International Journal of Mathematics and Soft Computing
Vol.5, No.2 (2015), 69 - 74.

ISSN Print : 2249 - 3328
ISSN Online: 2319 - 5215

Greek parameters of nonlinear Black-Scholes equation

Purity J. Kiptum[1], Joseph E. Esekon[2], Owino Maurice Oduor[3]

[1] Department of Pure and Applied Mathematics
Maseno University, Box 333, Maseno-Kenya.
pkiptum@gmail.com

[2] Department of Statistics and Actuarial Science
Maseno University, Box 333, Maseno-Kenya.
esekon2001@yahoo.com

[3] Department of Mathematics and Computer Science
University of Kabianga, Box 2030, Kericho-Kenya.

Abstract

Derivatives are used in hedging European options against risks. The partial derivatives of the solution to either a variable or a parameter in the Black-Scholes model are called risk (Greek) parameters or simply the Greeks. Nonlinear versions of the standard Black-Scholes Partial Differential Equations have been introduced in financial mathematics in order to deal with illiquid markets. In this paper we derive the Greek parameters of a nonlinear Black-Scholes Partial Differential Equation whose nonlinearity is as a result of transaction costs for modeling illiquid markets. We compute the Greek parameters of a European call option price from the nonlinear equation $u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) = 0$. All these Greeks were of the form $a + \dfrac{1}{\rho} f(S,t)$. The methodology involved deriving the Greek parameters from the formula of the equation by differentiating the formula with respect to either a variable or a parameter. These Greeks may help a trader to hedge risks in a non-ideal market situation.

Keywords: Greek parameters, nonlinear Black-Scholes equation, transaction cost model.
AMS Subject Classification(2010): 35K10, 35K55.

1 Introduction

The famous Black and Scholes equation provides markets with a way of pricing options. The derivation of this equation is based on the assumption that markets are complete, frictionless and perfectly liquid. In a frictionless market, there are no transaction costs and restrictions on trade. In perfectly liquid markets investors can trade large volumes of stock without affecting their prices. Risk management is concerned with controlling three financial risks that is market risk, credit risk and liquidity risk. Due to the introduction of liquidity risk in the market, the Greek parameters derived from the Black-Scholes formulae in the classical theory become unrealistic. The Greeks resulting from the Black-Scholes formula for valuing options in illiquid markets therefore, are appropriate in explaining this liquidity risk. These Greek parameters were obtained by differentiating the formula with respect to either a variable or a parameter. These derivatives are important in the hedging of an option position and play key roles in risk management. One of these models is the transaction cost model put forward by Cetin *et al.* [2]. This model takes into account the illiquidities arising from transaction costs. The purpose of this paper is to compute and analyze the Greek parameters from the Black-Scholes formula of a non-linear equation. We consider the European call options only since the formula from which we used to derive the Greek parameters was for the European call option.

2 Transaction Cost Model

We consider a market with one share denoted by S_t, and a risk free money market account with spot rate of interest $r \geq 0$ whose value at time t is $B_t \equiv 1$. Stock is illiquid (its price is affected by trading) while money market account is assumed to be liquid. The model we focus on is the model due to Cetin *et al.* [2] where a fundamental stock price S_t^0 follows the dynamics

$$dS_t^0 = \mu S_t^0 + \sigma S_t^0 dW_t,\ 0 \leq t \leq T$$

where dS_t is the change in stock price S_t at time t, μ is the drift(the expected rate of return), σ is the volatility of the stock price and W_t is the Wiener process. An investor who wants to trade α shares at time t has to pay the transaction price S_t which is given by $\overline{S}_t(\alpha) = e^{\rho\alpha} S_t^0$ where ρ is the liquidity parameter with $\rho \geq 0$. This models a bid-ask-spread whose size depends on α (number of shares traded).

Consider a Markovian trading strategy (i.e. a strategy of the form $\Phi_t = \phi(t, S_t^0)$) for a smooth function $\phi = u_S$ where ϕ is the hedge ratio. Then we have $\phi_S = u_{SS}$. If the stock and bond positions are Φ_t and η_t respectively then the value of this strategy at time t is

$$V_t = \Phi_t S_t^0 + \eta_t.$$

Φ_t is a semi-martingale with quadratic variation of the form

$$[\Phi]_t = \int_0^t (\phi_S(\tau, S_\tau^0)\sigma S_\tau^0)^2 d\tau$$

whose change is given by $d[\Phi]_t = (u_{SS}(t, S_t^0)\sigma S_t^0)^2$ since $\phi_S = u_{SS}$.
Applying Itô formula to $u(t, S_t^0)$ gives

$$du(t, S_t^0) = u_S(t, S_t^0) dS_t^0 + \left(u_t(t, S_t^0) + \frac{1}{2}\sigma^2 (S_t^0)^2 u_{SS}(t, S_t^0) \right) dt \tag{1}$$

For a continuous semi-martingale Φ with quadratic variation $[\Phi]_t$ the wealth dynamics of a self-financing strategy becomes

$$dV_t = \Phi_t dS_t^0 - \rho S_t^0 d[\Phi]_t. \tag{2}$$

Substituting $d[\Phi]_t$ into (2) yields the following dynamics:

$$dV_t = \phi(t, S_t^0)dS_t^0 - \rho S_t^0(\phi_S(t, S_t^0)\sigma S_t^0)^2 dt. \tag{3}$$

Since $V_t = u(t, S_t^0)$, equating the deterministic components of (1) and (3) and taking $\phi_S = u_{SS}$ gives the nonlinear PDE

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) = 0,\ u(S_T^0) = h(S_T^0) \tag{4}$$

where $h(S_T^0)$ is the payoff of the value claim at maturity time T.

3 Solution of the Nonlinear Black-Scholes Equation

Theorem 3.1. If $V(x,t)$ is any positive solution to the porous medium type equation $V_t + \frac{\sigma^2}{2}(VV_x + \frac{1}{2}V^2)_x = 0$, then

$$u(S,t) = S - \frac{\sqrt{S_0}}{\rho}\left(\sqrt{S}e^{\frac{\sigma^2 t}{8}} + \frac{\sqrt{S_0}}{4}e^{\frac{\sigma^2 t}{4}}\right) \tag{5}$$

solves the nonlinear Black-Scholes equation $u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) = 0$ for $S_0, S, \sigma, \rho > 0$ and $t \geq 0$.

Equation (5) is the formula for the European call option with $r = 0$ with the proof of this theorem found in [4].

4 The Greek Parameters

Partial derivatives of prices of options with respect to parameters or variables are called "*Greek letters*", or simply "*Greeks*". These derivatives are important for hedging. Each Greek measures a different dimension to the risk in an option position and a trader's aim is to manage these Greeks so that all risks are acceptable [6]. We explain some of these derivatives and how they are used.

Delta

The *delta* of a portfolio of options (or of an option) is the sensitivity of the portfolio (or option) to the underlying asset's price. It is the rate of change of the option price with respect to the price of the underlying asset. It is the slope of the price curve of the option to the market price of the underlying asset. obtain the Greek parameter *delta* we differentiate the solution of the

European call option with respect to the spatial variable S. We obtain *delta* by differentiating equation (5) with respect to S to obtain

$$u_S = 1 - \frac{1}{2\rho}\sqrt{\frac{S_0}{S}}e^{\frac{\sigma^2 t}{8}} \tag{6}$$

for $\rho, S, \sigma > 0$ and $S_0, t \geq 0$. In this case, $u(S,t)$ can be the value of one contract or of a whole portfolio of contracts. Summing up the deltas of all the individual positions gives rise to the delta of a portfolio of options.

Gamma

An option's *gamma*, Γ, is the rate at which the delta of a portfolio (or the option) changes with respect to the underlying asset's price. It is the second partial derivative of the position with respect to the price of the underlying asset. To obtain the Greek parameter *gamma* we differentiate equation (6) with respect to S to get

$$u_{SS} = \frac{1}{4\rho S^{\frac{3}{2}}}\sqrt{S_0}e^{\frac{\sigma^2 t}{8}} \tag{7}$$

for $\rho, S_0, S, \sigma > 0$ and $t \geq 0$.

Speed

The rate at which gamma changes with respect to the price of the stock is called the option's **speed**. Differentiating gamma with respect to the spatial variable S we get the option's speed as

$$u_{SSS} = -\frac{3}{8\rho S^{\frac{5}{2}}}\sqrt{S_0}e^{\frac{\sigma^2 t}{8}} \tag{8}$$

for $\rho, S_0, S, \sigma > 0$ and $t \geq 0$. Gamma is used by traders to estimate how much they will rehedge by if the stock price moves. The delta may change by more or less the amount the traders have approximated the value of the stock price to change. If it is by a large amount that the stock price moves, or the option nears the strike and expiration, the delta becomes unreliable and hence the use of the speed.

Theta

Theta of portfolio of options is the rate of change of the value of the portfolio with respect to the passage of time with all else remaining the same. The value of an option is the combination of time value and stock value. When time passes the value of the option decreases. Thus the rate of change of the option price with respect to the passage of time, (that is, *theta*) is usually negative. To obtain the Greek parameter *theta* we differentiate the solution $u(S,t)$ (5) with respect to time t to get

$$u_t = -\frac{\sigma^2\sqrt{S_0}}{8\rho}\left\{\sqrt{S}e^{\frac{\sigma^2 t}{8}} + \frac{\sqrt{S_0}}{2}e^{\frac{\sigma^2 t}{4}}\right\} \tag{9}$$

for $\rho, S_0, S, \sigma > 0$ and $t \geq 0$.

Vega

During the derivation of the Black-Scholes formula, the volatility σ of the asset underlying a derivative is assumed to be constant. In practice, volatility changes with time, which means that a derivative's value is liable to change due to movements in volatility and also due to changes in the price of the asset and the passage of time. *Vega* is the sensitivity of the price of the option to volatility. This risk parameter is a partial derivative of the option's price with respect to a parameter unlike the others mentioned above which are partial derivatives with respect to a variable. To obtain *vega* for $r = 0$ we differentiate equation (5) with respect to σ to obtain

$$u_\sigma = -\frac{\sigma t\sqrt{S_0}}{4\rho}\left\{\sqrt{S}e^{\frac{\sigma^2 t}{8}} + \frac{\sqrt{S_0}}{2}e^{\frac{\sigma^2 t}{4}}\right\} \tag{10}$$

for $\rho, S_0, S, \sigma > 0$ and $t \geq 0$.

5 Conclusion

This paper deals with the Greek parameters of a European call option which have been derived from a nonlinear Black-Scholes formula. These risk parameters are for a call option in illiquid markets whose illiquidity is arising from transaction costs. We have computed the Greek parameters derived from a nonlinear Black-Scholes formula in (5)(Cetin *et al.* model). All the Greek parameters are of the form $a + \frac{1}{\rho}f(S,t)$ where $a \in \mathbb{R}$.

References

[1] F. Black and M. Scholes, *The Pricing of Options and Corporate Liabilities*, The Journal of Political Economy, 81, 3 (1973), 637-654.

[2] U. Cetin, R. Jarrow and P. Protter, *Liquidity Risk and Arbitrage Pricing Theory*, Finance and Stochastics, 8 (2004), 311-341.

[3] J. E. Esekon, *Analytic solution of a nonlinear Black-Scholes equation*, International Journal of Pure and Applied Mathematics, 82, 4 (2013), 547-555.

[4] J. E. Esekon, *A particular solution of a nonlinear Black-Scholes Partial Differential Equation*, International Journal of Pure and Applied Mathematics, 81, 5 (2012), 715-721.

[5] J. E. Esekon, *The Black-Scholes formula and the Greek parameters for a nonlinear Black-Scholes equation*, International Journal of Pure and Applied Mathematics, 76 (2012), 167-171.

[6] J. C. Hull, *Options, Futures and other Derivatives*, Fifth Edition, Prentice Hall, New Jersey, USA, 2005.

[7] P. Wilmott, *Quantitative Finance*, Second Edition, John Wiley & Sons Ltd, West Sussex, England, 2006.