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The b-chromatic number of some graphs

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Abstract

A proper coloring of a graph G is called b-coloring if each color class contains a vertex which is adjacent to at least one vertex of every other color classes. We investigate b-chromatic number of some cycle and path related graphs.

Keywords: Graph coloring, b-coloring, b-vertex.
AMS Subject Classification(2010): 05C15, 05C76.

1 Introduction

We begin with finite, simple, connected and undirected graph. A proper k-coloring of a graph G is an assignment of k colors $1, 2, ..., k$ to the vertices such that no two adjacent vertices share the same colors. The color of a vertex v is denoted by $c(v)$. The chromatic number $\chi(G)$ is the minimum number k for which G admits a proper k-coloring.

A b-coloring is a proper coloring of vertices of a graph G such that each color class contains a b-vertex which has at least one neighbor in all the other color classes. The b-chromatic number $\varphi(G)$ is the largest integer k such that G admits a b-coloring with k colors.

The concept of b-coloring was introduced by Irving and Manlove [5] and they proved that determining $\varphi(G)$ is NP-hard for general graphs. The theory of b-coloring attracted many researchers. For eg., the b-chromatic number for Peterson graph and power of a cycle is discussed by Chandrakumar and Nicholas [1]. They also investigated the b-chromatic number of the square of cartesian product of two cycles [2]. The b-chromatic numbers of some path related graphs are investigated by Vaidya and Rakhimol [9]. The same authors have discussed the b-chromatic number of some degree splitting graphs [10].

Definition 1.1. [4] The m-degree of a graph G, denoted by $m(G)$, is the largest integer m such that G has at least m vertices of degree at least $m - 1$.

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Proposition 1.2. [3] For any graph G, $\chi(G) \geq 3$ if and only if G has an odd cycle.

Proposition 1.3. [5] If G admits a b-coloring with m colors, then G must have at least m vertices with degree at least $m - 1$.

Proposition 1.4. [6] $\chi(G) \leq \varphi(G) \leq m(G)$.

It is obvious that if $\chi(G) = k$, then every coloring of a graph G by k colors is a b-coloring of G.

2 Main Results

Definition 2.1. The one point union $C_n^{(k)}$ of k-copies of cycle C_n is the graph obtained by taking v as a common vertex such that any two distinct cycles $C_n^{(i)}$ and $C_n^{(j)}$ are edge disjoint and do not have any vertex in common except v.

Theorem 2.2. $\phi(C_n^{(k)}) = 3$ for all $n \geq 3$.

Proof: Let v_1^p, v_2^p,....,v_n^p be the vertices in the p^{th} copy of C_n. Take $v_1^1 = v_1^2 = ... = v_1^k = v$ in $C_n^{(k)}$. In the graph $C_n^{(k)}$, the vertex v is of degree $2k$ and the remaining vertices are of degree 2. Hence $m(C_n^{(k)})$=3. Thus $\phi(C_n^{(k)}) \leq 3$.

When n is odd, the graph $C_n^{(k)}$ contains odd cycles. Then by Proposition 1.2, $\phi(C_n^{(k)}) \geq 3$. Thus $\phi(C_n^{(k)}) = 3$. But when n is even, assign the colors as $c(v) = 1$, $c(v_2^{2i-1}) = 2$, $c(v_3^{2i-1}) = 3$, $c(v_4^{2i-1}) = 2$, $c(v_5^{2i-1}) = 3$,..., $c(v_n^{2i-1}) = 2$, $c(v_2^{2i}) = 3$, $c(v_3^{2i}) = 2$, $c(v_4^{2i}) = 3$,..., $c(v_n^{2i}) = 3$ with $i \in$ N. Consequently v, v_2^{2i-1} and v_2^{2i} are the b-vertices for the color classes 1, 2 and 3 respectively. Hence $\phi(C_n^{(k)}) = 3$. ■

Illustration 2.3. $C_6^{(3)}$ and $C_3^{(4)}$ and their b-coloring are shown in figure 1 and figure 2 respectively.

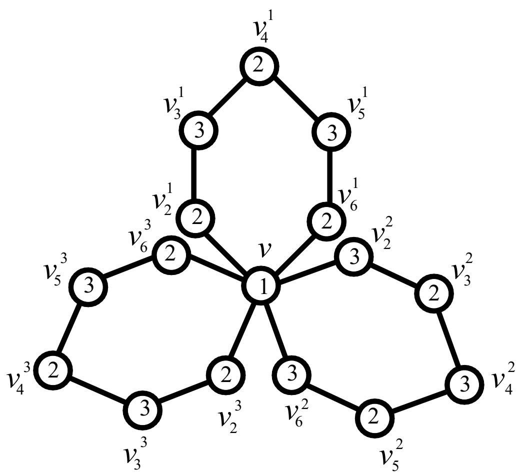

Figure 1

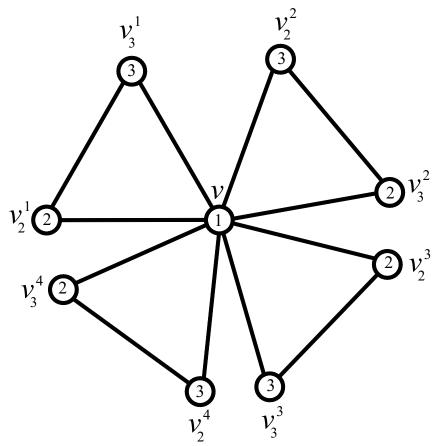

Figure 2

Definition 2.4. ([8]) Let G_1, G_2,..., G_k be k copies of a graph G where $k \geq 2$. $G(k)$ is the graph obtained by adding an edge from G_i to G_{i+1}; $i = 1, 2, ..., k-1$ and we call $G(k)$ the path union of k copies of the graph G.

Theorem 2.5. $\phi(C_n(k)) = \begin{cases} 3, & k = 2, 3 \\ 4, & k = 4, 5, 6 \\ 5, & k \geq 7. \end{cases}$

Proof: Let $C_n(k)$ denotes the path union of k copies of the cycle C_n with vertices v_i^p in the p^{th} copy of C_n, where $1 \leq i \leq n$ and $1 \leq p \leq k-1$. For obtaining $C_n(k)$ we join v_1^p and v_1^{p+1} by an edge. The vertices v_1^1 and v_1^k are of degree 3, the vertices $v_1^p; 2 \leq p \leq k-1$ are of degree 4 and the remaining vertices are of degree 2.
To prove the result we consider following three cases:

Case (i): When $k = 2, 3$.
The graph $C_n(k)$ has m-degree 3. Then by Proposition 1.4, $\phi(C_n(k)) \leq 3$. If we assign the colors as $c(v_1^1) = 1$, $c(v_2^1) = 2$, $c(v_n^1) = 3$, $c(v_1^2) = 3$, $c(v_2^2) = 2$, $c(v_3^2) = 1$ then v_1^1, v_1^2 and v_2^2 are the b-vertices for the color classes 1, 3 and 2 respectively. Thus $\phi(C_n(k)) = 3$.

Case (ii): When $k = 4, 5, 6$.
The graph $C_n(k)$ has m-degree 4. Then by Proposition 1.4, $\phi(C_n(k)) \leq 4$. If we assign the colors as $c(v_1^1) = 1$, $c(v_2^1) = 2$, $c(v_n^1) = 3$, $c(v_1^2) = 4$, $c(v_2^2) = 2$, $c(v_n^2) = 3$, $c(v_1^3) = 2$, $c(v_2^3) = 1$, $c(v_n^3) = 4$, $c(v_1^4) = 3$, $c(v_2^4) = 1$, $c(v_n^4) = 4$ then v_1^1, v_1^2, v_1^3 and v_1^4 are the b-vertices for the color classes 1, 4, 2 and 3 respectively. Thus $\phi(C_n(k)) = 4$.
Case (iii): When $k \geq 7$.
The graph $C_n(k)$ has m-degree 5. Then by Proposition 1.4, $\phi(C_n(k)) \leq 5$. If we assign the colors as $c(v_1^1) = 3$, $c(v_1^2) = 1$, $c(v_2^2) = 4$, $c(v_n^2) = 5$, $c(v_1^3) = 2$, $c(v_2^3) = 4$, $c(v_n^3) = 5$, $c(v_1^4) = 3$, $c(v_2^4) = 1$, $c(v_n^4) = 5$, $c(v_1^5) = 4$, $c(v_2^5) = 1$, $c(v_n^5) = 2$, $c(v_1^6) = 5$, $c(v_2^6) = 1$, $c(v_n^6) = 2$, $c(v_1^7) = 3$ then v_1^2, v_1^3, v_1^4, v_1^5 and v_1^6 are the b-vertices for the color classes 1, 2, 3, 4 and 5 respectively. Thus $\phi(C_n(k)) = 5$. ■

Illustration 2.6. The b-coloring of $C_5(4)$ is shown in Figure 3.

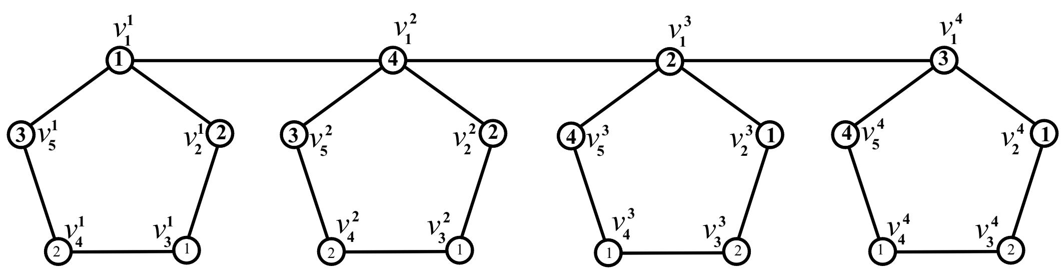

Figure 3: b-coloring of $C_5(4)$.

Definition 2.7. [7] A t-ply $P_t(u,v)$ is a graph with t paths, each of length at least two and such that no two paths have a vertex in common except the end vertices u and v.

The maximum over the length of all the paths with end points u and v in a t-ply $P_t(u,v)$ is denoted by *max($l(P)$)*.

Theorem 2.8. For a path $P \equiv \{u, v_1, v_2, ..., v_n, v\}$ with end points u and v in a t-ply $P_t(u,v)$,

$$\phi(P_t(u,v)) = \begin{cases} 2, & if\ max(l(P)) = 2 \\ 3, & if\ max(l(P)) \geq 3. \end{cases}$$

Proof: Consider a t-ply graph $P_t(u,v)$ with max($l(P)$)=$n+1$. The vertices u and v are of degree t and the remaining vertices are of degree 2.

When max($l(P)$)=2, $P_t(u,v)$ contains a P_3. Then obviously $\phi(P_t(u,v)) \geq 2$. As $P_t(u,v)$ has the m-degree 3, by Proposition 1.4, $\phi(P_t(u,v)) \leq 3$. If possible let $\phi(P_t(u,v)) = 3$ and $c(u) = 1$, $c(v_1) = 2$, $c(v) = 3$. But such b-coloring does not give the b-vertices for the color classes 1 and 3. Consequently $\phi(P_t(u,v)) \neq 3$. Thus in turn $\phi(P_t(u,v)) = 2$.

In the case when max($l(P)$)≥ 3, then m-degree of $P_t(u,v)$ is 3. Thus by Proposition 1.4, $\phi(P_t(u,v)) \leq 3$. Also $P_t(u,v)$ contains a cycle of length greater than or equal to 5. Thus $\phi(P_t(u,v)) \geq 3$. Clearly $\phi(P_t(u,v)) = 3$. ■

Illustration 2.9. $P_3(u,v)$ with *max($l(P)$)*=3 and its b-coloring is shown in Figure 4.

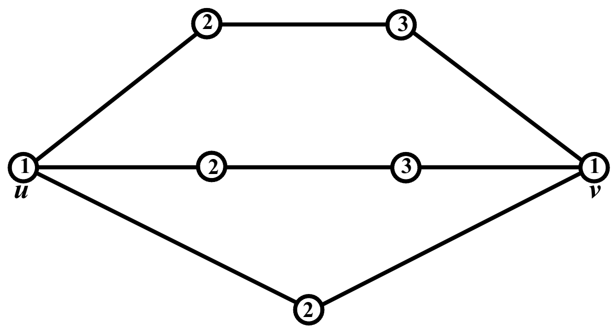

Figure 4: b-coloring of $P_3(u,v)$ with *max($l(P)$)*=3.

3 Concluding Remarks

The b-chromatic number for cycle and path are known. But we have explored the concept of b-coloring for the larger graphs obtained from these standard graphs. We have investigated the b-chromatic number for one point union of cycles, path union of cycles and t-ply graphs.

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