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Graceful labeling of bi-partite related graphs

S. Sudha, V. Kanniga

Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai-05, Tamilnadu, India.
sudhasubramanian1955@gmail.com, kan15gemini@gmail.com

Abstract

In this paper we have shown that the the splitting graph of the complete bi-partite graph $K_{m,\,n}$ is graceful and the tensor product of the complete bi-partite graph $K_{m,\,n}$ and a path graph $P_k(k > 1)$ is also graceful.

Keywords: Graceful labeling, odd-even graceful labeling, splitting graph, tensor product.
AMS Subject Classification(2010): 05C78.

1 Introduction

In 1967, Rosa[3] introduced the concept of labeling the edges and Golomb[2] gave the name graceful for such labelings. Gallian[1] has given a dynamic survey of graph labeling. Many graceful graphs are constructed from standard graphs by using various operations. Sekar[4] proved that the splitting graph of a path P_k and the splitting graph of even cycle C_n are odd graceful graphs. Vaidya et al.[6] proved that the splitting graph of $K_{1,\,n}$ as well as the tensor product of $K_{1,\,n}$ and P_2 admits odd graceful labeling. Sudha et al.[5] proved that the splitting graph of $K_{1,\,n}$ admits graceful labeling and the tensor product of $K_{1,\,n}$ and P_2 admit odd-even graceful labeling.

In this paper we show that the splitting graph of the complete bi-partite graph $K_{m,\,n}$ and the Tensor product of the complete bi-partite graph $K_{m,\,n}$ and the path P_k for any integer values of $m, n, k > 1$ are graceful. Also we establish that the odd-even graceful labeling of the tensor product of $K_{1,\,n}$ and P_k which is the generalization of the result in [5].

2 Basic definitions

Definition 2.1. A graph $G = (V(G), E(G))$ with p vertices and q edges is said to admit *graceful labeling* if $f : V(G) \longrightarrow \{0, 1, 2, \ldots, q\}$ such that distinct vertices receive distinct numbers and $\{\mid f(u) - f(v) \mid \ /uv \in E(G)\} = \{1, 2, 3, \ldots, q\}$.

Definition 2.2. A graph $G = (V(G), E(G))$ with p vertices and q edges is said to admit *odd-even graceful labeling* if $f : V(G) \longrightarrow \{0, 1, 2, \ldots, 2q\}$ is injective and the induced function, $f^* : E(G) \longrightarrow \{2, 4, 6, \ldots, 2q\}$ defined as $f^*(uv) = \mid f(u) - f(v) \mid$ is bijective.

It should be noted that the vertices take both odd and even labelings whereas the edges take only even labelings. That is the reason we call it as odd-even gracefulness.

Definition 2.3. For any graph G, *the splitting graph* is obtained by adding to each vertex v, a new vertex v' so that v' is adjacent to each and every vertex that is adjacent to v in G.

Definition 2.4. The *tensor product of two graphs* G_1 and G_2 denoted by $G_1 \otimes G_2$ has the vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and the edge set $E(G_1 \otimes G_2) = \{(u_1, v_1)(u_2, v_2)/u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$.

3 The splitting graph of the complete bi-partite graph $K_{m,\,n}$ is graceful

Theorem 3.1. The splitting graph of the complete bi-partite graph $K_{m,\,n}$ is graceful.

Proof: Let $u_1,\ u_2, \ldots, u_m$ and $v_1,\ v_2, \ldots, v_n$ be the vertices of the complete bi-partite graph $K_{m,\,n}$.

Let G be the splitting graph of $K_{m,\,n}$. Let $u_1',\ u_2', \ldots, u_m'$ and $v_1',\ v_2', \ldots, v_n'$ be the newly added vertices in $K_{m,\,n}$ to form G. G has $2(m+n)$ vertices and $3mn$ edges.

Define $f : V(G) \longrightarrow \{0, 1, 2, \ldots, 3mn\}$ as follows:

$$\begin{aligned}
f(u_i') &= i-1, && 1 \leq i \leq m\\
f(v_i) &= (3n+1-i)m, && 1 \leq i \leq n\\
f(u_i) &= mn+(i-1), && 1 \leq i \leq m\\
f(v_i') &= (2n+1-i)m, && 1 \leq i \leq n
\end{aligned}$$

The above defined function f is a graceful labeling for the splitting graph of the complete bi-partite graph $K_{m,\,n}$.

Hence, the splitting graph of the complete bi-partite graph $K_{m,\,n}$ is graceful. ∎

Illustration 3.2. The splitting graph of the complete bi-partite graph $K_{5,7}$ consists of 24 vertices and 105 edges. A graceful labeling of $K_{5,7}$ is given in Figure 1.

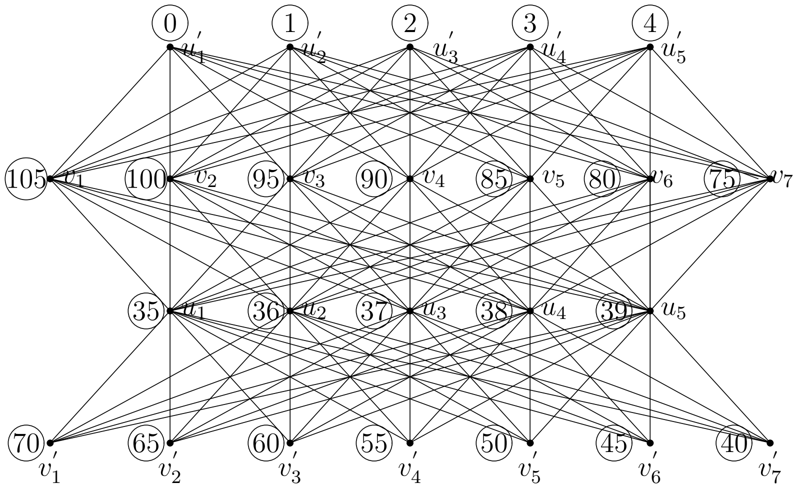

Figure 1. Splitting graph of $K_{5,7}$

4 Gracefulness of the tensor product of the complete bi-partite graph $K_{m,\,n}$ with the path P_2 and P_3

Theorem 4.1. The tensor product of the complete bi-partite graph $K_{m,\,n}$ for $m,\ n > 1$ and the path P_2 admits graceful labeling.

Proof: Let $u_1,\ u_2, \ldots, u_{m+n}$ be the vertices of the complete bi-partite graph $K_{m,\,n}$ and let $v_1,\ v_2$ be the vertices of the path graph P_2. The tensor product of the complete bi-partite graph $K_{m,\,n}$ and the path P_2 is denoted by $K_{m,\,n} \otimes P_2$.

Let $G = K_{m,\,n} \otimes P_2$. Then G consists of $2(m+n)$ vertices and $2mn$ edges. We divide the vertices of $K_{m,\,n} \otimes P_2$ into two disjoint sets

$$V_1 = \{(u_i,\ v_1)\ /\ i = 1, 2, \ldots, m+n\}$$

$$V_2 = \{(u_j,\ v_2)\ /\ j = 1, 2, \ldots, m+n\}$$

Define $f : V(G) \longrightarrow \{0, 1, 2, \ldots, 2mn\}$ by

$$f(u_i,\ v_1) = \begin{cases} (i-1)n, & 1 \le i \le m \\ m(n+1) + 2 - i, & m+1 \le i \le m+n \end{cases}$$

$$f(u_j,\ v_2) = \begin{cases} (j-1)n + 1, & 1 \le j \le m \\ q + m - j + 1, & m+1 \le j \le m+n \end{cases}$$

The function f defined above is a graceful labeling for the tensor product of complete bi-partite graph $K_{m,\,n}$ and the path P_2. Hence, $K_{m,\,n} \otimes P_2$ is a graceful graph. ■

Illustration 4.2. Consider the complete bi-partite graph $K_{3,\,4}$ and the path P_2. The resultant graph $K_{3,\,4} \otimes P_2$ consists of 14 vertices and 24 edges. A graceful labeling of $K_{3,\,4} \otimes P_2$ is given in Figure 2.

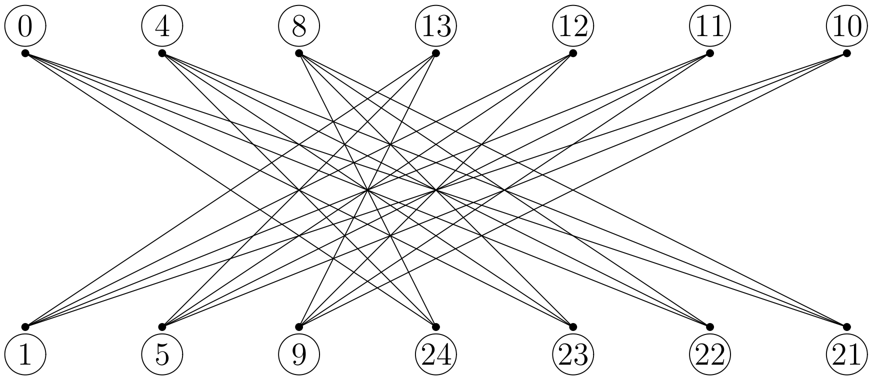

Figure 2. Tensor product of $K_{3,\,4}$ and P_2

Theorem 4.3. The tensor product of the complete bi-partite graph $K_{m,\,n}$ for $m,\ n > 1$ and the path P_3 admits graceful labeling.

Proof: Let $u_1,\ u_2, \ldots, u_{m+n}$ be the vertices of the complete bi-partite graph $K_{m,\,n}$ and let $v_1,\ v_2$ and v_3 be the vertices of the path graph P_3. The tensor product of the complete bi-partite graph $K_{m,\,n}$ and the path P_3 is denoted by $K_{m,\,n} \otimes P_3$.

Let $G = K_{m,\,n} \otimes P_3$. Then G consists of $3(m+n)$ vertices and $4mn$ edges. We divide the vertices of $K_{m,\,n} \otimes P_3$ into three disjoint sets for each vertex in P_3 as V_1, V_2 and V_3 and they are given as

$$V_1 = \{(u_i,\ v_1)\ /\ i = 1, 2, \ldots, m+n\}$$

$$V_2 = \{(u_j,\ v_2)\ /\ j = 1, 2, \ldots, m+n\}$$

$$V_3 = \{(u_k,\ v_3)\ /\ k = 1, 2, \ldots, m+n\}$$

Define $f : V(G) \longrightarrow \{0, 1, 2, \ldots, 4mn\}$ by

$$f(u_i,\ v_1) = \begin{cases} m + ni + 1, & 1 \le i \le m \\ q - m(i - m - 1), & m + 1 \le i \le m + n \end{cases}$$

$$f(u_j,\ v_2) = \begin{cases} j - 1, & 1 \le j \le m \\ j, & m + 1 \le j \le m + n \end{cases}$$

$$f(u_k,\ v_3) = \begin{cases} n(m + k) + (m + 1), & 1 \le k \le m \\ q - mn - m(k - m - 1), & m + 1 \le k \le m + n \end{cases}$$

The function f defined above provides graceful labeling for the tensor product of the complete bi-partite graph $K_{m,\,n}$ and the path P_3 that is, $K_{m,\,n} \otimes P_3$ is a graceful graph. ∎

Illustration 4.4. Consider the complete bi-partite graph $K_{2,\,3}$ and the path P_3. The resultant graph $K_{2,\,3} \otimes P_3$ consists of 15 vertices and 24 edges. A graceful labeling of $K_{2,\,3} \otimes P_3$ is shown in Figure 3.

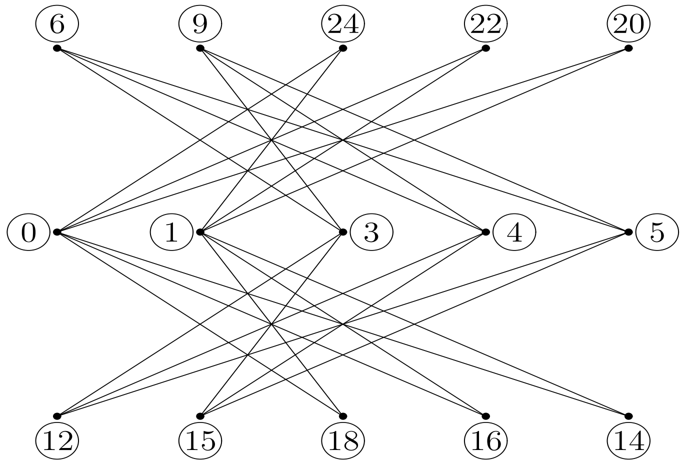

Figure 3. Tensor product of $K_{2,\,3}$ and P_3

5 Generalization of the tensor product of $K_{m,\,n}$ with P_k

Theorem 5.1. The tensor product of the complete bi-partite graph $K_{m,\,n}$ for $m,\ n > 1$ and the path P_k where $k > 3$ admits graceful labeling.

Proof: Let $u_1,\ u_2, \ldots, u_{m+n}$ be the vertices of the complete bi-partite graph $K_{m,\,n}$ and let $v_1,\ v_2, \ldots, v_k$ be the vertices of the path graph P_k. The tensor product of the complete bi-partite graph $K_{m,\,n}$ and the path P_k is denoted by $K_{m,\,n} \otimes P_k$.

Let $G = K_{m,\,n} \otimes P_k$. Then G consists of $k(m+n)$ vertices and $2(k-1)mn$ edges. We divide the vertices of $K_{m,\,n} \otimes P_k$ into k disjoint sets

$$V_j = \{(u_i,\ v_j)\ /\ i = 1, 2, \ldots, m+n \text{ and } j = 1, 2, \ldots, k\}$$

We prove the theoren in two cases.

Case 1: k is even.

Define $f : V(G) \longrightarrow \{0, 1, 2, \ldots, 2(k-1)mn\}$ as follows:
For odd j,

$$f(u_i,\ v_j) = \begin{cases} q + m - \left(\frac{k}{2}\right) mn - \left(\frac{j-1}{2}\right) mn - \left(\frac{k}{2} - 1\right) mn - n(i-1), & 1 \le i \le m, \\ q + m - \left(\frac{j-1}{2}\right) mn - m(i-m), & m+1 \le i \le m+n. \end{cases}$$

For even j,

$$f(u_i, v_j) = \left(\frac{j-2}{2}\right) mn + (i-1); \quad 1 \le i \le m+n.$$

Then f is a graceful labeling for G and hence G is graceful.

Case 2: k is odd.

Define $f : V(G) \longrightarrow \{0, 1, 2, \ldots, 2(k-1)mn\}$ as follows:

For odd j,

$$f(u_i,\ v_j) = \begin{cases} q + m - \left(\frac{k+1}{2}\right) mn - \left(\frac{j-1}{2}\right) mn - \left(\frac{k-1}{2} - 1\right) mn - n(i-1), & 1 \le i \le m, \\ q + m - \left(\frac{j-1}{2}\right) mn - m(i-m), & m+1 \le i \le m+n. \end{cases}$$

For even j,

$$f(u_i, v_j) = \left(\frac{j-2}{2}\right) mn + (i-1); \quad 1 \le i \le m+n.$$

Then f is a graceful labeling for G.

From both the cases, we have the tensor product of the complete bi-partite graph $K_{m,\,n}$ and the path P_k is graceful. ∎

Illustration 5.2. A graceful labeling of the tensor product $K_{4,\,4} \otimes P_4$ is shown in Figure 4.

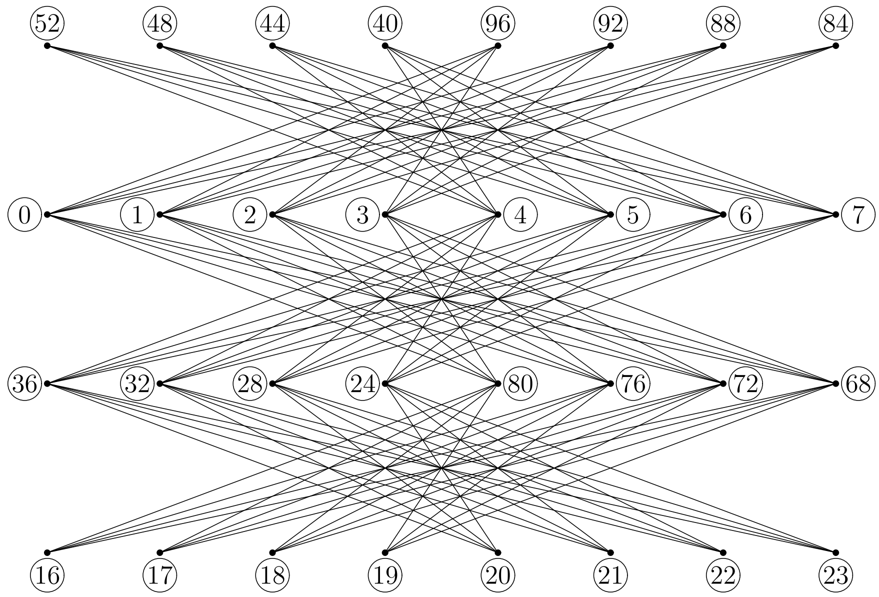

Figure 4. Tensor product of $K_{4,\,4}$ and P_4

Illustration 5.3. A graceful labeling of the tensor product $K_{4,\,5} \otimes P_5$ is shown in Figure 5.

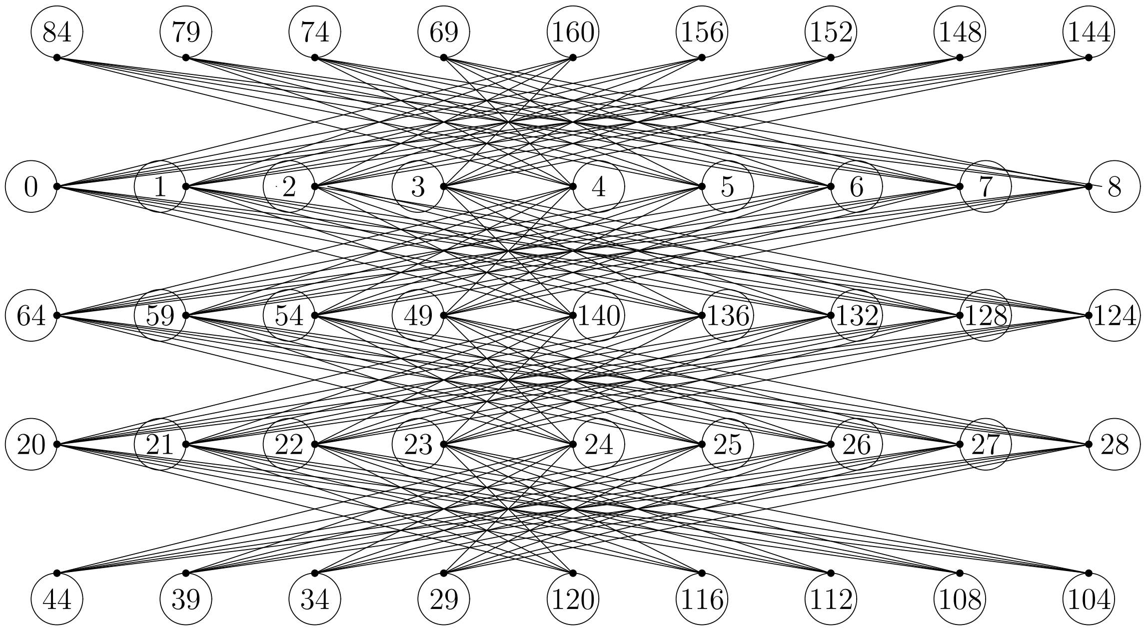

Figure 5. Tensor product of $K_{4,\,5}$ and P_5

6 Odd-even graceful labeling of the tensor product of $K_{1,\,n}$ with P_k

Theorem 6.1. The tensor product of the complete bi-partite graph $K_{1,\,n}$ and the path P_k where $n,\ k > 1$ admits odd-even graceful labeling.

Proof: Let $u_1,\ u_2, \ldots, u_{n+1}$ be the vertices of the complete bi-partite graph $K_{1,\,n}$ and let $v_1,\ v_2, \ldots, v_k$ be the vertices of the path graph P_k. The tensor product of the complete bi-partite graph $K_{1,\,n}$ and the path P_k is denoted by $K_{1,\,n} \otimes P_k$.

Let $G = K_{1,\,n} \otimes P_k$. Then G consists of $k(n+1)$ vertices and $2n(k-1)$ edges. We divide the vertices of $G = K_{1,\,n} \otimes P_k$ into k disjoint sets

$$V_j = \{(u_i,\ v_j)\ /\ i = 1, 2, \ldots, n+1 \text{ and } j = 1, 2, \ldots, k\}$$

.

We prove the theorem in two cases.

Case 1: k is odd.

Define $f : V(G) \longrightarrow \{0, 2, 4, \ldots, 2q\}$ where $q = 2n(k-1)$ as follows:

$$f(u_1, v_{2j-1}) = 2n(j-1) + 1, \qquad 1 \leq j \leq \frac{k+1}{2}$$

$$f(u_1, v_{2j}) = 2n(j-1), \qquad 1 \leq j \leq \frac{k-1}{2}$$

For $2 \leq i \leq n+1,\ 1 \leq j \leq k$,

$$f(u_i,\ v_j) = \begin{cases} 2q - n(j-1) + 2 - 2(i-1), & \text{for odd j} \\ 2n(k-1) + 3 - n(j-2) - 2(i-1), & \text{for even j} \end{cases}$$

Then f is a graceful labeling for G. **Case 2:** k is even.

Define $f : V(G) \longrightarrow \{0, 2, 4, \ldots, 2q\}$ where $q = 2n(k-1)$ as follows:

$$f(u_1, v_{2j-1}) = 2n(j-1) + 1, 1 \le j \le \frac{k}{2}$$
$$f(u_1, v_{2j}) \quad = 2n(j-1), 1 \le j \le \frac{k}{2}$$

For $2 \le i \le n+1,\ 1 \le j \le k$,

$$f(u_i,\ v_j) = \begin{cases} 2q - n(j-1) + 2 - 2(i-1), & \text{for odd j,} \\ 2n(k-1) + 3 - n(j-2) - 2(i-1), & \text{for even j.} \end{cases}$$

Then f is a graceful labeling for G. From both the cases, tensor product of the complete bi-partite graph $K_{1,\,n}$ and the path P_k where $n,\ k > 1$ admits an odd-even graceful labeling. ■

Illustration 6.2. Consider the complete bi-partite graph $K_{1,\,4}$ and the path P_7. The resultant graph $K_{1,\,4} \otimes P_7$ consists of 35 vertices and 48 edges. An odd-even graceful labeling of $K_{1,\,4} \otimes P_7$ is shown in Figure 6.

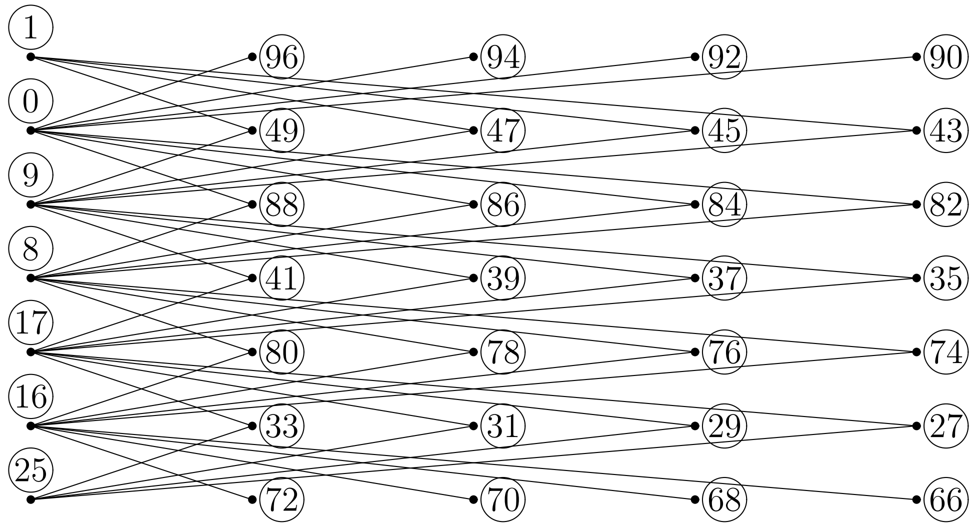

Figure 6. Tensor product of $k_{1,4}$ and P_7

Illustration 6.2 Consider the complete bi-partite graph $K_{1,\,5}$ and the path P_6. The resultant graph $K_{1,\,5} \otimes P_6$ consists of 36 vertices and 50 edges. An odd-even graceful labeling of $K_{1,\,5} \otimes P_6$ is shown in Figure 7.

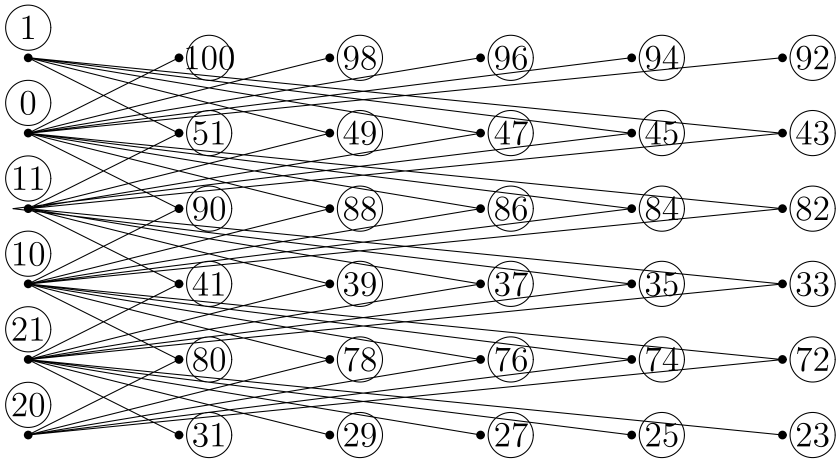

Figure 7. Tensor product of $K_{1,5}$ and P_6

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