International Journal of Mathematics and Soft Computing
Vol.5, No.2 (2015), 09 - 14.

ISSN Print : 2249 - 3328
ISSN Online: 2319 - 5215

The l_1 - Convexity number of a graph

S.V. Padmavathi*, **V. Swaminathan**

Ramanujan Research Centre in Mathematics
Saraswathi Narayanan College, Perungudi
Madurai-625 022, Tamilnadu, India.
svpadhma@yahoo.co.in

Abstract

Let G be a simple connected graph. A subset S of vertices of G is said to be a convex set if for any two vertices u, v of S, S contains all the vertices of every $u-v$ shortest path in G. The convexity number $con(G)$ of G is the maximum cardinality of a proper convex set of G. The local convexity number of a graph denoted by $l_1con(G)$ is defined as the maximum of $\{con(< N[x] >)/x \in V(G)$ and $con(< N[x] >)$ set is a proper convex set of $G\}$. For a connected graph G of order $n \geq 3$, we have $2 \leq l_1con(G) \leq n-1$. Local convex set for which its cardinality is same as $l_1con(G)$ is called a maximum local convex set. Local clique number denoted by $\omega_1(G)$ is the cardinality of a maximum clique in the set of all maximum local convex sets of G. Here we present characterisation of graphs for which $l_1con(G) = con(G)$, equivalent condition in graphs for which $N[S]$ is convex for any connected subgraph $< S >$ of G is presented. Interesting results and construction of graphs with prescribed $l_1con(G)$, $\omega_1(G)$ are also presented.

Keywords: Convex set, local convexity number, local clique number.
AMS Subject Classification(2010): 05C12.

1 Introduction

Throughout this paper G denotes a finite, connected, undirected graph without loops or multiple edges. In the study of convexity in graphs ([1], [5], [6], [7], [8], [9], [12], [13], [14], [15], [17]), two types of convexity played a vital role. A set S of vertices in G is g-convex (respectively, m-convex), if, for any pair of vertices u, v in S, all vertices on all shortest (respectively, induced) paths from u to v also lie in S.

For two vertices u and v in a connected graph G, the distance d(u, v) is the length of a shortest $u-v$ path in G referred to as a $u-v$ geodesic. For any set S of vertices in G and any integer $j \geq 0$, the closed neighborhood of radius j about S, denoted $N^j[S]$, is $\{ x/d(x,u) \leq j$ for some u in S $\}$. We write $N[S]$ instead of $N^1[S]$, and $N^j[x_1, \cdots, x_n]$ if the elements of S are explicitly given.
The local convexity may be distinguished in atleast four ways.

‡ This research work is supported by University Grants Commission of India.
∗ Corresponding author email:svpadhma@yahoo.co.in

(i): $N[v]$ is convex for every vertex v of G.
(ii): $N^j[v]$ is convex for every vertex v of G and for every $j \geq 0$.
(iii):$N[S]$ is convex for every vertex subset S of G.
(iv):$N^j[S]$ is convex for every vertex subset S of G and for every $j \geq 0$. A paper by Farber and Jamison [9] contains a class of graphs which are characterised by certain local convexity conditions with respect to geodesic convexity, in particular, those graphs in which balls around vertices are convex, and those graphs in which neighborhoods of convex sets are convex. Also paper by Soltan and Chepoi [17] contains some overlapping results to that of [18] but proofs are different.

In [9] equivalent condition in graphs for which $N[S]$ is convex for any convex subgraph $< S >$ of G is presented. Here we present characterisation of graphs for which $l_1con(G) = con(G)$, equivalent condition in graphs for which $N[S]$ is convex for any connected subgraph $< S >$ of G is presented. Also we present results related to local convexity numbers of G. Convexity number $con(G)$ of G is the maximum cardinality of a proper convex set of G. For local convexity number we consider cardinality of a maximum convex set among all $N[v]$ for any v in $V(G)$. If $N[v]$ is convex for every vertex v of G then local convexity number of G denoted by $l_1con(G)$ is the maximum cardinality of $N[v]$ with $N[v] \neq V(G)$. Otherwise $l_1con(G)$ is maximum of $\{$ $con(< N[v] >)/v \text{in} V(G)$ and $con(< N[v] >)$ set is a proper subset of $V(G)$ $\}$. From the above definition we observe that $con(< N[v] >) = |N[v]|$ if $N[v]$ is convex and equal to maximum cardinality of a convex set in $N[v]$ otherwise. Thus for a connected graph G of order $n \geq 3$, we have $2 \leq l_1con(G) \leq n - 1$. If G is a non-complete graph containing a complete subgraph H, then the vertex set $V(H)$ is convex in G thus $V(H)$ is local convex and so $l_1con(G) \geq |V(H)|$. The clique number $\omega(G)$ of a graph is the maximum order of a complete subgraph in G. Local clique number of a graph is defined as the maximum clique in $\langle l_1con(G)\rangle$. Clearly $l_1con(G) \geq \omega_1(G)$ for a non-complete connected graph. For a complete graph we have $\omega_1(K_n) = n-1$, $l_1con(K_n) = n-1$ and $\omega(K_n) = n$. But if G is non-complete then $2 \leq \omega_1(G) \leq \omega(G) \leq l_1con(G) \leq n - 1$.

Illustration 1.1. For the graph given in Figure 1, $l_1con(G) = 4 = \{d, e, k, j\}$, $con(G) = 9 = \{a, d, e, f, g, h, i, j, k\}$, $\omega_1(G) = 2 = |\{a, d\}|$ and $\omega(G) = 3 = |\{a, g, i\}|$. Hence, $\omega_1(G) < \omega(G)$.

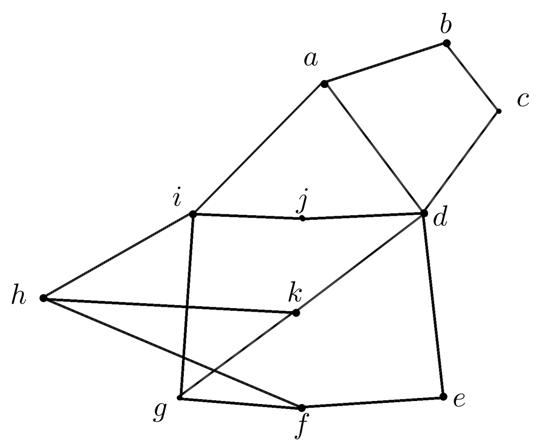

Figure 1: A graph G with $\omega_1(G) < \omega(G)$.

We observe that every m - convex set is a convex set and every convex set is a weak convex or isometric convex set. Also l_1-convex set is a convex set but not always m - convex set. In figure 1, $\{d, e, k, j\}$ is a l_1 - convex set which is a convex set but not m - convex set whereas $\{a, i, g\}$ is l_1 - convex as well as m - convex set. Also $\{a, b, c, d\}$ is a m - convex set which is not a l_1 - convex set. Therefore, set of all l_1 - convex set intersect both convex and m - convex regions.

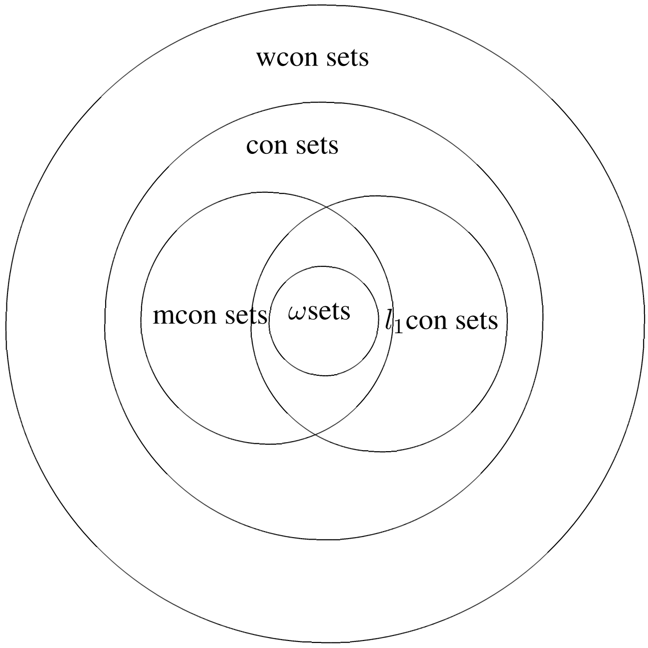

Figure 2

Observation 1.2. Let G be a non-complete connected graph. Then the largest convex set S which has a vertex adjacent to rest of the vertices of S is the maximum local convex set of G.

Corrolary 1.3. For every tree T of order $n \geq 3$,

$$l_1con(T) = \begin{cases} \Delta + 1 \text{ if } \ \Delta \neq n-1 \\ \\ n-1 \text{ if } \ \Delta = n-1 \end{cases}$$

Theorem 1.4. Let G be a non-complete connected graph of order n. Then $l_1con(G) = n - 1$ iff $\Delta = n - 2$ with no C_4 as induced subgraph containing the remaining vertex or $\Delta = n - 1$ with a complete vertex.

Proof: Suppose $l_1con(G) = n-1$ then let S be a maximum local convex set of cardinality $n-1$. Also let $u \in S$. Therefore, degree of u is $n-2$ in $< S >$ and G or degree of u is $n-2$ in $< S >$ and $n-1$ in G. If $\Delta = n-2$ then clearly G has no C_4 containing the remaining vertex. If degree of u is $n-1$ in G then G has a complete vertex.
Converse is obvious. ∎

Observation 1.5. For $n \geq 3$,

$$l_1con(C_n) = \begin{cases} 2 \text{ if } & n = 3, 4 \\ \\ 3 \text{ if } & n \geq 5 \end{cases}$$

2 Graphs with prescribed clique number, l_1-Convexity number and order

If G is a non-complete connected graph of order n such that $\omega_1(G) = l_1^{'}$ and $l_1con(G) = k_1^{'}$, then G is called an $(l_1^{'}, k_1^{'}, n)$ graph. Now we show that $(2, 3, 5)$ is either C_5 or has a pendant.

Theorem 2.1. The $(2, 3, 5)$ graph is either C_5 or has a pendant with $l_1con(G) = 3$ and $\omega_1(G) = 2$.

Proof: Let G be a connected graph of order 5 with $\omega_1(G) = 2$ and $l_1con(G) = 3$. Let $S = \{u, v, w\}$ be a maximum local convex set in G and let $u - v - w$ be a path of length 2. From hypothesis we observe the following.

(i) G has no triangles, since $\omega_1(G) = 2$.

(ii) Suppose G has a C_4 then G is C_4 with a pendant. Otherwise G is C_5 or P_5 or $K_{1,3}$ with a pendant.

Hence the theorem is proved. ∎

3 Realisation Problem

Lemma 3.1. For every pair $k_1^{'}, n$ of integers with $n \geq 3$, $2 \leq k_1^{'} \leq n - 1$ there exists a non-complete connected graph such that $\omega_1(G) = l_1con(G) = k_1^{'}$.

Required graph F is obtained as follows. F = $(K_{k_1^{'}-1} \bigcup \bar{K}_{n-k_1^{'}-1})$ + $\bar{K}_2$. $l_1con(G)$ set has $k_1^{'} - 1$ vertices with a vertex of $\bar{K}_2$. If atleast one vertex of $\bar{K}_{n-k_1^{'}-1}$ is included then convexity is violated.

Theorem 3.2. For every triple $l_1^{'}, k_1^{'}, n$ with $2 \leq l_1^{'} \leq k_1^{'} \leq n - 1$ there exists a non-complete connected graph of order n having $\omega_1(G) = l_1^{'}$, $l_1con(G) = k_1^{'}$.

Proof: If $\omega_1(G) = l_1con(G) = k_1^{'}$ then by Lemma 3.1 we get the result.
Assume $l_1^{'} < k_1^{'}$. Consider $F = (K_{l_1^{'}-1} + \bar{K}_2)$ where $V(\bar{K}_2) = \{u_1, u_2\}$. Consider $(\bar{K}_{k_1^{'}-l_1^{'}}) + u_1$, $u_2 + \bar{K}_2$ where $V(\bar{K}_2) = \{v_1, v_2\}$ and remaining $(n - k_1^{'} - 3)$ vertices as isolates are joined to $\{v_1, v_2\}$. ∎

Corrolary 3.3. For every three integers $l_1^{'}, k_1^{'}, N$ such that $2 \leq l_1^{'} \leq k_1^{'}$ and $N \geq 2$ there exists a connected graph G with $\omega_1(G) = l_1^{'}$, $l_1con(G) = k_1^{'}$ whose vertices can be partitioned into N maximum l_1 convex sets.

Proof: If $N = 2$, consider two copies of F = $(K_{l_1^{'}-1} \bigcup \bar{K}_{k_1^{'}-l_1^{'}})$ + K_1.Let $K_1 = u_1$. Let $V(\bar{K}_{k_1^{'}-l_1^{'}}) = \{w_1, w_2 \cdots w_{k_1^{'}-l_1^{'}}\}$. Join two $u_1^{'}s$ and $w_i^{'}s$ for $i = 1 \text{to} k_1^{'} - l_1^{'}$. Also join each vertex of $K_{l_1^{'}-1}$ to w_1 of next copy. For $N = 3$, join as in $N = 2$ and for third copy join u_1 to second copy of u_1 and $w_i^{'}s$. By the very construction the theorem is true. Repeat the same for large values of N changing $w_i^{'}s$. ∎

Theorem 3.4. For every five positive integers a, b, c, d, n with $2 \leq a \leq b \leq c \leq d \leq n-1$ there exists a connected graph G of order n and $l_1con(G) = a$, $mcon(G) = b$, $con(G) = c$, $wcon(G) = d$.

Proof: Consider $K_{1,a-1}$. Form a path on $b-a$ vertices. Let $V(K_{1,a-1}) = \{u_1, u_2, \cdots, u_a\}$ centered at u_1. Let $V(P_{b-a}) = \{u_{a+1}, u_{a+2}, \cdots, u_b\}$. Join u_{a+1} to u_2, u_{a+2} to u_3 and so on untill all the vertices of P_{b-a} exhaust. Next consider $\bar{K}_{c-b}$. Let $V(\bar{K}_{c-b}) = \{u_{b+1}, u_{b+2}, \cdots, u_c\}$. New edges are formed by joining each vertex of $\bar{K}_{c-b}$ to u_{a+2} except u_{b+1}, instead $u_{b+1}u_{a+1}$ edge is formed. Also $\{u_{b+2}, u_{b+3}, \cdots, u_c\}$ vertices are joined to u_{b+1}. Form a path on $d-c$ vertices. Let $V(P_{d-c}) = \{u_{c+1}, u_{c+2}, \cdots, u_d\}$. Form new edges $u_{c+1}u_{b+1}$, $u_{c+2}u_{b+2}$, $u_{c+4}u_{b+3}$, $u_{c+6}u_{b+4}$, $u_{c+8}u_{b+5}$ and so on until vertices of P_{d-c} exhausts. Suppose $u_{c+y \neq d}$ is joined to u_c then u_{c+y+2} is joined to u_{b+2}, and the process repeated. If u_{d-1} is joined to some vertex in $\{u_{b+2}, u_{b+3}, \cdots, u_c\}$ then by our construction u_d does not form an edge with any of the vertex in $\{u_{b+2}, u_{b+3}, \cdots, u_c\}$. Since $con(G)$ by this construction is d we join u_d to u_{b+1}. Rest of $n-d$ vertices form a path. Now consider the cycle C formed in the construction by vertices $u_{c+1}u_{c+2}u_{c+3}u_{c+4}u_{b+3}u_{b+1}$. Let $V(P_{n-d}) = \{u_{d+1}, u_{d+2}, \cdots, u_n\}$. Join u_{d+1} to u_{c+1}, u_{d+2} to u_{c+3}, u_{d+3} to u_{b+3}, u_{d+4} to u_{c+1} until all the vertices of P_{n-d} exhaust. Also join u_{d+1} to u_{b+2}. By the construction we can easily check for $l_1con(G) = a$, $mcon(G) = b$, $con(G) = c$, $wcon(G) = d$. ∎

Theorem 3.5. For every five positive integers a, b, c, d, n with $2 \leq a \leq b \leq c \leq d \leq n-1$ there exists a connected graph G of order n and $mcon(G) = a$, $l_1con(G) = b$, $con(G) = c$, $wcon(G) = d$.

Proof: Consider $K_{1,b}$. Let $V(K_{1,b}) = \{u_1, u_2, \cdots, u_a, u_{a+1}, \cdots, u_b\}$ centered at u_1. Form a path on $\{u_2, u_3, \cdots, u_a\}$. Join u_2 to u_{a+1}. Form a path on $c-b$ vertices. Let $V(P_{c-b}) = \{u_{b+1}, u_{b+2}, \cdots, u_c\}$. Join u_{b+1} to u_{a+1}, u_{b+2} to u_{a+2} and so on until all the vertices in P_{c-b}. Also join u_a to u_{b+3} to make u_a non-complete vertex. Next form a path on $d-c$ vertices. Let $V(P_{d-c}) = \{u_{c+1}, u_{c+2}, \cdots, u_d\}$. Join u_{c+1} to u_{b+1}, u_{c+2} to u_{b+2} and the process repeated until u_d is joined to some vertex in $\{u_{b+1}, u_{b+2}, \cdots, u_c\}$. Rest of $n-d$ vertices are also formed as path. Join u_{d+1} to u_{c+1} and u_{b+2}. Consider the cycle C formed in this construction on the vertices u_{c+1}, u_{c+2}, u_{c+3}, u_{b+3}, u_{b+2}, u_{b+1}. Join u_{d+2} to u_{c+3}, u_{d+3} to u_{b+2}, u_{d+4} to u_{c+1} and the process repeated until u_n is joined to a vertex among $\{u_{c+1}, u_{c+3}, u_{b+2}\}$. By the construction we can easily check for $mcon(G) = a$, $l_1con(G) = b$, $con(G) = c$, $wcon(G) = d$. ∎

References

[1] L.M. Batten, *Geodesic Subgraphs*, J. Graph Theory, 7(1983), 159-163.

[2] F. Buckley and F. Harary, *Distance in Graphs*, Redwood City, Addison - Wesley, 1990.

[3] G. Chartrand, Curtiss.E.Wall and Ping Zhang, *The Convexity Number of a Graph*, Graphs and combinatorics, 18(2002), 209-217.

[4] Douglas B. West, *Introduction to Graph Theory*, Second Edition Prentice - Hall, 2001.

[5] P. Duchet, *Convex sets in Graphs II Minimal path convexity*, J.Comb.Theory ser-B, 44(1988), 307-316.

[6] P. Duchet, and H, Meyniel, *Ensemble convexes dans les graphes I*, European J. Combin., 4(1983), 127-132.

[7] M. Farber, *Bridged graphs and geodesic convexity*, Discrete Mathematics, 66, 3(1987), 249-257.

[8] M. Farber and R.E. Jamison, *Convexity in Graphs and Hypergraphs*, SIAM J. Alg. Disc. Math., 7(1986), 433-444.

[9] M. Farber and R.E. Jamison, *On Local Convexity in Graphs*, Discrete Mathematics, 66(1987), 231-247.

[10] J. Gimbel, *Some Remarks on the Convexity Number of a Graph*, Graphs and Combinatorics, 19(2003), 351-361.

[11] F. Harary, *Graph Theory*, Addison Wesley, Reading Mass, 1969.

[12] F. Harary and J. Nieminen, *Convexity in Graphs*, J.Differ. Geom., 16(1981), 185-190.

[13] S.P.R. Hebbare, *A Class of Distance Simple Graphs*, Ars Combinatorics, 7(1979), 19-26.

[14] R.E. Jamison, *Partition numbers for Trees and Ordered sets*, Pac.J.Math., 96(1981), 115-140.

[15] H.M. Mulder, *The Interval Function of a Graph*, Math.Centre Tracts 132, Amsterdam, 1980.

[16] Ralph J. Faudree, Zdenek Ryjacek and Richard H.Schelp, *On local and global independence numbers of a graph*, Discrete Applied Mathematics, 132(2004), 79-84.

[17] V.P. Soltan, *d-convexity in graphs*, Soviet Math. Dokl., 28(1983), 419-421.

[18] V.P. Soltan and V.D. Chepoi, *Conditions for Invariance of set diameters under d-convexification in a Graph*, Cybernetics, 19(1983), 750-756.

[19] M. Van de Vel, *Theory of convex structures*, North - Holland, Amsterdam, MA, 1993.