On Face Magic Labeling of Duplication Graphs

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Abstract

A labeling of a graph G of type (a, b, c) assigns labels from the set {1, 2, …, aυ + be + cf} to the vertices, edges and faces of G such that each vertex receives a label, each edge receives b label and each face receives c label and each number is used exactly once as a label. We restrict a, b and c to be not greater than one. The weight of face w(f) under a labeling is the sum of labels of the face itself together with labels of vertices and edges surrounding that face. A labeling is said to be magic, if for every positive integer s, all s-sided faces have the same weight. In this paper, we examine whether vertex and edge duplication on families of graphs admit face magic labeling of types (1, 0, 1), (1, 1, 0), (0, 1, 1) and (1, 1, 1) or not.

Keywords: Face magic labeling, Bijective function, Duplication graphs.

1. Preliminary

Throughout this paper, all graphs are finite plane graphs without loops and multiple edges. Let $G = (V, E, F)$ be a finite plane graph where V, E and F are its vertex set, edge set and set of interior face with $|V| = \upsilon$, $|E| = e$ and $|F| = f$. A labeling of type (a, b, c) assigns labels from the set $\{1, 2, \ldots, a\upsilon + be + cf\}$ to the vertices, edges and faces of G such that each vertex receives a label, each edge receives b label and each face receives c label and each number is used exactly once as a label. We restrict a, b and c to be not greater than one. Labeling of type $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are called vertex, edge and face labeling, respectively. The weight of face $w(f)$ under a labeling is the sum of labels of the face itself together with labels of vertices and edges surrounding that face. A labeling is said to be magic, if for every positive integer s, all s-sided faces have the same weight. We allow different weights for different s. A labeling is said to consecutive if, for every positive integer s the weight of all s-sided faces constitute a set of consecutive integers. For standard terminology and notation we follow *Bondy* and *Murty* [6].

The notions of magic and consecutive labeling of plane graphs were defined by *Ko-Wei Lih* in [8]. However, the subject of magic labeling can be traced back to the 13th century when similar notions were investigated by the *Chinese mathematician Yang Hui* (1275). This concept was further developed by *Chang Chhao* (1670). *Ko-Wei Lih* [8] clarified the concepts after *Pao Chhi-Shou's* labeling by using modern notions of the graph theory and extend these classical labelings of platonic polyhedral to certain families of plane graphs. *Lih* [8] described face-magic labeling of type *(1, 1, 0)* for the wheels, the friendship graphs, the prisms and some of the platonic polyhedra. *Martin Baca* [2] has described the magic and consecutive labeling for fans, planar pyramids and ladders. The face-magic labeling of type *(1, 1, 1)* for Mobius ladder L_n^m, $n \geq 3$ odd, $m \geq 1$, grid graphs G_n^m, $n \geq 2$, $m \geq 1$, $n + m \neq 3$ and the hexagonal planar graphs H_n^m (honeycomb) are proved in [3], [5] and [4], respectively. In [9] the face-magic labeling of type *(1, 1, 1)* for special families of planar graphs with 3-sided faces, 5-sided faces, 6-sided faces and one external infinite face are shown. In [10], magic labeling of type *(a, b, c)* for families of wheels are proved. *Martin Baca* and others investigated the face magic labeling for some planar graphs. For further details we refer the recent survey of graph labeling by *Gallian* [7]. In this paper, we investigate the results of face magic labeling of types *(1, 0, 1)*, *(1, 1, 0)*, *(0, 1, 1)* and *(1, 1, 1)* on duplication graphs.

Definition 1.1 *Duplication of a vertex* v_k *by a new edge* $e = v'v''$ *in a graph G produces a new graph G' such that* $N(v') = \{v_k, v''\}$ *and* $N(v'') = \{v_k, v'\}$.

Definition 1.2 *Duplication of an edge* $e = v_iv_{i+1}$ *by a vertex* v' *in a graph G produces a new graph G' such that* $N(v') = \{v_iv_{i+1}\}$.

2. Main Result

In this section, face magic labeling of vertex and edge duplication of graphs are discussed.

Theorem 2.1 *Let G be a tree of order* $n \geq 2$. *Then the vertex by edge duplication at all vertices simultaneously on G admits face magic labeling of types (1, 0, 1), (1, 1, 0), (0, 1,1) and (1, 1, 1).*

Proof

Let *G (V, E, F)* be an arbitrary tree of order n and let $V = \{v_i : 1 \leq i \leq n\}$, $E = \{e_i : 1 \leq i \leq n-1\}$.

Let *G′ (V′, E′, F′)* be the graph obtained by duplicating all vertices in G by edges and let $V' = V \cup \{v_i', v_i'': 1 \leq i \leq n\}$, where v_i', v_i'' are adjacent to v_i, edge set $E' = E \cup \{v_i v_i', v_i v_i'', v_i' v_i'': 1 \leq i \leq n\}$ and face set $F' = \{f_i : v_i v_i' v_i'': 1 \leq i \leq n\}$.

The four types of labeling are discussed separately.

Type (i): *(1, 0, 1)-Face magic*

Define a mapping $f_1 : V' \cup F' \to \{1, 2, \cdots, 4n\}$ in the following way:

For i = 1 to n; $f_1(v_i) = i,\ f_1(v_i') = 2n+1-i,\ f_1(v_i'') = 2n+i,\ f_1(f_i) = 4n+1-i.$

Clearly, the common weight of all 3-sided faces is $w(f_i) = 8n+2$.

Type (ii): *(1, 1, 0)-Face magic*

We construct a mapping $f_2 : V' \cup E' \to \{1, 2, \cdots, 7n-1\}$ as follows:

For i = 1 to n; $f_2(v_i) = i,\ f_2(v_i') = 2n+1-i,\ f_2(v_i'') = 2n+i,$

$f_2(v_i v_i') = 4n+1-i,\ f_2(v_i v_i'') = 4n+i,\ f_2(v_i' v_i'') = 6n+1-i.$

For i = 1 to n-1; $f_2(e_i) = 6n+i.$

Clearly, the common weight of all 3-sided faces is $w(f_i) = 18n+3$.

Type (iii): *(0, 1, 1)-Face magic*

Define a bijective mapping $f_3 : E' \cup F' \to \{1, 2, \cdots, 5n-1\}$ as below:

For i = 1 to n; $f_3(v_i v_i') = i,\ f_3(v_i v_i'') = 2n+1-i,\ f_3(v_i' v_i'') = 2n+i,\ f_3(f_i) = 5n-i.$

For i = 1 to n-1; $f_3(e_i) = 3n+i.$

The common weight for all 3-sided faces is $w(f_i) = 9n+1$.

Type (iv): *(1, 1, 1)-Face magic*

Define a bijective mapping $f_4 : V' \cup E' \cup F' \to \{1, 2, \cdots, 8n-1\}$ as follows:

For i = 1 to n; $f_4(v_i) = i,\ f_4(v_i') = 2n+1-i,\ f_4(v_i'') = 2n+i,$

$f_4(v_i v_i') = 4n+1-i,\ f_4(v_i v_i'') = 4n+i,\ f_4(v_i' v_i'') = 7n+1-2i,\ f_4(f_i) = 7n-1+i.$

For i = 1 to n-1; $f_4(e_i) = 7n-2i.$

The common weight for all 3-sided faces is $w(f_i) = 26n+2$.

Illustration 1: The graph obtained by vertex by edge duplication at all vertices on T_7 and its face magic labeling is shown in figure 1.

Illustration 2: The graph obtained by edge by vertex duplication at all edges on T_8 and its face magic labeling of type (0, 1, 1) is shown in figure 2.

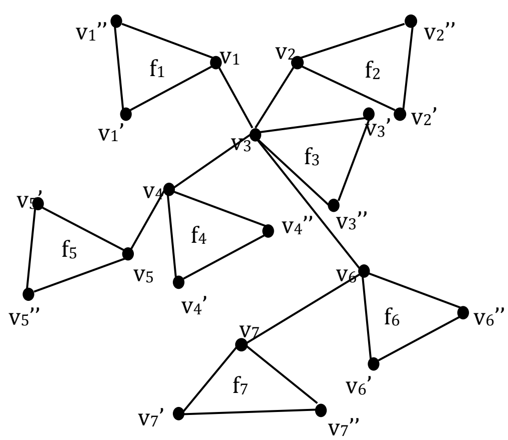

Figure 1:

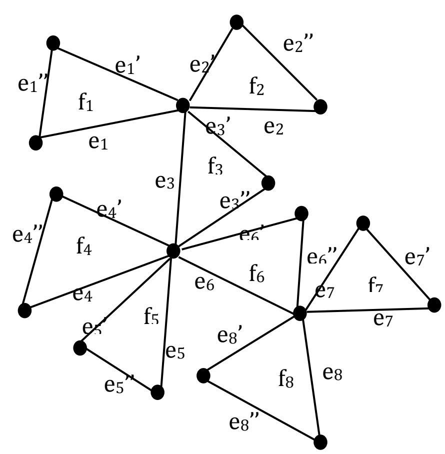

Figure 2:

Theorem 2.2 *Let G be a tree of order* $n \geq 2$. *Then the edge by vertex duplication at all edges simultaneously on G admits face magic labeling of type (0, 1, 1).*

Proof

Let *G (V, E, F)* be an arbitrary tree of order *n* and assign the labels $u_1, u_2, \cdots, u_n$ to vertices and $e_1, e_2, \cdots, e_{n-1}$ to edges. Let *G′ (V′, E′, F′)* be the graph obtained by "*edge by vertex duplication"* at all edges in G simultaneously. Let $E' = E \cup \{ e_i', e_i''; 1 \le i \le n-1 \}$, where e_i' and e_i'' are adjacent to e_i, face set $F' = \{ f_i : e_i e_i' e_i'' : 1 \le i \le n-1 \}$.

We construct a mapping $g : E' \cup F' \to \{1, 2, \cdots, 4n\}$ in the following way:

For i = 1 to n-1; $g(e_i) = i,\ g(e_i') = 2n-1-i,\ g(e_i'') = 2(n-1)+i,\ g(f_i) = 4n-3-i.$

We compute the common weight of all 3- sided faces is

$w(f_i) = g(e_i) + g(e_i') + g(e_i'') + g(f_i) = i + (2n-1-i) + (2(n-1)+i) + (4n-3-i) = 8n-6.$

Theorem 2.3 *For $n, m \ge 3$, the graph G' obtained by duplicating all vertices by edges on mP_n admits face magic labeling of types (1, 0, 1), (1, 1, 0) and (0, 1, 1).*

Proof

Let mP_n be the disjoint union of m path graph P_n with $V = \{v_i^j : 1 \le i \le m, 1 \le j \le n\}$,

$E = \{e_i^j : 1 \le i \le m, 1 \le j \le n-1\}$. Let G' be the graph obtained by duplicating all vertices in mP_n by edges simultaneously. Let the vertex set $V' = V \cup \{u_i^j, w_i^j : 1 \le i \le m, 1 \le j \le n\}$, edge set $E' = E \cup \{u_i^j v_i^j, v_i^j w_i^j, u_i^j w_i^j : 1 \le i \le m, 1 \le j \le n\}$ and face set $F' = \{f_i^j : 1 \le i \le m, 1 \le j \le n\}$.

The three types of labeling are discussed below:

Type (i): *(1, 0, 1)-Face magic*

Define a mapping $f_1 : V' \cup F' \to \{1, 2, \cdots, 4mn\}$ in the following way:

For i = 1 to m, j = 1 to n; $f_1(v_i^j) = (i-1)n + j,\ f_1(u_i^j) = 2mn + (i-1)n + j,$

$f_1(w_i^j) = 2mn + 1 - (i-1)n - j,\ f_1(f_i^j) = 4mn + 1 - (i-1)n - j.$

Clearly, the weight of all 3-sided faces is $w(f_i) = 8mn + 2.$

Type (ii): *(1, 1, 0)-Face magic*

Define a bijective mapping $f_2 : V' \cup E' \to \{1, 2, \cdots, m(7n-1)\}$ as follows:

For i = 1 to m, j = 1 to n; $f_2(v_i^j) = (i-1)n + j,\ f_2(u_i^j) = 2mn + (i-1)n + j,$

$f_2(w_i^j) = 2mn+1-(i-1)n-j,\ f_2(u_i^j w_i^j) = 4mn+1-(i-1)n-j,$

$f_2(v_i^j w_i^j) = 4mn+(i-1)n+j,\ f_2(u_i^j v_i^j) = 6mn+1-(i-1)n-j.$

For i = 1 to m, j = 1 to n-1; $f_2(e_i^j) = 6mn+(j-1)m+i.$

Clearly, the weight of all 3-sided faces is $w(f_i) = 18mn+3.$

Type (iii): *(0, 1, 1)-Face magic*

A bijective mapping $f_3 : E' \cup F' \to \{1, 2, \cdots, m(5n-1)\}$ is given below:

For i = 1 to m, j = 1 to n; $f_3(v_i^j w_i^j) = (i-1)n+j,\ \ f_3(u_i^j w_i^j) = 2mn+1-(i-1)n-j,$

$f_3(u_i^j v_i^j) = 2mn+(i-1)n+j,\ f_3(f_i^j) = 5mn-m+1-(i-1)n-j.$

For i = 1 to m, j = 1 to n-1; $f_2(e_i^j) = 6mn+(j-1)m+i.$

Clearly, the weight of all 3-sided faces is $w(f_i) = 9mn-m+2.$

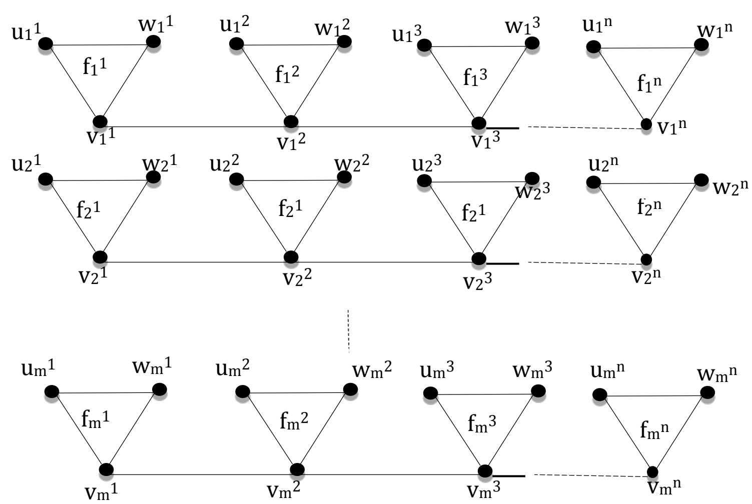

Figure 3: Labeled graphs for vertex by edge duplication on mP_n

Theorem 2.4 *The graph G′ obtained by vertex by edge duplication at all vertices on* mP_n *which admits face magic labeling of type (1, 1, 1), where both m and n are not simultaneously even.*

Proof

Let mP_n be the disjoint union of m path graph P_n with $V=\{v_i^j : 1\le i\le m, 1\le j\le n\}$ and $E=\{e_i^j : 1\le i\le m, 1\le j\le n-1\}$. Let G' be the graph obtained by duplicating all vertices by edges in mP_n simultaneously. Let the vertex set $V' = V\cup\{u_i^j, w_i^j : 1\le i\le m, 1\le j\le n\}$, edge set $E' = E\cup\{u_i^j v_i^j, v_i^j w_i^j, u_i^j w_i^j : 1\le i\le m, 1\le j\le n\}$ and face set $F'=\{f_i^j : 1\le i\le m, 1\le j\le n\}$.

Define a bijective mapping $f_4 : V'\cup E'\cup F' \to \{1, 2, \cdots, 8mn-m\}$ as follows:

We consider two cases depending on the values of m and n.

Case (i): when $m, n\equiv 1 \,(\mathrm{mod}\,2)$, $m, n\ge 3$

For i = 1 to m, j = 1 to n;

$f_4(v_i^j)=(i-1)n+j$, $f_4(u_i^j)=2mn+(i-1)n+j$, $f_4(w_i^j)=2mn+1-(i-1)n-j$,

$f_4(u_i^j w_i^j)=4mn+1-(i-1)n-j$, $f_4(v_i^j w_i^j)=4mn+(i-1)n+j$,

$f_4(e_i^j)=6mn+(i-1)(n-1)+j.$

$$f_4(u_i^j v_i^j)=\begin{cases} 5mn+\left\lceil \dfrac{mn}{2}\right\rceil+1-(i-1)\left(\dfrac{n}{2}\right)-\left(\dfrac{j+1}{2}\right), & \text{if } i \text{ and } j \text{ are odd}, 1\le i\le m, 1\le j\le n,\\ 6mn-(i-1)\left(\dfrac{n}{2}\right)+1-\left(\dfrac{j}{2}\right), & \text{if } i \text{ is odd and } j \text{ is even}, 1\le i\le m, 2\le j\le n-1,\\ 6mn-\left(\dfrac{n-1}{2}\right)-(i-2)\left(\dfrac{n}{2}\right)+1-\left(\dfrac{j+1}{2}\right), & \text{if } i \text{ is even and } j \text{ is odd}, 2\le i\le m-1, 1\le j\le n,\\ 5mn+\left\lceil \dfrac{mn}{2}\right\rceil-\left(\dfrac{n-1}{2}\right)-(i-2)\left(\dfrac{n}{2}\right)-\dfrac{j}{2}, & \text{if } i \text{ and } j \text{ are even}, 2\le i\le m-1, 2\le j\le n-1. \end{cases}$$

$$f_4(f_i^j)=\begin{cases}8mn-m+1-(i-1)\left(\frac{n}{2}\right)-\left(\frac{j+1}{2}\right), & \text{if } i \text{ and } j \text{ are odd}, 1\le i\le m, 1\le j\le n,\\ 7mn-m+\left\lceil\frac{mn}{2}\right\rceil-(i-1)\left(\frac{n}{2}\right)-\left(\frac{j}{2}\right), & \text{if } i \text{ is odd and } j \text{ is even}, 1\le i\le m, 2\le j\le n-1,\\ 7mn-m+\left\lceil\frac{mn}{2}\right\rceil-\left(\frac{n-1}{2}\right)-(i-2)\left(\frac{n}{2}\right)-\left(\frac{j+1}{2}\right), & \text{if } i \text{ is even and } j \text{ is odd},\\ & 2\le i\le m-1, 1\le j\le n,\\ 8mn-m-\left(\frac{n-1}{2}\right)-(i-2)\left(\frac{n}{2}\right)-\frac{j}{2}, & \text{if } i \text{ and } j \text{ are even}, 2\le i\le m-1, 2\le j\le n-1.\end{cases}$$

Clearly, the common weight of all 3-sided faces is $w(f_i)=25mn-m+\left\lceil\frac{mn}{2}\right\rceil+3.$

Case (ii): When (a) $m\equiv 1\,(\text{mod}2)$ *and* $n\equiv 0\,(\text{mod}2)$

(b) $m\equiv 0\,(\text{mod}2)$ *and* $n\equiv 1\,(\text{mod}2)$

In this case, labels are assigned to the vertices as in case (i). We define the edge and face labeling as follows;

For i = 1 to m, j = 1 to n; $f_4(u_i^j w_i^j)=4mn+1-(i-1)n-j,$

$f_4(v_i^j w_i^j)=4mn+(i-1)n+j,\;\; f_4(e_i^j)=6mn+(i-1)(n-1)-1+j.$

When m is odd and n is even, let

$$f_4(u_i^j v_i^j)=\begin{cases}5mn-\frac{1}{2}(in-n+j+1)+1, & \text{if } j \text{ is odd}, 2\le i\le m, 1\le j\le n-1\\ 5mn+\frac{mn}{2}+1-\frac{1}{2}(in-n+j), & \text{if } j \text{ is even}, 1\le i\le m, 2\le j\le n.\end{cases}$$

$$f_4(f_i^j)=\begin{cases}7mn-m-\frac{1}{2}(in-n+j-mn+1)+2, & \text{if } j \text{ is odd}, 2\le i\le m, 1\le j\le n-1\\ 8mn-m+1-\frac{1}{2}(in-n+j), & \text{if } j \text{ is even}, 1\le i\le m, 2\le j\le n.\end{cases}$$

When m is even and n is odd, let

$$f_4(f_i^{\,j}) = \begin{cases} 6mn + m(n-1)+1, & if \;\; i=1 \;\; and \;\; j=1, \\ 7mn + \frac{3}{2}(mn) - \frac{1}{2}\left(in - n + j + 1\right) - (m+n-2), & if \; i \; and \; j \; are \; odd, \; 3 \le i \le m-1, 1 \le j \le n, \\ 8mn - \frac{1}{2}\left(in - n + j\right) - m + 1, & if \; i \; is \; odd \; and \; j \; is \; even, 1 \le i \le m-1, \; 2 \le j \le n-1, \\ 8mn - \frac{1}{2}\left(in - n + j\right) - m + 1, & if \; i \; is \; even \; and \; j \; is \; odd, \; 2 \le i \le m, 1 \le j \le n, \\ 6mn + \frac{mn}{2} - \frac{1}{2}\left(in + n + j + 1\right) - m + n + 1, & f \; i \; and \; j \; are \; even, \; 2 \le i \le m, \; 2 \le j \le n-1. \end{cases}$$

$$f_4(u_i^j v_i^j) = \begin{cases} 6mn + 1 - \left(\frac{j+1}{2}\right), & if \;\; j \;\; is \; odd, \;\; i=1 \;\; and \;\; j \ne 1, \\ 6mn + \left(\frac{mn}{2}\right), & if \;\; i=1 \;\; and \;\; j=1, \\ 6mn - \frac{1}{2}\left(in + n + j + 1\right) + 1, & if \; i \; and \; j \; are \; odd, 3 \le i \le m-1, 1 \le j \le n, \\ 5mn + \frac{mn}{2} - \frac{1}{2}\left(in - n + j\right) + 1, & if \; i \; is \; odd \; and \; j \; is \; even, 1 \le i \le m-1, \; 2 \le j \le n-1, \\ 5mn + \frac{mn}{2} - \frac{1}{2}\left(i + n + j\right) + 1 + n, & if \; i \; is \; even \; and \; j \; is \; odd, \; 2 \le i \le m, 1 \le j \le n, \\ 6mn - \frac{1}{2}\left(in + n + j + 1\right) + n, & f \; i \; and \; j \; are \; even, \; 2 \le i \le m, \; 2 \le j \le n-1. \end{cases}$$

Clearly, the common weight of all 3-sided faces is $w(f_i) = 25mn + \frac{mn}{2} - m + 4.$

Theorem 2.5 *For* $n \ge 4$ *and* $m \ge 2$, *the graph G obtained by edge by vertex duplication at all edges on* mP_n *admits face magic labeling of type (0, 1, 1) and also G admits face magic labeling of types (1, 0, 1) and (1, 1, 0) if n is even.*

Proof

Let *G (V, E, F)* be the graph obtained by duplicating all edges in mP_n by vertices respectively.

Let $V = \{v_i^j : 1 \le i \le m, 1 \le j \le n\} \cup \{u_i^j : 1 \le i \le m, 1 \le j \le n-1\}$, $E = \{u_i^j v_i^{j+1}, u_i^j v_i^j, v_i^j v_i^{j+1} :$

$1 \le i \le m, 1 \le j \le n-1\}$ and face set $F = \{ f_i^j : 1 \le i \le m, 1 \le j \le n-1\}$. The three types of magic labeling are discussed below:

Type (i): *(1, 0, 1)-Face magic (n is even)*

Define a bijective mapping $g_1 : V \cup F \to \{1, 2, \cdots, 3mn-2m\}$ in the following way:

$$g_1(v_i^j) = \begin{cases} (i-1)\left(\frac{n}{2}\right) + \left(\frac{j+1}{2}\right), & \text{if } 1 \le i \le m \text{ and for odd } j, 1 \le j \le n-1, \\ mn - i\frac{n}{2} + \frac{j}{2}, & \text{if } 1 \le i \le m \text{ and for even } j, 2 \le j \le n. \end{cases}$$

$$g_1(u_i^j) = \begin{cases} mn + i\left(\frac{n}{2}\right) + 2 - \frac{1}{2}(i+j+2), & \text{if } i \text{ and } j \text{ are odd}, 1 \le i \le m, 1 \le j \le n-1, \\ mn + (i+1)\left(\frac{n}{2}\right) + 1 - \frac{1}{2}(i+j+1), & \text{if } i \text{ is odd and } j \text{ is even}, 1 \le i \le m, 2 \le j \le n-2, \\ 2mn - m - (i-1)\left(\frac{n}{2}\right) + \frac{1}{2}(i-j-1) + 1, & \text{if } i \text{ is even and } j \text{ is odd}, 2 \le i \le m-1, 1 \le j \le n-1, \\ 2mn - m - (i-2)\left(\frac{n}{2}\right) + \frac{1}{2}(i-j), & \text{if } i \text{ and } j \text{ are even}, 2 \le i \le m-1, 2 \le j \le n-2. \end{cases}$$

$$g_1(f_i^j) = \begin{cases} 3mn - 2m - (i-1)\left(\frac{n}{2}\right) + \frac{1}{2}(i-j), & \text{if } i \text{ and } j \text{ are odd}, 1 \le i \le m, 1 \le j \le n-1, \\ 3mn - 2m - i\left(\frac{n}{2}\right) + \frac{1}{2}(i-j-1), & \text{if } i \text{ is odd and } j \text{ is even}, 1 \le i \le m, 2 \le j \le n, \\ 2mn - m + (i)\left(\frac{n}{2}\right) + 1 - \frac{1}{2}(i+j+1), & \text{if } i \text{ is even and } j \text{ is odd}, 2 \le i \le m, 1 \le j \le n-1, \\ 2mn - m + (i-1)\left(\frac{n}{2}\right) - \frac{1}{2}(i+j) + 1, & \text{if } i \text{ and } j \text{ are even}, 2 \le i \le m, 2 \le j \le n. \end{cases}$$

We can easily see that all 3-sided faces have the equal weight $w(f_i) = 5mn-2m+2$.

Type (ii): *(1, 1, 0)-Face magic (n is even)*

Define a bijective mapping $g_2 : V \cup E \to \{1, 2, \cdots, 5mn-4m\}$ as follows:

Here, labels are assigned to the vertices as in type (i).

$$g_2(v_i^j u_i^j) = 2mn - m + (i-1)(n-1) + j, \quad \text{if } 1 \le i \le m \text{ and } 1 \le j \le n-1.$$

$g_2(v_i^{j+1}u_i^j) = 4mn - 3m + 1 - (i-1)(n-1) - j, \quad if\ 1 \le i \le m\ and\ 1 \le j \le n-1.$

$$g_2(v_i^j v_i^{j+1}) = \begin{cases} 5mn - 4m - (i-1)\left(\frac{n}{2}\right) + \left(\frac{i-j}{2}\right), & if\ i\ and\ j\ are\ odd, 1 \le i \le m, 1 \le j \le n-1, \\ 5mn - 4m - \frac{n}{2} - (i-1)\left(\frac{n}{2}\right) + \left(\frac{i+1-j}{2}\right), & if\ i\ is\ odd\ and\ j\ is\ even, 1 \le i \le m, 2 \le j \le n-2, \\ 4mn - 3m + i\left(\frac{n}{2}\right) - \left(\frac{i+j+1}{2}\right) + 1, & if\ i\ is\ even\ and\ j\ is\ odd, 2 \le i \le m-1, 1 \le j \le n-1, \\ 4mn - 3m + (i-1)\left(\frac{n}{2}\right) + 1 - \left(\frac{i+j}{2}\right), & if\ i\ and\ j\ are\ even, 2 \le i \le m-1, 2 \le j \le n-2. \end{cases}$$

Clearly, the common weight for all 3-sided faces is $w(f_i) = 13mn - 8m + 3$.

Type (iii): *(0, 1, 1)-Face magic*

A bijective mapping $g_3 : E \cup F \to \{1, 2, \cdots, 4m(n-1)\}$ is given below:

$For\ 1 \le i \le m\ and\ 1 \le j \le n-1;\ \ g_3(v_i^j u_i^j) = (i-1)(n-1) + j,$

$g_3(v_i^{j+1}u_i^j) = 2m(n-1) + 1 - (i-1)(n-1) - j, \quad g_3(v_i^j v_i^{j+1}) = 2m(n-1) + (i-1)(n-1) + j,$

$g_3(f_i^j) = 4m(n-1) + 1 - (i-1)(n-1) - j.$

Clearly, the common weight for all 3-sided faces is $w(f_i) = 8m(n-1) + 2$.

Theorem 2.6 *Let G be the graph obtained by edge by vertex duplication at all edges on* mP_{n+1}, *then the graph G has face magic labeling of type (1, 1, 1) when* $n, m \equiv 1 (\mathrm{mod} 2)$.

Proof

Let *G (V, E, F)* be the graph obtained by duplicating all edges in mP_{n+1} by vertices respectively.

Let $V = \{v_i^j : 1 \le i \le m, 1 \le j \le n+1\} \cup \{u_i^j : 1 \le i \le m, 1 \le j \le n\}$,

$E = \{u_i^j v_i^{j+1}, u_i^j v_i^j, v_i^j v_i^{j+1} : 1 \le i \le m, 1 \le j \le n\}$ and face set $F = \{f_i^j : 1 \le i \le m, 1 \le j \le n\}$.

Define the vertex labeling $g_4 : V(G) \to \{1, 2, \cdots, 2mn + m\}$ as follows:

$$g_4(v_i^j)=\begin{cases}(i-1)\left(\dfrac{n+1}{2}\right)+\left(\dfrac{j+1}{2}\right), & \text{if } j \text{ is odd}, 1\le i\le m, 1\le j\le n,\\ m(n+1)-i\left(\dfrac{n+1}{2}\right)+\dfrac{j}{2}, & \text{if } j \text{ is even}, 1\le i\le m,\ 2\le j\le n+1.\end{cases}$$

$$g_4(u_i^j)=\begin{cases}m(n+1)+i\left(\dfrac{n+1}{2}\right)+2-\left(\dfrac{i+j+2}{2}\right), & \text{if } i \text{ and } j \text{ are odd}, 1\le i\le m, 1\le j\le n,\\ m(n+1)+(i+1)\left(\dfrac{n+1}{2}\right)+1-\left(\dfrac{i+j+1}{2}\right), & \text{if } i \text{ is odd and } j \text{ is even}, 1\le i\le m,\ 2\le j\le n-1,\\ 2mm+m-(i-1)\left(\dfrac{n+1}{2}\right)+\left(\dfrac{i-j-1}{2}\right)+1, & \text{if } i \text{ is even and } j \text{ is odd},\ 2\le i\le m-1, 1\le j\le n,\\ 2mn+m-(i-2)\left(\dfrac{n+1}{2}\right)+\left(\dfrac{i-j}{2}\right), & \text{if } i \text{ and } j \text{ are even},\ 2\le i\le m-1,\ 2\le j\le n-1.\end{cases}$$

We construct the edge labeling $f_2 : E(G) \to \{1, 2, \cdots, 3mn\}$ in the following way;

$$g_4(v_i^j u_i^j)=2m(n+1)-m+(i-1)n+j, \quad \text{if } 1\le i\le m \text{ and } 1\le j\le n.$$

$$g_4(v_i^{j+1} u_i^j)=\begin{cases}5mn+m-(i-1)\left(\dfrac{n}{2}\right)+1-\left(\dfrac{j+1}{2}\right), & \text{if } i \text{ and } j \text{ are odd}, 1\le i\le m, 1\le j\le n,\\ 5mn+m-\left\lceil\dfrac{mn}{2}\right\rceil-(i-1)\left(\dfrac{n}{2}\right)+1-\left(\dfrac{j}{2}\right), & \text{if } i \text{ is odd and } j \text{ is even}, 1\le i\le m,\ 2\le j\le n-1,\\ 5mn+m-\left\lceil\dfrac{mn}{2}\right\rceil-i\left(\dfrac{n}{2}\right)+\dfrac{n+1}{2}-\left(\dfrac{j+1}{2}\right)+1, & \text{if } i \text{ is even and } j \text{ is odd},\ 2\le i\le m-1, 1\le j\le n,\\ 5mn+m-i\left(\dfrac{n}{2}\right)+\left(\dfrac{n+1}{2}\right)-\dfrac{j}{2}, & \text{if } i \text{ and } j \text{ are even},\ 2\le i\le m-1,\ 2\le j\le n-1.\end{cases}$$

$$g_4(v_i^j v_i^{j+1})=\begin{cases}4mn+m-\left\lceil\dfrac{mn}{2}\right\rceil-(i-1)\left(\dfrac{n}{2}\right)+2-\left(\dfrac{j+1}{2}\right), & \text{if } i \text{ and } j \text{ are odd}, 1\le i\le m, 1\le j\le n,\\ 4mn+m-(i-1)\left(\dfrac{n}{2}\right)+1-\left(\dfrac{j}{2}\right), & \text{if } i \text{ is odd and } j \text{ is even}, 1\le i\le m,\ 2\le j\le n-1,\\ 4mn+m-i\left(\dfrac{n}{2}\right)+\dfrac{n+1}{2}-\left(\dfrac{j+1}{2}\right)+1, & \text{if } i \text{ is even and } j \text{ is odd},\ 2\le i\le m-1, 1\le j\le n,\\ 4mn+m-\left\lceil\dfrac{mn}{2}\right\rceil-i\left(\dfrac{n}{2}\right)+\left(\dfrac{n+1}{2}\right)-\dfrac{j}{2}+1, & \text{if } i \text{ and } j \text{ are even},\ 2\le i\le m-1,\ 2\le j\le n-1.\end{cases}$$

Define the face labeling $f_3 : F(G) \to \{1, 2, \cdots, mn\}$ as follows:

$$g_4(f_i^j) = \begin{cases} 6mn + m - (i-1)\left(\dfrac{n+1}{2}\right) + \left(\dfrac{i-j}{2}\right), & \textit{if } i \textit{ and } j \textit{ are odd}, 1 \le i \le m, 1 \le j \le n, \\ 6mn + m - i\left(\dfrac{n+1}{2}\right) + \left(\dfrac{i+1-j}{2}\right), & \textit{if } i \textit{ is odd and } j \textit{ is even}, 1 \le i \le m, 2 \le j \le n-1, \\ 5mn + m + i\left(\dfrac{n+1}{2}\right) + 1 - \left(\dfrac{i+j+1}{2}\right), & \textit{if } i \textit{ is even and } j \textit{ is odd}, 2 \le i \le m-1, 1 \le j \le n, \\ 5mn + m + (i-1)\left(\dfrac{n+1}{2}\right) - \left(\dfrac{i+j}{2}\right) + 1, & \textit{if } i \textit{ and } j \textit{ are even}, 2 \le i \le m-1, 2 \le j \le n-1. \end{cases}$$

By direct computation that following, we observe that all 3-sided faces have the same weight

(a) If i and j are odd, $1 \le i \le m, 1 \le j \le n,$ then the weight of 3-sided faces is

$$= \left((i-1)\left(\frac{n+1}{2}\right) + \left(\frac{j+1}{2}\right)\right) + \left(m(n+1) - i\left(\frac{n+1}{2}\right) + \frac{j+1}{2}\right) + \left(m(n+1) + i\left(\frac{n+1}{2}\right) + 2 - \left(\frac{i+j+2}{2}\right)\right)$$

$$+ \left(4mn + m - \left\lceil \frac{mn}{2} \right\rceil + 2 - (i-1)\left(\frac{n}{2}\right) - \frac{j+1}{2}\right) + \left(5mn + m - (i-1)\left(\frac{n}{2}\right) + 1 - \left(\frac{j+1}{2}\right)\right)$$

$$+ (2mn + m + (i-1)n + j) + \left(6mn + m - (i-1)\left(\frac{n+1}{2}\right) + \left(\frac{i-j}{2}\right)\right) = 19mn + 6m - \left\lceil \frac{mn}{2} \right\rceil + 4.$$

(b) If i is odd and j is even, $1 \le i \le m, 2 \le j \le n-1$, then the weight of 3-sided faces is

$$w(f_{i,j}) = g_4(v_i^j) + g_4(v_i^{j+1}) + g_4(u_i^j) + g_4(v_i^j v_i^{j+1}) + g_4(u_i^j v_i^{j+1}) + g_4(u_i^j v_i^j) + g_4(f_i^j)$$

$$= \left(m(n+1) - i\left(\frac{n+1}{2}\right) + \frac{j}{2}\right) + \left((i-1)\left(\frac{n+1}{2}\right) + \left(\frac{j+2}{2}\right)\right) + \left(m(n+1) + (i+1)\left(\frac{n+1}{2}\right) + 1 - \left(\frac{i+j+1}{2}\right)\right)$$

$$+ \left(4mn + m - (i-1)\left(\frac{n}{2}\right) + 1 - \frac{j}{2}\right) + \left(5mn + m - \left\lceil \frac{mn}{2} \right\rceil - (i-1)\left(\frac{n}{2}\right) + 1 - \left(\frac{j}{2}\right)\right)$$

$$+ (2mn + m + (i-1)n + j) + \left(6mn + m - i\left(\frac{n+1}{2}\right) + \left(\frac{i+1-j}{2}\right)\right) = 19mn + 6m - \left\lceil \frac{mn}{2} \right\rceil + 4.$$

(c) If i is even and j is odd, $2 \le i \le m-1, 1 \le j \le n$, then the weight of 3-sided faces is

$$w(f_{i,j}) = g_4(v_i^j) + g_4(v_i^{j+1}) + g_4(u_i^j) + g_4(v_i^j v_i^{j+1}) + g_4(u_i^j v_i^{j+1}) + g_4(u_i^j v_i^j) + g_4(f_i^j)$$

$$=\left((i-1)\left(\frac{n+1}{2}\right)+\left(\frac{j+1}{2}\right)\right)+\left(m(n+1)-i\left(\frac{n+1}{2}\right)+\frac{j+1}{2}\right)+\left(2mm+m-(i-1)\left(\frac{n+1}{2}\right)+\left(\frac{i-j-1}{2}\right)+1\right)$$

$$+\left(4mn+m-i\left(\frac{n}{2}\right)+\left(\frac{n+1}{2}\right)+1-\frac{j+1}{2}\right)+\left(5mn+m-\left\lceil\frac{mn}{2}\right\rceil+(1-i)\left(\frac{n}{2}\right)+1-\left(\frac{j+1}{2}\right)\right)$$

$$+(2mn+m+(i-1)n+j)+\left(5mn+m+i\left(\frac{n+1}{2}\right)-\left(\frac{i+1+j}{2}\right)+1\right)=19mn+6m-\left\lceil\frac{mn}{2}\right\rceil+4.$$

(d) If i and j are even, $2\le i\le m-1,\ 2\le j\le n-1$, then the weight of 3-sided faces is

$$w(f_{i,j})=g_4(v_i^j)+g_4(v_i^{j+1})+g_4(u_i^j)+g_4(v_i^j v_i^{j+1})+g_4(u_i^j v_i^{j+1})+g_4(u_i^j v_i^j)+g_4(f_i^j)$$

$$=\left(m(n+1)-i\left(\frac{n+1}{2}\right)+\frac{j}{2}\right)+\left((i-1)\left(\frac{n+1}{2}\right)+\left(\frac{j+2}{2}\right)\right)+\left(2mm+m-(i-2)\left(\frac{n+1}{2}\right)+\left(\frac{i-j}{2}\right)\right)$$

$$+\left(4mn+m-\left\lceil\frac{mn}{2}\right\rceil-i\left(\frac{n}{2}\right)+\left(\frac{n+1}{2}\right)+1-\frac{j}{2}\right)+\left(5mn+m-i\left(\frac{n}{2}\right)+\left(\frac{n+1}{2}\right)-\left(\frac{j}{2}\right)\right)$$

$$+(2mn+m+(i-1)n+j)+\left(5mn+m+(i-1)\left(\frac{n+1}{2}\right)+1-\left(\frac{i+j}{2}\right)\right)=19mn+6m-\left\lceil\frac{mn}{2}\right\rceil+4.$$

Theorem 2.7 *The G graph obtained by duplicating all vertices by edges in* $C_n,\ n\ge 3,$ *admits face magic labeling of types (1, 0, 1), (1, 1, 0), (0, 1, 1) and (1, 1, 1).*

Proof

Let *G (V, E, F)* be the graph obtained by duplicating all vertices in $C_n: u_1, u_2, \cdots, u_n,\ n\ge 3$ by edges. Let $V=\{u_i, v_i, w_i : 1\le i\le n\}$, $E=\{u_iv_i, u_iw_i, v_iw_i : 1\le i\le n\}\cup\{u_iu_{i+1} : 1\le i\le n-1\}$ $\cup\{u_1u_n\}$ and $F=\{f_i : u_iv_iw_i : 1\le i\le n\}\cup\{f_{n+1} : u_1u_2\cdots u_n\}$.

The four types of labeling are discussed below:

Type (i): *(1, 0, 1)-Face magic*

Define a mapping $f_1: V\cup F\to\{1, 2, \cdots, 4n+1\}$ in the following way:

For i = 1 to n; $f_1(u_i)=i,\ f_1(v_i)=2n+1-i,\ f_1(w_i)=2n+i,$

$f_1(f_i)=4n+1-i,\ f_1(f_{n+1})=4n+1.$

Clearly, the weight of all 3-sided faces is $w_1(f_i) = 8n+2.$

The weight of n-side face is

$$w_2(f_i) = \sum_{i=1}^{n} f_1(u_i) + f_1(f_{n+1}) = \sum_{i=1}^{n} i + (4n+1) = \frac{n(n+1)}{2} + (4n+1) = \frac{n^2+9n+2}{2}.$$

Type (ii): *(1, 1, 0)-Face magic*

We define a bijective function $f_2 : V \cup E \to \{1, 2, \cdots, 7n\}$ as follows:

For i = 1 to n; $f_2(u_i) = i,\ f_2(v_i) = 2n+1-i,\ f_2(w_i) = 2n+i,\ f_2(v_i w_i) = 4n+1-i,$

$f_2(u_i v_i) = 4n+i,\ f_2(u_i w_i) = 6n+1-i,\ f_2(u_1 u_n) = 6n+1.$

For i = 1 to n-1; $f_2(u_i u_{i+1}) = 6n+1+i.$

Clearly, the weight of all 3-sided faces is $w_1(f_i) = 18n+3$ and the weight of n-side face is

$$w_2(f_i) = \sum_{i=1}^{n} f_2(u_i) + \sum_{i=1}^{n-1} f_2(u_i u_{i+1}) + f_2(u_1 u_n) = \sum_{i=1}^{n} i + \sum_{i=1}^{n-1} (6n+1+i) + (6n+1)$$

$$= \frac{n(n+1)}{2} + (6n+1)(n-1) + \frac{(n-1)n}{2} + (6n+1) = 7n^2 + n.$$

Type (iii): *(0, 1, 1)-Face magic*

We construct a bijective mapping $f_3 : E \cup F \to \{1, 2, \cdots, 5n+1\}$ as below:

For i = 1 to n; $f_3(u_i w_i) = n+i,\ \ f_3(u_i v_i) = 3n+1-i,\ f_3(v_i w_i) = 3n+i,\ f_3(f_i) = 5n+2-i.$

For i = 1 to n-1; $f_3(u_i u_{i+1}) = i,\ f_3(u_1 u_n) = n,\ f_3(f_{n+1}) = 4n+1.$

Clearly, the weight of all 3-sided faces is $w_1(f_i) = 12n+3$ and the weight of n-side face is

$$w_2(f_i) = \sum_{i=1}^{n-1} f_3(u_i u_{i+1}) + f_3(u_1 u_n) + f_3(f_{n+1}) = \sum_{i=1}^{n-1} i + n + (4n+1)$$

$$= \frac{n(n-1)}{2} + (5n+1) = \frac{n^2+9n+2}{2}.$$

Type (iv): *(1, 1, 1)-Face magic*

Define a bijective mapping $f_4 : V \cup E \cup F \to \{1, 2, \cdots, 8n+1\}$ as follows:

For i = 1 to n; $f_4(u_i) = i,\ f_4(v_i) = 2n+i,\ f_4(w_i) = 2n+1-i,\ f_4(v_i w_i) = 4n+1-i,$

$f_4(u_i w_i) = 4n + i,\;\; f_4(u_i v_i) = 7n + 1 - 2i,\;\; f_4(f_i) = 7n + 1 + i.$

For i = 1 to n-1; $f_4(u_i u_{i+1}) = 7n + 2 - 2i,\;\; f_4(u_1 u_n) = 5n + 2,\;\; f_4(f_{n+1}) = 7n + 1.$

Clearly, the weight of all 3-sided faces is $w_1(f_i) = 26n + 4$ and the weight of n-side face is

$$w_2(f_i) = \sum_{i=1}^{n} f_4(u_i) + \sum_{i=1}^{n-1} f_4(u_i u_{i+1}) + f_4(u_1 u_n) + f_4(f_{n+1})$$

$$= \sum_{i=1}^{n} i + \sum_{i=1}^{n-1} (7n + 2 - 2i) + (5n + 2) + (7n + 1)$$

$$= \frac{n(n+1)}{2} + (7n+2)(n-1) - 2\frac{n(n-1)}{2} + (12n+3) = \frac{13n^2 + 17n + 2}{2}.$$

The labeling technique used in the proof of the theorems 2.7 and 2.8 are exhibited in the following figures.

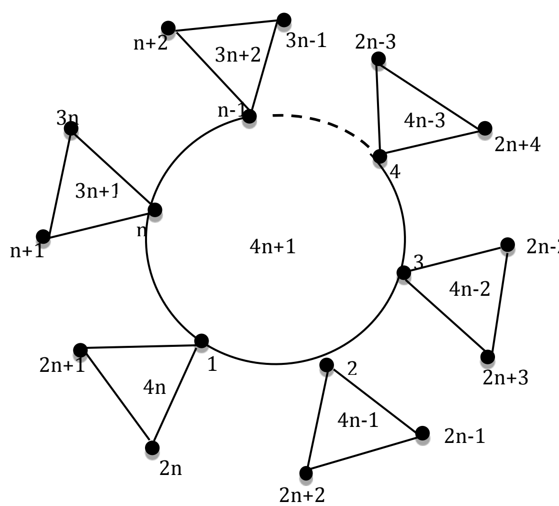

Figure 4:

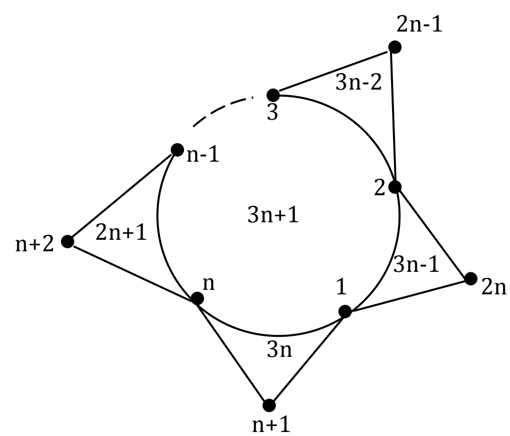

Figure 5:

Figure 4: Face magic labeling of type (1, 0, 1) for vertex by edge duplication on C_n

Figure 5: Face magic labeling of type (1, 0, 1) for edge by vertex duplication on C_n

Theorem 2.8 *The G graph obtained by duplicating all edges by vertex in outside the cycle C_n, $n \geq 3$ admits face magic labeling of types (1, 0, 1) and (0, 1, 1). Also G admits face magic labeling of types (1, 1, 0) and (1, 1, 1) if n is odd.*

Proof

Let *G (V, E, F)* be the graph obtained by duplicating all edges by vertex in

$C_n : v_1, v_2, \cdots, v_n,\ n \geq 3.$ Let $V = \{v_i, u_i : 1 \leq i \leq n\},\ E = \{v_i v_{i+1}, u_i v_{i+1} : 1 \leq i \leq n-1\} \cup$

$\{v_1 u_n, v_1 v_n\} \cup \{u_i v_i : 1 \leq i \leq n\}$ and $F = \{f_i : 1 \leq i \leq n\} \cup \{f_{n+1} : v_1 v_2 \cdots v_n\}$

The four types of labeling are discussed separately below:

Type (i): *(1, 0, 1)-Face magic*

Define a mapping $g_1 : V \cup F \to \{1, 2, \cdots, 3n+1\}$ in the following way:

$For\ 1 \leq i \leq n;\ g_1(v_i) = i,\ g_1(u_i) = 2n+1-i.$

$$g_1(f_i) = \begin{cases} 3n - i, \text{ if } 1 \leq i \leq n-1, \\ 3n, \quad \text{if } i = n, \\ 3n+1, \text{ if } i = n+1. \end{cases}$$

Clearly, the weight of all 3-sided faces is $w_1(f_i) = 5n+2$ and the weight of n-side face is

$$w_2(f_i) = \sum_{i=1}^{n} g_1(v_i) + g_1(f_{n+1}) = \sum_{i=1}^{n} i + (3n+1) = \frac{n(n+1)}{2} + (3n+1) = \frac{n^2 + 7n + 2}{2}.$$

Type (ii): *(1, 1, 0)-Face magic (n is odd)*

We define a bijective function $g_2 : V \cup E \to \{1, 2, \cdots, 5n\}$ as follows:

$$g_2(v_i) = \begin{cases} \dfrac{i+1}{2}, \quad \text{if } i \text{ is odd}, 1 \leq i \leq n, \\ \left\lceil \dfrac{n}{2} \right\rceil + \dfrac{i}{2}, \quad \text{if } i \text{ is even}, 2 \leq i \leq n-1. \end{cases}$$

$$g_2(v_i v_{i+1}) = \begin{cases} 4n + \left\lceil \dfrac{n}{2} \right\rceil - 1 + \dfrac{i+1}{2}, \text{ if } i \text{ is odd}, 1 \leq i \leq n-2, \\ 4n + \dfrac{i}{2}, \quad \text{if } i \text{ is even}, 2 \leq i \leq n-1. \end{cases}$$

$$g_2(u_i\, v_{i+1}) = \begin{cases} 3n + \dfrac{i+1}{2}, & \text{if } i \text{ is odd}, 1 \le i \le n-2, \\ 3n + \left\lceil \dfrac{n}{2} \right\rceil + \dfrac{i}{2}, & \text{if } i \text{ is even}, 2 \le i \le n-1. \end{cases}$$

$For\, 1 \le i \le n;\;\; g_2(u_i v_i) = 3n+1-i,\;\; g_2(v_1 v_n) = 5n,\; g_2(v_1 u_n) = 3n + \left\lceil \frac{n}{2} \right\rceil.$

$$g_2(u_i) = \begin{cases} 2n-i, \text{ if } 1 \le i \le n-1, \\ 2n, \text{ if } i = n. \end{cases}$$

Clearly, the weight of all 3-sided faces is $w_1(f_i) = 2\left(6n + \left\lceil \frac{n}{2} \right\rceil + 1\right)$.

The weight of n-side face is

$$w_2(f_i) = \sum_{i=odd} g_2(u_i) + \sum_{i=odd} g_2(v_i v_{i+1}) + \sum_{i=even} g_2(u_i) + \sum_{i=even} g_2(v_i v_{i+1}) + g_2(v_1 v_n)$$

$$= \sum_{i=1}^{n}\left(\frac{i+1}{2}\right) + \sum_{i=1}^{n-2}\left(4n + \left\lceil \frac{n}{2} \right\rceil - 1 + \frac{i+1}{2}\right) + \sum_{i=2}^{n-1}\left(4n + \frac{i}{2}\right) + \sum_{i=2}^{n-1}\left(\left\lceil \frac{n}{2} \right\rceil + \frac{i}{2}\right) + 5n$$

$$= \frac{1}{2}\left(\frac{n+1}{2}\right)\left(\frac{n+3}{2}\right) + \left(4n + \left\lceil \frac{n}{2} \right\rceil - 1\right)\left(\frac{n-1}{2}\right) + \frac{1}{2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) + 4n\left(\frac{n-1}{2}\right)$$

$$+ \frac{1}{2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) + \left\lceil \frac{n}{2} \right\rceil\left(\frac{n-1}{2}\right) + \frac{1}{2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) + 5n$$

$$= \frac{3}{2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) + \left(\frac{n-1}{2}\right)\left(2\left\lceil \frac{n}{2} \right\rceil + 8n - 1\right) + \frac{1}{2}\left(\frac{n+3}{2}\right)\left(\frac{n+1}{2}\right) + 5n.$$

Type (iii): *(0, 1, 1)-Face magic*

We construct a mapping $g_3 : E \cup F \to \{1, 2, \cdots, 4n+1\}$ in the following way:

For i = 1 to n; $g_3(u_i v_i) = 2n+1-i,\; g_3(f_i) = 4n+1-i,\; g_3(f_{n+1}) = 4n+1.$

For i = 1 to n-1; $g_3(v_i v_{i+1}) = i,\; g_3(u_i v_{i+1}) = 2n+i.\;\; g_3(v_1 v_n) = n,\; g_3(u_n v_1) = 3n,$

Clearly, the weight of all 3-sided faces is $w_1(f_i) = 8n+2.$

The weight of n-side face is

$$w_2(f_i) = \sum_{i=1}^{n-1} g_3(v_i v_{i+1}) + g_3(f_{n+1}) = \sum_{i=1}^{n-1} i + (4n+1) = \frac{n(n-1)}{2} + (4n+1) = \frac{n^2+7n+2}{2}.$$

Type (iv): *(1, 1, 1)-Face magic (n is odd)*

A bijective function $g_4 : V \cup E \cup F \to \{1, 2, \cdots, 6n+1\}$ is given below:

$$g_4(f_i) = \begin{cases} 5n + \left\lceil \frac{n}{2} \right\rceil + \frac{i+1}{2}, & \text{if } i \text{ is odd}, 1 \le i \le n-2, \\ 5n + 1 + \frac{i}{2}, & \text{if } i \text{ is even}, 2 \le i \le n-1, \\ 6n+1, & \text{if } i = n+1. \end{cases}$$

$$g_4(u_i v_{i+1}) = \begin{cases} 3n + \frac{i+1}{2}, & \text{if } i \text{ is odd}, 1 \le i \le n-2, \\ 3n + \left\lceil \frac{n}{2} \right\rceil + \frac{i}{2}, & \text{if } i \text{ is even}, 2 \le i \le n-1. \end{cases}$$

For i = 1 to n; $g_4(v_i) = i,\ g_4(u_i) = 3n+1-i,\ g_4(u_i v_i) = 5n - i.$

For i = 1 to n-1; $g_4(v_i v_{i+1}) = 2n - i.\ \ g_4(v_1 u_n) = 3n + \left\lceil \frac{n}{2} \right\rceil,\ g_4(v_1 v_n) = 2n.$

Clearly, the weight of all 3-sided faces is $w_1(f_i) = 18n + \left\lceil \frac{n}{2} \right\rceil + 3.$

The weight of n-side face is

$$w_2(f_i) = \sum_{i=1}^{n} g_4(w_i) + \sum_{i=1}^{n-1} g_4(u_i u_{i+1}) + g_4(u_1 u_n) + g_4(f_{n+1})$$

$$= \sum_{i=1}^{n} i + \sum_{i=1}^{n-1} (2n - i) + 2n + (6n+1) = \frac{n(n+1)}{2} + 2n(n-1) - \frac{n(n-1)}{2} + 8n + 1 = 2n^2 + 7n + 1.$$

Remark

We have considered unique structure of G in the above theorems 2.7 and 2.8. That is, G has exactly n+1 interior faces of which one is an n-sided face and the others are 3-sided faces.

Conclusion

In this paper, we have studied face magic labeling for vertex and edge duplication on some families of graphs. In future, we are interested in extending this idea to some graph operations. We propose the following problem for further study in this area.

Problem 1 *Whether the edge by vertex duplication at all edges simultaneously on an arbitrary tree has face magic labeling of types (1, 0, 1), (1, 1, 0) and (1, 1, 1) or not.*

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