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Steiner domination number of some wheel related graphs

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Abstract

For a non - empty set W of vertices in a connected graph G, the steiner distance $d(W)$ of W is the minimum size of a connected subgraph G containing W. Necessarily, each such subgraph is a tree and is called a steiner tree or a steiner W - tree. The set of all vertices of G that lie on some steiner W - tree is denoted by $S(W)$. If $S(W) = V(G)$ then W is called a steiner set for G. The *steiner number* $s(G)$ is the minimum cardinality of a steiner set. The minimum cardinality of a steiner dominating set is called the steiner domination number of graph.

Keywords: Domination number, steiner number and steiner domination number.
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1 Introduction

We begin with simple, finite, connected and undirected graph G with vertex set $V(G)$ and edge set $E(G)$. For standard terminology and notation in graph theory we rely upon West [6] while the concepts related to theory of domination we refer Haynes *et al* [2]. We give brief summary of definitions and other informations related to present work.

The concept of steiner number was introduced by Chartrand *et al.* [1] and further explored in Hernando *et al.* [3] as well as in Santhakumaran and John [5]. The concept of steiner domination was introduced by John *et al.* [4] in which they have investigated steiner domination number for K_n, tree and $K_{m,n}$.

Definition 1.1. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of shortest $u - v$ path in G.

Definition 1.2. For a non - empty set W of vertices in connected graph G, the *steiner distance* $d(W)$ of W is the minimum size of a connected subgraph G containing W. Necessarily, each such subgraph is a tree and is called a *steiner tree* or a *steiner W - tree*. The set of all vertices of G that lie on some steiner W - tree is denoted by $S(W)$. If $S(W) = V(G)$ then W is called a *steiner set* for G. The *steiner number* $s(G)$ is the minimum cardinality of a steiner set.

Definition 1.3. A set $S \subseteq V(G)$ of vertices in graph $G = (V(G), E(G))$ is called a *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. The *domination number* $\gamma(G)$ is the minimum cardinality of minimal dominating set of G.

Definition 1.4. For a connected graph $G = (V(G), E(G))$, a set of vertices W in G is called a *steiner dominating set* if W is both steiner and dominating while the *steiner domination number* $\gamma_s(G)$ is the minimum cardinality of a steiner dominating set of G.

Definition 1.5. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is a complete graph.

Definition 1.6. A systematic visit of each vertex of a tree is called a *tree traversal*.

Definition 1.7. Let G_1 and G_2 be two graphs with no vertex in common. The *join* of G_1 and G_2, denoted by $G_1 + G_2$ is the graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$, $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup J$, where $J = \{x_1x_2 : x_1 \in V(G_1), x_2 \in V(G_2)\}$.

Definition 1.8. The *wheel graph* W_n with n vertices is defined to be the join of K_1 and C_n. The vertex corresponding to K_1 is known as apex while the vertices corresponding to C_n are known as rim vertices.

Definition 1.9. The *helm graph* H_n is a graph obtained from wheel graph W_n by attaching a pendant edge to each rim vertex. It contains three types of vertices, the vertex of degree n called apex, n pendant vertices and n vertices of degree four.

Definition 1.10. The *flower graph* Fl_n is a graph obtained from a helm graph H_n by joining each pendant vertex to the apex of the helm graph. There are three types of vertices, the apex of degree $2n$, n vertices of degree four and n vertices of degree two.

Definition 1.11. The *sunflower graph* Sf_n is the graph obtained from the flower graph Fl_n by attaching n pendant edges to the apex vertex.

Proposition 1.12. [4] Each extreme vertex of a connected graph G belongs to every steiner dominating set of G.

Proposition 1.13. [4] If G is a connected graph of order n, then $2 \leq max\{s(G), \gamma(G)\} \leq \gamma_s(G) \leq n$.

In the present work, we investigate steiner domination number of wheel, helm and flower graphs.

2 Main Results

Theorem 2.1. For $n \geq 5$, $\gamma_s(W_n) = n - 2$.

Proof: For the wheel graph W_n, let $v_1, v_2, \ldots, v_n$ be the rim vertices and v be the apex. As the apex v is of maximum degree, each steiner tree traversal must contain it. Therefore, v is an internal vertex of some steiner tree. So, $v \in S(W)$ which implies that any steiner dominating set W contains only rim vertices.

Without loss of generality we begin with $W = \{v_1, v_3\}$. Then any tree traversal corresponding to v_1 and v_3 of size two must contain v_2 and v as internal vertices. Therefore $S(W) = \{v_1, v_3, v_2, v\} \neq V(W_n)$. If v_3 is replaced by v_{n-1}, that is, if $W = \{v_1, v_{n-1}\}$ then any tree traversal corresponding to v_1 and v_{n-1} of size two must contain v_n and v as internal vertices. Hence $S(W) = \{v_1, v_{n-1}, v_n, v\} \neq V(W_n)$. Even if v_1 and v_3 are replaced by any other vertices v_i and v_j where $1 \leq i, j \leq n$ then any tree traversal corresponding to v_i and v_j of size two does not contain all the vertices of W_n. Hence $S(W) \neq V(W_n)$. Thus $\gamma_s(W_n) > 2$.

Now if $v_{n-1} \in W$, then $W = \{v_1, v_3, v_{n-1}\}$ and any tree traversal corresponding to v_1, v_3 and v_{n-1} of size three contains only v as an internal vertex. Consequently, $S(W) = \{v_1, v_3, v_{n-1}, v\} \neq V(W_n)$. Even if v_1, v_3 and v_{n-1} are replaced by any other vertices v_i, v_j and v_k where $1 \leq i, j, k \leq n$ then any tree traversal corresponding to v_i, v_j and v_k of size three does not contain all the vertices of W_n. Hence $S(W) \neq V(W_n)$. Thus, $\gamma_s(W_n) > 3$.

From the above argument it is clear that if we initiate with v_1 then v_2, v_n and v are internal vertices. Hence all other vertices apart from v_2, v_n and v are not traversed by elements of W, implying $v_4, v_5 \ldots\ldots v_{n-2} \in W$. Therefore $W = \{v_1, v_3, v_4, v_5, \ldots, v_{n-2}, v_{n-1}\}$ then $S(W) = \{v_1, v_2, \ldots, v_n, v\} = V(W_n)$. Hence, W is a steiner set.

Moreover the vertices of W_n are either in W or adjacent to the vertices of W. In other words W is a steiner dominating set of minimum $n-2$ vertices. Hence, for $n \geq 5$, $\gamma_s(W_n) = n-2$. ■

Theorem 2.2. For $n \geq 4$, $\gamma_s(H_n) = n + 1$.

Proof: For helm graph H_n, let $u_1, u_2, \ldots, u_n$ be the pendant vertices and $v_1, v_2, \ldots, v_n$ be the rim vertices and v be the apex. So, $V(H_n) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, v\}$. The vertices $u_1, u_2, \ldots, u_n$ being extreme vertices, they must be in every steiner dominating set W according to proposition 1.12. Moreover we claim that every steiner dominating set W must contain the apex vertex v. If possible let $v \notin W$ then according to the definition of a steiner set $v \in S(W)$. That is, v must be an internal vertex that lies on some steiner $W-$tree. But v cannot belong to the path $u_1 - u_n$ otherwise the path $u_1 - u_n$ will not be of minimum size. Consequently $v \notin S(W)$. Hence $v \in W$.

Thus minimum $n + 1$ vertices, namely $u_1, u_2, \ldots, u_n$ and v are essential for any steiner set corresponding to H_n. Therefore $W = \{u_1, u_2, \ldots, u_n, v\}$. Then steiner $W-$tree induced by the vertices $u_1, u_2, \dots, u_n, v$ contains $v_1, v_2, \dots, v_n$ as internal vertices. That is, $S(W) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, v\} = V(H_n)$. Hence W is a steiner set.

Thus the vertices of H_n are either in W or adjacent to the vertices of W. In other words W is a steiner dominating set of minimum $n+1$ vertices. Therefore for $n \geq 4$, $\gamma_s(H_n) = n+1$. ∎

Theorem 2.3. For $n \geq 3$, $\gamma_s(Fl_n) = 2n$.

Proof: For the flower graph Fl_n, let $u_1, u_2, \dots, u_n$ be the vertices of degree two, $v_1, v_2, \dots, v_n$ be the vertices of degree four and v be the apex of degree $2n$. So, $V(Fl_n) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, v\}$. As $u_1, u_2, \dots, u_n$ are extreme vertices, by Proposition 1.12 they must be in every steiner dominating set W . Moreover we claim that $v_1, v_2, \dots, v_n$ must be in every steiner dominating set W. Since the degree of apex vertex is $2n$, that is, v is adjacent to every vertex of Fl_n. Hence there exists only one steiner $W-$tree of minimum size such that v is the only internal vertex. This implies that $v_1, v_2, \dots, v_n$ are not the internal vertices of a steiner $W-$tree. Hence, $v_1, v_2, \dots, v_n \notin S(W)$.

Therefore by the definition of steiner set $v_1, v_2, \dots, v_n \in W$. Hence $W = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$. So, $S(W) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, v\} = V(Fl_n)$. Therefore W is a steiner set.

The vertices of Fl_n are either in W or adjacent to the vertices of W. In other words W is a steiner dominating set of minimum $2n$ vertices. Hence, for $n \geq 3$, $\gamma_s(Fl_n) = 2n$. ∎

Corollary 2.4. $\gamma_s(Sf_n) = 3n$, for $n \geq 3$.

3 Concluding Remarks

The concept of steiner distance and dominating sets in graphs are explored. The steiner domination number is a combination of these two concepts. The steiner domination number is known for very few graphs and we have investigated steiner domination number of wheel and two other graphs obtained from wheel.

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References

[1] G. Chartrand and P. Zhang, *The Steiner Number of a Graph*, Discrete Mathematics, 242, (2002), 41 - 54.

[2] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[3] C. Hernando, T. Jiang, M. Mora, I. M. Pelayo and C. Seara, *On the Steiner, Geodetic and Hull Number of Graphs*, Discrete Mathematics, 293, (2005), 139 - 154.

[4] J. John, G. Edwin and P. Arul Paul Sudhahar, *The Steiner Domination Number of a Graph*, International Journal of Mathematics and Computer Application Research, 3(3), (2013), 37 - 42.

[5] A. P. Santhakumaran and J. John, *The Forcing Steiner Number of a Graph*, Discussiones Mathematicae Graph Theory, 31, (2011), 171 - 181.

[6] D. B. West, *Introduction to Graph Theory, 2/e*, Prentice - Hall of India, New Delhi, 2003.