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Efficient secure domination in graphs

P. Roushini Leely Pushpam[1], Chitra Suseendran[2]

[1] Department of Mathematics
D.B. Jain College, Chennai 600 097, Tamil Nadu, India.
roushinip@yahoo.com

[2] Department of Mathematics
Ethiraj College for Women, Chennai 600 008, Tamil Nadu, India.
chitrasuseendran@gmail.com

Abstract

Let $G = (V, E)$ be a graph. A dominating set $S \subseteq V(G)$ is an *efficient dominating set* if S is a 2-packing set. The set S is a *secure dominating set* of G if for each $u \in V \backslash S$ there exists a vertex $v \in S$ such that uv belongs to the edge set of G and $(S\backslash\{v\}) \cup \{u\}$ is a dominating set of G. In this paper we introduce *efficient secure domination* in graphs. We define and study the parameter *efficient secure domination number* of G.

Keywords: Efficient domination, secure domination, efficient secure domination.
AMS Subject Classification(2010): 05C69.

1 Introduction

We consider only finite simple undirected graphs $G = (V, E)$ of order $|V| = n$. For any vertex $v \in V$, the *open neighborhood of* v is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the *open neighborhood* is $N(S) = \bigcup_{v \in S} N(v)$ and the *closed neighborhood* is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V\backslash S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G, and a dominating set S of minimum cardinality is called a γ-set of G. A set S of vertices is called *2-packing*, if for every pair of vertices $u, v \in S$, $N[u] \cap N[v] = \phi$.

Bange, Barkauskas and Slater [3, 4] introduced *efficiency measure* for a graph G. The *efficient domination number* of a graph, denoted by $F(G)$, is the maximum number of vertices that can be dominated by a set S, that dominates each vertex at most once. A graph G of order $n = |V(G)|$ has an *efficient dominating set* if and only if $F(G) = n$. A vertex v of $deg(v) = |N(v)|$ dominates $|N[v]| = 1 + deg(v)$ vertices. Grinstead and Slater [13], defined the *influence* of a set of vertices S to be $I(S) = \sum_{v \in S}(1 + deg(v))$, the total amount of domination being done by S. Because S does not dominate any vertex more than once if and only if any two vertices in S are at a distance at least 3 (that is, S is a 2-packing), we have $F(G) = \max\{I(S) : S$

is a 2-packing}. A set S is an efficient dominating set if and only if $|N(v) \cap S| = 1$ for all vertices $v \in V(G)$, or equivalently, S is an efficient dominating set if and only if S is a 2-packing with $I(S) = n = F(G)$. A graph G has an efficient dominating set if and only if $F(G) = n$.

For results in efficient domination one can refer to [1–4, 11–13, 16]. The set S is a *secure dominating set* of G if for each $u \in V \backslash S$ there exists a vertex $v \in S$ such that $uv \in E(G)$ and $(S \backslash \{v\}) \cup \{u\}$ is a dominating set of G. In this case we say v-S defends u or v is an S-defender. Secure domination in graphs has been studied in [5–10, 15, 17, 18].

Now we introduce the concept of efficient secure domination in graphs. Let S be a 2-packing. We define the *secured influence of* S as $I_s(S) = \sum_{v \in S} I_v$, where I_v is the order of a maximum clique containing v. The efficient secure domination number of a graph G denoted by $F_s(G)$, is the maximum number of vertices of G that can be defended by S. Hence $F_s(G) = max\{I_s(S) : S$ is a 2-packing$\}$, called the *efficient secure domination number of* G. If $F_s(G) = n$, then G is said to be *efficiently secure dominatable* or simply *ESD*. A 2-packing set S with *maximum secured influence* $I_s(S)$ is called an $F_s(G)$-set.

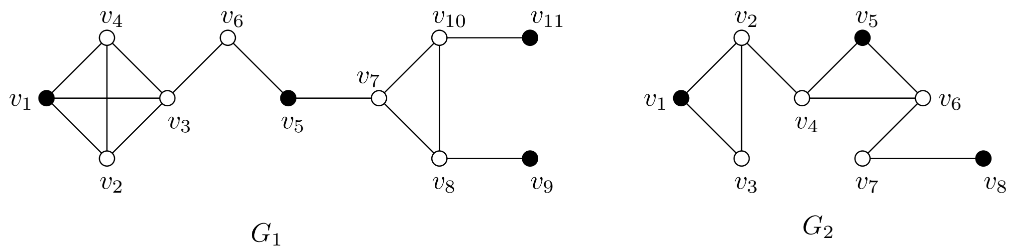

Figure 1: $F_s(G_1) = 10$ and $F_s(G_2) = 8$.

For the graphs G_1 and G_2 given in Figure 1, the set $\{v_1, v_5, v_9, v_{11}\}$ is an $F_s(G_1)$-set and the set $\{v_1, v_5, v_8\}$ is an $F_s(G_2)$-set. Now $F_s(G_1) = 10$ and $F_s(G_2) = 8$. Thus G_1 is *not ESD*, but G_2 is *ESD*.

For any graph G, $F_s(G) = n$ implies $F(G) = n$ but not conversely. (Refer Figure 2).

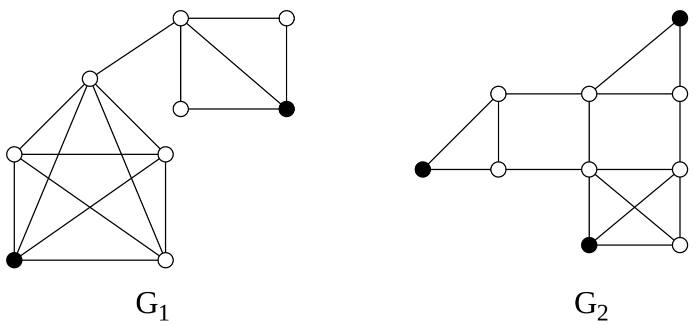

Figure 2: $F(G_1) = n = 9$, $F_s(G_1) = 8$ and $F(G_2) = F_s(G_2) = n = 10$.

In the following Section 2 we list out some definitions required for our study. In Section 3 certain classes of graphs which are efficiently secure dominatable are characterized. Finally in Section 4 bounds in terms of edge independence number are obtained for triangle free graphs.

2 Notations

For notation and graph theory terminology, we follow [14]. Specifically, let $G = (V, E)$ be a *simple* and *connected graph* with the vertex set V of order n and edge set E of size m. We denote the *degree of* v in G by $deg_G(v)$ or simply $deg(v)$, if the graph G is clear from context. A vertex of degree 0 is called an *isolated* vertex. A *leaf* u or a pendant vertex u of G is a vertex of degree one and the *support* vertex of the leaf u is the unique vertex v such that $uv \in E$. A support with one leaf is called a *weak support*, where as with at least two leaves is called a *strong support*. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$.

A *unicyclic graph* is a connected graph that contains precisely one cycle. A *split graph* is a graph in which the vertices can be partitioned into a *clique* and an *independent* set. The *edge independence number* $\beta'(G)$ is the maximum cardinality among the independent sets of edges of G. A *complete binary tree* is a rooted tree in which all leaves have the same depth and all internal vertices have degree three except the root, which is of degree two. If T is a complete binary tree with root vertex v, the set of all vertices with depth k are called *vertices at level* k. A *subtree* of a rooted tree T is a tree S consisting of a vertex in T and all of its descendants in T. The subtree corresponding to the root vertex is the *entire tree* and the subtree corresponding to any other vertex is called a *proper subtree*.

A set $S \subseteq V(G)$ is called a *packing* in G if $N[u] \cap N[v] = \phi$ for every pair $u, v \in S$. In other words, the shortest path between any pair of vertices in S is at least 3 in G. A *perfect matching* of a graph G, if it exists is a matching of G containing all the vertices of G. A matching M is a *maximum matching* of G if G has no matching M' with $|M'| > |M|$. The vertices of a graph which are not incident to the edges of a matching M are said to be *unsaturated* or M-*unsaturated* vertices. The *distance* from a vertex u to a vertex v in a graph G is defined as the length of a shortest $u - v$ path and is denoted by $d(u, v)$.

3 Efficiently Secure Dominatable Graphs

In this section we characterize triangle free graphs, split graphs and unicylic graphs which are efficiently secure dominatable (ESD).

Theorem 3.1. For any graph G of order n, G is ESD if and only if $V(G)$ can be partitioned into cliques such that for each clique K_t, $1 < t \leq n$, there is a vertex v in K_t such that $deg_G(v) = t - 1$.

Proof: If G is a complete graph, then $F_s(G) = n$. Otherwise, assume that $V(G)$ is partitioned into cliques K_t satisfying the hypothesis of the theorem.

Let $S = \{v : v \in V(K_t) \text{ and } deg_G(v) = t-1, 1 < t \leq n\}$. To prove $F_s(G) = n$, consider any two vertices $v_1, v_2 \in S$. We now claim that $d(v_1, v_2) \geq 3$. There exist cliques K_{t_1} and K_{t_2} containing the vertices v_1 and v_2, respectively. Now $deg_G(v_i) = t_i - 1$, $i = 1, 2$.

Let Q be the shortest (v_1, v_2) path in G. Then there exist vertices w_1, w_2 in K_{t_1} and K_{t_2}, respectively, such that $w_1, w_2 \in Q$ $(w_i \neq v_i, i = 1, 2)$. If $w_1 \neq w_2$ then the result is obvious. If $w_1 = w_2$ and since $deg_G(v_i) = t_i - 1$, w_1 is not adjacent to both v_1 and v_2. Therefore, $d(v_1, v_2) \geq 3$. Hence S is a 2-packing. Further, it is clear that $I_s(S) = n$. Therefore, $F_s(G) = n$.

Conversely, let $F_s(G) = n$. Then there exists a 2-packing S, such that $I_s(S) = n$. This implies that $\sum_{v \in S} I_v = n$, where I_v is the order of a maximum clique containing v. Now $\sum_{i=1}^{|S|} I_{v_i} = n$. Therefore, $V(G)$ can be partitioned into cliques say K_{t_i}, $1 \leq i \leq |S|$. Now we claim that for each clique K_{t_i} there exists a $v \in K_{t_i}$ such that $deg(v) = t_i - 1$. If $|S| = 1$, G is a complete graph, otherwise there exists a clique K_{t_j}, such that $deg_G(z) \geq t_j$ for all $z \in K_{t_j}$. Since $F_s(G) = n$, some member say x_j in K_{t_j} belongs to S. Also there exists a vertex y in some K_{t_ℓ} such that x_j and y are adjacent. Let y_ℓ (the possibility that $y_\ell = y$ is not ruled out) be the vertex in K_{t_ℓ} such that $y_\ell \in S$. Then since, y and y_ℓ are adjacent, we see that $d(x_j, y_\ell) \leq 2$, which is a contradiction. ■

Theorem 3.2. Let G be a triangle free graph. Then G is ESD if and only if every vertex v of degree at least two is a weak support.

Proof: Assume that every vertex v of $deg_G(v) > 1$ is a weak support. By Theorem 3.1, $F_s(G) = n$.

Conversely, let $F_s(G) = n$. Again by Theorem 3.1, $V(G)$ can be partitioned into cliques K_{t_i} such that each K_{t_i} contains a vertex v_i with $deg_G(v_i) = t_i - 1$. Since G is triangle free, a maximum clique in it is a K_2. Hence $t_i = 2$ for all $i = 1, 2, \ldots, k$,$1 < k < n$ and the vertices $v_1, v_2, \ldots, v_k$ are of degree one. Thus each v_i is a pendant vertex. Hence the theorem. ■

Theorem 3.3. Let G be a split graph (X, Y), where X is independent and $G[Y]$ is a clique. Then G is ESD if and only if $deg_G(y) = |Y|$ for all $y \in Y$.

Proof: Let G be ESD and S be an $F_s(G)$-set. Assume to the contrary that there exists a vertex $w \in Y$ such that $deg_G(w) \neq |Y|$. Then either $deg_G(w) > |Y|$ or $deg_G(w) = |Y| - 1$.

If $deg_G(w) > |Y|$, then there exists at least two neighbors say u_1, u_2 of w in X. Either $w \in S$ or $w \in I(z)$ for some $z \in S$. If $w \in S$ then clearly u_1 or u_2 does not belong to the influence of any member of S, which is a contradiction. If $w \in I(z)$ then either $z \in Y$ or $z \in X$, which is a contradiction.

If $deg_G(w) = |Y| - 1$, then either $w \in S$ or $w \in I(z)$ for some $z \in S$. Hence each member of Y has exactly one neighbor in X. In both the cases we see that $F_s(G) < n$, which is a contradiction.

Conversely, let $deg_G(y) = |Y|$ for all $y \in Y$. Then every vertex in Y has exactly one neighbor in X and corresponding to each vertex of Y, the leaf vertex belongs to S. Hence $F_s(G) = n$. Therefore, G is ESD. ■

Corollary 3.4. Let G be a triangle free ESD graph. Then G has a perfect matching.

Proof: The proof follows from Theorem 3.2. ■

Corollary 3.5. A triangle free k-regular graph is ESD if and only if $k = 1$.

Proof: Let G be a triangle free k-regular graph. By Theorem 3.1, $V(G)$ can be partitioned into cliques K_t, $t = 2$ such that one of the vertices of each of K_t's is of degree one. As G is k-regular, $k = 1$. Conversely, if $k = 1$, $F_s(G) = n$. Therefore, G is ESD. ■

As the proofs are straight forward, we state the following corollaries.

Corollary 3.6. Let G be a 4-regular circulant graph $C_n(1, k)$, $1 < k \leq \lfloor \frac{n}{2} \rfloor$, without triangle. Then $F_s(G) < n$.

Corollary 3.7. Complete graphs are ESD.

Corollary 3.8. A triangle free ESD graph is of even order.

Corollary 3.9. A triangle free graph G, with $\delta(G) > 1$ is not efficiently secure dominatable.

Proof: We have to prove $F_s(G) < n$. On contrary, assume that $F_s(G) = n$. Then by Theorem 3.1, G gets partitioned into cliques K_t, with each K_t containing a vertex of degree $t - 1$. Since G is triangle free, $t = 2$. Also by Theorem 3.1, G contains a vertex of degree $t - 1$, which is a contradiction to the hypothesis that $\delta(G) > 1$. Hence $F_s(G) < n$. ■

Theorem 3.10. A unicyclic graph G with a cycle C_3 is ESD if and only if one of the following conditions hold.

(i) Every vertex of degree at least 2 is a weak support.

(ii) At least one vertex of C_3 is of degree 2, the vertices in C_3 of degree > 2 are not supports and every vertex not in C_3 of degree > 1 is a weak support.

Proof: Let G be an ESD graph. By Theorem 3.1, there exists a partition $W = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ such that each $G[V_i]$ is a complete graph K_t, $t > 1$ with at least one vertex of degree $t - 1$. Since G is unicyclic with a cycle C_3 one of the following conditions hold.
(a) Each $G[V_i]$ is a K_2, $2 \leq i \leq k$
(b) $G[V_1]$ is a K_3 and $G[V_i]$, $2 \leq i \leq k$ is a K_2.

If (a) holds then condition (i) follows. If (b) holds, by Theorem 3.1, at least one vertex of C_3 is of degree 2.

Now we claim that the vertices in C_3, which are of degree > 2 are not supports. Let a vertex of C_3, say x be a support and z be the leaf adjacent to x. Then for some j, $V_j = \{z\}$, which is a contradiction, since $t > 1$.

Conversely, let the given conditions (i) and (ii) be true. If every vertex of degree at least two is a weak support, then clearly G is ESD. Now assume that the condition (ii) holds, then $S = \{x : x \text{ is a leaf of } G\} \cup \{w\}$, where w is the vertex of degree two in C_3. Hence S is an $F_s(G)$-set. Therefore, G is ESD. ■

4 Bounds for Triangle Free Graphs in terms of Edge Independence Number of the Graph

In a triangle free graph, a maximum clique is K_2. Hence an attempt is made to give a bound for $F_s(G)$ of a triangle free graph G in terms of the edge independence number $\beta'(G)$, as any $F_s(G)$ set is a 2-packing.

Theorem 4.1. For a triangle free graph G, $2 \leq F_s(G) \leq 2\beta'(G)$, where $\beta'(G)$ is the edge independence number of the graph.

Proof: Let S be a 2-packing, such that $I_s(S) = F_s(G)$. Since G is triangle free, any maximum clique contained in it is a K_2. Let M be a maximum matching in G. Consider two elements say e_1 and e_2 in M such that e_1 and e_2 have a common neighbor say e. Then either an end vertex of e_1 and an end vertex of e_2 are in S or at most one end vertex of e_1 or one end vertex of e_2 is in S. Hence $F_s(G) \leq 2\beta'(G)$. ■

Now we characterize trees and unicylic graphs with cycle C_k, $k \geq 4$ for which $F_s(G) = 2\beta'(G)$. For convenience we define two supports x_1 and x_2 of G to be *consecutive* if at least one (x_1, x_2) path in G does not contain any support (Refer Figure 3).

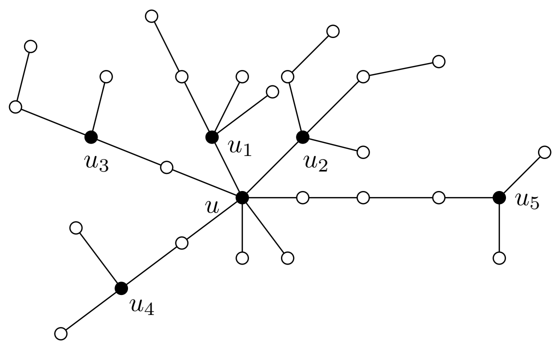

Figure 3: u_1, u_2, u_3, u_4, u_5 are consecutive supports of u.

Theorem 4.2. Let G be a tree. Then $F_s(G) = 2\beta'(G)$ if and only if the distance between any two consecutive supports is 1, 2 or 4.

Proof: Assume that $F_s(G) = 2\beta'(G)$. Let u be an arbitrary support and x be a leaf which is adjacent to u. Let $u_1, u_2, \ldots, u_k$ be the consecutive supports of u. We claim that $\ell(Q_i) =$ 1, 2 or 4, $0 \leq i \leq k$ where Q_i is the (u, u_i) path in T and $\ell(Q_i)$ denotes the length of Q_i. Suppose not, then there exists some j such that $\ell(Q_j) = 3$ or $\ell(Q_j) > 4$. Let $w_1, w_2, \ldots, w_k$ be the internal vertices of Q_j. Let y be a leaf adjacent to u_j and H be the subgraph induced by $V(G) \backslash \{x, u, w_1, w_2, \ldots, w_k, u_j, y\}$.

Now $\beta'(G) = 4 + \beta'(H)$ and $F_s(G) = 6 + F_s(H)$. Hence $F_s(G) = 2\beta'(G)$ implies that $F_s(H) = 2 + 2\beta'(H)$, which is a contradiction. Hence the distance cannot be more than 4 or equal to 3.

Conversely, let the distance between the consecutive supports be 1, 2 or 4. Let u_1 and u_2 be any two consecutive supports and x_1 and x_2 be the leaves adjacent to u_1 and u_2 respectively.

Case (i): $d(u_1 u_2) \leq 2$.

Clearly x_i, $i = 1, 2$ belong to some $F_s(G)$ set and the edges $u_i x_i$ belong to some $\beta'(G)$ set. (Refer Figure 4).

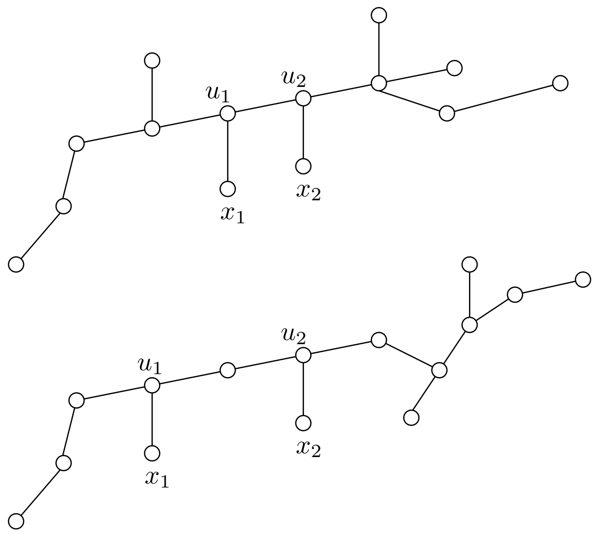

Figure 4

Case (ii): $d(u_1 u_2) = 4$.

Let w_1, w_2, w_3 be the internal vertices in the path joining u_1 and u_2. Then the edges $x_1 u_1, w_1 w_2, w_3 u_2$ belong to any $\beta'(G)$ set and the vertices x_1, w_2, x_2 belong to any $F_s(G)$ set. We see that $F_s(G) = 2\beta'(G)$. (Refer Figure 5). ■

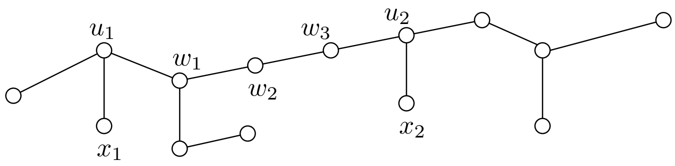

Figure 5

Corollary 4.3. Let G be a complete binary tree of level k. Then $F_s(G) = 2\beta'(G)$ if and only if $k \leq 3$.

Proof: Let $k > 3$. In a binary tree we see that any two supports are consecutive. Let v_1 and v_2 be the two descendants of the root vertex of the tree and T_1 and T_2 be the two subtrees corresponding to the vertices v_1 and v_2 respectively. Now we see that $d(x, y) \geq 6$ where x and y are supports of T_1 and T_2 respectively, which is a contradiction to Theorem 4.2.

Conversely, let $k \leq 3$. Again by Theorem 4.2, it follows that $F_s(G) = 2\beta'(G)$, since the distance between any two consecutive supports is 2 or 4. (Refer Figure 4).

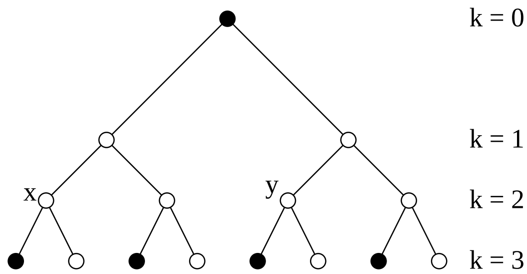

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Figure 6: A complete binary tree with $F_s(G) = 2\beta'(G)$.

Theorem 4.4. Let G be a unicyclic graph with a cycle C_k, $k \geq 4$ and $\Delta(G) \geq 3$. Then $F_s(G) = 2\beta'(G)$ if and only if the following conditions hold.

(a) Between every pair of consecutive supports, the length of every path is 1, 2 or 4.

(b) When $k > 4$, C_k contains at least 2 vertices of degree at least 3 and when $k = 4$, C_k contains at least 1 vertex of degree at least 3.

Proof: Let $F_s(G) = 2\beta'(G)$. To prove condition (a), consider x_1 and x_2 to be any two consecutive supports. If there is a unique path between x_1 and x_2, then as in Theorem 4.2, $\ell =$ 1, 2 or 4, where ℓ is the length of the unique path between x_1 and x_2. If there are 2 paths say, Q_1 and Q_2 between x_1 and x_2 with lengths ℓ_1 and ℓ_2 respectively, we claim ℓ_1 and $\ell_2 = 1$, 2 or

4. Assume that there exists at least one of ℓ_1 or ℓ_2, such that either $\ell_i = 3$ or $\ell_i > 4$, $i = 1, 2$. Without loss of generality, let $\ell_1 = 3$ or $\ell_1 > 4$. Let w_1 and w_2 be leaves which are adjacent to x_1 and x_2 respectively. Consider (w_1, w_2) path containing Q_1, say R. Then $F_s(R) < 2\beta'(R)$, which is a contradiction. Hence condition (a) holds.

To prove condition (b) we have two cases.

Case (i): When $k > 4$. Suppose C_k contains exactly one vertex of degree at least 3, $F_s(C_k) < 2\beta'(C_k)$. Also any subgraph H in G not contained in C_k is a tree. By Theorem 4.2 $F_s(H) < 2\beta'(H)$. Hence we get a contradiction.

Case (ii): When $k = 4$. Suppose C_k does not contain any vertex of degree more than 2, then G is isomorphic to a cycle C_4, which is a contradiction to $\Delta(G) \geq 3$.

Converse follows from the conditions (a) and (b). ∎

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