit an ADD.

For, if we have $q = \frac{n^2+3n-8}{2} - 1$ or $q = \frac{n^2+3n-8}{2} - 2$, then we remove 1 or 2 edges in one or two $G_i{}^{'}s$. Then we have $\gamma(G_i) \neq i$ for one or two $i^{'}s$. Even if the edges are arranged in all possible ways, it would not admit an ADD.
This gives a contradiction to our assumption that $q \geq \frac{n^2+3n-8}{2}$.
Thus, $S(P_p)$ admits an ADD. ■

Note 2.2. If we add more than 3 edges, then the upper bound for q becomes $\frac{n^2+5n-10}{2} + 3 = \frac{n^2+5n-4}{2} = \frac{(n+1)^2+3(n+1)-8}{2}$, which is the lower bound for q in which $S(P_p)$ admits ADD into $(n+1)$- parts.

Theorem 2.3. Let C_p be a (p,q)- cycle. $S(C_p)$ has an ADD $\psi = \{G_1, G_2, \ldots, G_n\}$ if and only if $\frac{n^2+3n-8}{2} \leq q \leq \frac{n^2+5n-10}{2}$.

Proof: The proof is same as in Theorem 2.1 ■

Theorem 2.4. If $p = \frac{n^2+n}{2}$, then $S(K_p)$ admits ADD into n - parts.

Proof: Let $V(K_p) = \{v_1, v_2, \ldots, v_p\}$. Let v_1', v_2', …, $v'_{\frac{p(p-1)}{2}}$ be the subdivision of the edges of K_p.

Suppose $p = \frac{n(n+1)}{2}$

Let $G_1 = < N[v_1]$

$G_2 = < N[v_2, v_3] >$

$G_3 = < N[v_4, v_5, v_6] >$

⋮

$G_n = < N[v_l, \ldots v_p] >$, where $l = \frac{n^2-n+2}{2}$, $p = \frac{n(n+1)}{2}$

Here $\frac{n^2-n+2}{2}$ can be found using Newton's forward difference formula as follows.

Table 4

n	l	Δl	$\Delta^2 l$	$\Delta^3 l$
1	1			
		1		
2	2		1	
		2		0
3	4		1	
		3		0
4	7		1	
		4		
5	11			

$n = n_0 + xh$
$= x + 1$
$x = n - 1$
$l = l_0 + x \ \Delta l_0 + \frac{x(x-1)}{2!} \Delta^2 l_0 + \ldots$
$= 1 + (n-1)\ (1) + \frac{(n-1)(n-2)}{2!}\ (1)$
$= \frac{n^2-n+2}{2}$.

Here G_n is a neighbourhood of $(\frac{n^2+n}{2})$-$(\frac{n^2-n+2}{2} - 1) = n$ vertices.

From the above decomposition, we see that $\gamma(G_i) = i$ and $\psi = \{G_1, G_2, \ldots, G_n\}$ is an ADD. ■

Example 2.5. ADD of subdivision of a complete graph is given as follows.

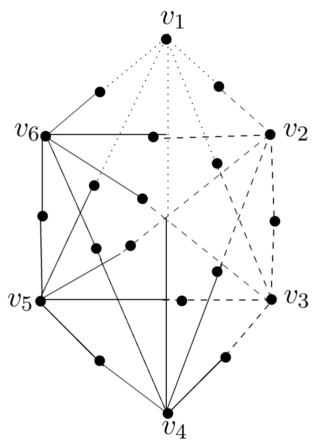

Figure 3: ADD of SK_p.

Here,

G_1 is an edge induced subgraph whose edges are denoted by dotted lines,

G_2 is an edge induced subgraph whose edges are denoted by dashed lines,

G_3 is an edge induced subgraph whose edges are denoted by dash parameters and

G_4 is an edge induced subgraph whose edges are denoted by plain lines.

The converse of Theorem 2.4 is not true. It is explained in the following example.

Example 2.6. $S(K_p)$ admits ADD into n - parts even if $p \neq \frac{n(n+1)}{2}$.

Theorem 2.7. If $p = \frac{n^2+n}{2}$, then $S(W_p)$ admits ADD into n - parts.

Proof: Let $V(W_p) = \{v_1, v_2, \ldots, v_p\}$. Let v'_1, $v'_2, \ldots,$ $v'_{2(p-1)}$ be the subdivision of the edges of W_p.

Suppose $p = \frac{n(n+1)}{2}$

Let $G_1 = < N[v_1] >$

$G_2 = < N[v_2, v_3] >$

$G_3 = < N[v_4, v_5, v_6] >$

$\vdots$

$G_n = < N[v_l, \ldots v_p] >$, where $l = \frac{n^2-n+2}{2}$, $p = \frac{n(n+1)}{2}$

Here $\frac{n^2-n+2}{2}$ can be found using Newton's forward difference formula as in Table-4.

Here G_n is a neighbourhood of $(\frac{n^2+n}{2})$-$(\frac{n^2-n+2}{2} - 1) = n$ vertices.

From the above decomposition, we see that $\gamma(G_i) = i$ and $\psi = \{G_1, G_2, \ldots, G_n\}$ is an ADD. ∎

Example 2.8. ADD of subdivision of a wheel graph.

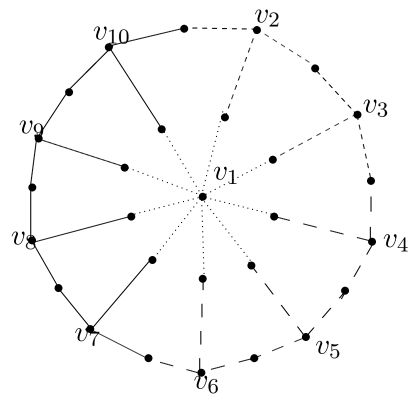

Figure 4: ADD of SW_p.

Here

G_1 is an edge induced subgraph whose edges are denoted by dotted lines,

G_2 is an edge induced subgraph whose edges are denoted by dashed lines,

G_3 is an edge induced subgraph whose edges are denoted by dash parameters and

G_4 is an edge induced subgraph whose edges are denoted by plain lines.

The converse of Theorem 2.7 is not true. It is explained in the following example.

Example 2.9. $S(W_p)$ admits ADD into n - parts even if $p \neq \frac{n(n+1)}{2}$.

Theorem 2.10. Let P_p be a (p,q)- path. If $p = \frac{2n^2-5n+7}{2}$ and $n \cong 1(mod4)$ or $\frac{2n^2-7n+14}{2}$, $n \cong 2(mod4)$, $n > 2$ then the subdivision of P_p^+ admits ADD into n - parts.

Proof: Let $P_p : (v_1\ v_2\ v_3 \dots v_p)$ be a path. If we attach the vertices $v_1', v_2' \dots v_p'$ to $v_1, v_2, \dots, v_p$ respectively, then we get $P_p{}^+$ and $u_1, u_2, \dots, u_{\frac{n(n+1)}{2}}$ are subdivision vertices of P_p^+.

Let $G_1 = < N[u_1] >$

$G_2 = < N[u_2, u_3] >$

$G_3 = < N[u_4, u_5, u_6] >$

$G_4 = < N[u_7, u_8, u_9, u_{10}] >$

$G_5 = < N[u_{11}, u_{12}, u_{13}, u_{14}, u_{15}] >$

$\vdots$

$G_n = < N[u_k, \dots u_l] >$, where $k = \frac{n^2-n+2}{2}$, $l = \frac{n(n+1)}{2}$.

Here $\frac{n^2-n+2}{2}$ can be found as in Table - 4 and $\frac{n(n+1)}{2}$ can be found using Newton's forward difference formula as follows.

Table 5

n	l	Δl	$\Delta^2 l$	$\Delta^3 l$
1	1			
		2		
2	3		1	
		3		0
3	6		1	
		4		0
4	10		1	
		5		
5	15			

$$n = n_0 + xh$$
$$= 1 + x$$
$$x = n - 1$$
$$l = l_0 + x\ \Delta l_0 + \frac{x(x-1)}{2!}\ \Delta^2 l_0 + \dots$$
$$= 1 + (n-1)\ (2) + \frac{(n-1)(n-2)}{2!}\ (1)$$
$$= \frac{n(n+1)}{2}.$$

Here G_n is a neighbourhood of $(\frac{n^2+n}{2})$-$(\frac{(n^2-n+2)}{2} - 1) = n$ vertices.

From the above decomposition, we see that $\gamma(G_i) = i$, $i = 1, 2, \dots, n$ and $\psi = \{G_1, G_2, \dots, G_n\}$ is an ADD.

If $n \equiv 1(mod4)$ or $n \equiv 2(mod4)$, then P_p^+ admits ADD into n - parts. Otherwise p is not an integer.

For, $p = \frac{2n^2-5n+7}{2} = \frac{2(n-1)^2-5(n-1)+4}{2}$ is possible only if $n-1 \equiv 0(mod4)$ and $p = \frac{2n^2-7n+14}{2} = \frac{2(n-2)^2-5(n-2)+4}{2}$ is possible only if $n-2 \equiv 0(mod4)$. ■

Example 2.11. ADD of $S(P_p^+)$ is given as follows:

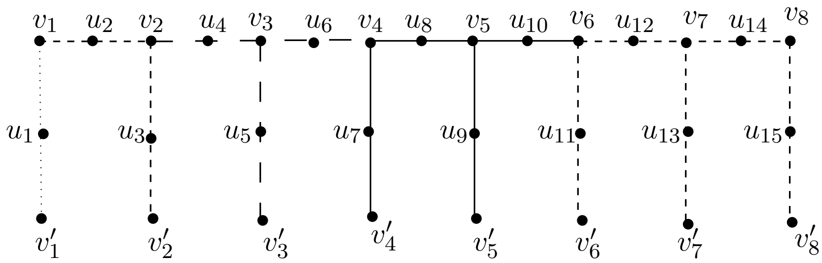

Figure 5: ADD of SP_p^+.

Here,

G_1 is an edge induced subgraph whose edges are denoted by dotted lines,
G_2 is an edge induced subgraph whose edges are denoted by dashed lines,
G_3 is an edge induced subgraph whose edges are denoted by dash parameters and
G_4 is an edge induced subgraph whose edges are denoted by plain lines.

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