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Neighborhood total domination and colouring in graphs

C. Sivagnanam
Department of General Requirements
College of Applied Sciences - Ibri
Sultanate of Oman.
choshi71@ gmail.com

Abstract

Let $G = (V, E)$ be a graph without isolated vertices. A dominating set S of G is a neighborhood total dominating set (ntd-set) if the induced subgraph $\langle N(S)\rangle$ of G has no isolated vertices. The neighborhood total domination number $\gamma_{nt}(G)$ is the minimum cardinality of a ntd-set. The minimum number of colours required to colour all the vertices such that no two adjacent vertices have same colour is the chromatic number $\chi(G)$ of G. In this paper we find an upper bound for sum of the ntd-number and chromatic number and characterize the corresponding extremal graphs.

Keywords: Neighborhood total domination, chromatic number.
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1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

Let $G = (V, E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. If $S \subseteq V$, then $N(S) = \bigcup_{v\in S} N(v)$ and $N[S] = N(S) \cup S$. A support is a vertex with at least one of its neighbor has degree one.

A subset S of V is called a dominating set of G if $N[S] = V$. The minimum cardinality of a dominating set is called a domination number of G and is denoted by $\gamma(G)$. S. Arumugam and C. Sivagnanm [1] introduced the concept of neighborhood total domination. A dominating set S of a graph G is called a neighborhood total dominating set (ntd-set) if the induced subgraph $\langle N(S)\rangle$ has no isolated vertices. The minimum cardinality of a ntd-set of G is called the neighborhood total domination number (ntd-number) of G and is denoted by $\gamma_{nt}(G)$. The chromatic number $\chi(G)$ of a graph G is defined to be the minimum number of colours required to colour all the vertices such that no two adjacent vertices receive the same colour.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterizing the corresponding extremal graphs. In [3], J. Paulraj Joseph and S. Arumugam proved that $\gamma + \chi \leq n + 1$. They also characterized the class of graphs for the upper bound is attained. In this paper, we obtain upper bounds for the sum of ntd-number and chromatic number and characterize the extremal graphs. We need the following theorems.

Theorem 1.1. [2] For any graph G, $\chi \leq 1 + \Delta$.

Theorem 1.2. [2] If G is a connected graph that is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

Theorem 1.3. [1]Let G be a graph with $\Delta = n - 1$. Then $\gamma_{nt}(G) = 1$ or 2. Further $\gamma_{nt}(G) = 2$ if and only if G has exactly one vertex v with $deg\ v = n - 1$ and v is adjacent to a vertex of degree 1.

Theorem 1.4. [1] For any path P_n,

$$\gamma_{nt}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{if} \quad n \equiv 1 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 1 & \text{otherwise.} \end{cases}$$

Theorem 1.5. [1] For the cycle C_n,

$$\gamma_{nt}(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1 & \text{if} \quad n \equiv 2 \pmod 3, \\ \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

Theorem 1.6. [1] For any graph G, $\gamma_{nt}(G) \leq \lceil \frac{n}{2} \rceil$.

Theorem 1.7. [1] For any graph G, $\gamma_{nt}(G) \leq n - \Delta + 1$.

Theorem 1.8. [1] Let G be a connected graph with $\Delta < n - 1$. Then $\gamma_{nt}(G) \leq n - \Delta$.

2 Definitions and Notations

Definition 2.1. $H(m_1, m_2, \cdots, m_n)$ denotes the graph obtained from the graph H by attaching m_i edges to the vertex $v_i \in V(H),\ 1 \leq i \leq n$.

Definition 2.2. $H(P_{m_1}, P_{m_2}, \cdots, P_{m_n})$ is the graph obtained from the graph H by attaching an end vertex of P_{m_i} to the vertex v_i in H, $1 \leq i \leq n$.

Definition 2.3. Let $H_1$and H_2 be two copies of C_3 with vertex sets $V(H_1) = \{v_1^{(1)}, v_2^{(1)}, v_3^{(1)}\}$ and $V(H_2) = \{v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\}$. Then the graph $C_3^{(2)}$ is obtained from $H_1 \cup H_2$ by joining the vertices $v_i^{(1)}$ and $v_i^{(2)}$, $1 \leq i \leq 3$, by an edge.

2.1 Graphs and Notations

We define the following graphs.

G_1 is the graph obtained from $C_3(P_3, P_2, P_2)$ by attaching a P_2 to the vertex of degree 2.

G_2 is the graph obtained from $C_3(P_4, P_2, P_1)$ by attaching a P_2 to the vertex $u \notin V(C)$ of degree 2, where C is a cycle in $C_3(P_4, P_2, P_1)$.

G_3 is the graph obtained from $C_3(P_3, P_2, P_1)$ by attaching a P_2 to the vertex $u \notin V(C)$ of degree 2,where C is a cycle in $C_3(P_3, P_2, P_1)$.

G_4 is the graph obtained from $C_3(P_4, P_1, P_1)$ by attaching a P_2 to the vertex $u \notin V(C)$ of degree 2,where C is a cycle in $C_3(P_4, P_1, P_1)$.

G_5 is the graph obtained from $C_3(P_3, P_1, P_1)$ by attaching a P_2 to the vertex $u \notin V(C)$ of degree 2, where C is a cycle in $C_3(P_3, P_1, P_1)$.

G_6 is the graph obtained from $C_3(P_4, P_1, P_1)$ by attaching a P_2 to the vertex of degree 3.

G_7 is the graph obtained from $C_3(P_3, P_1, P_1)$ by attaching a P_2 to the vertex of degree 3.

G_8 is the graph obtained from $C_3(P_3, P_1, P_1)$ by attaching a P_3 to the vertex of degree 3.

G_9 is the graph obtained from $C_3(2, 1, 0)$ by attaching a P_2 to a pendant vertex whose support has degree 4.

G_{10} is the graph obtained from $C_3(2, 0, 0)$ by attaching a P_3 to the vertex $u \in V(C)$ of degree 2, where C is a cycle in $C_3(2, 0, 0)$.

G_{11} is the graph obtained from $C_3(2, 1, 0)$ by attaching a P_2 to the vertices whose support has degree 4.

G_{12} is the graph obtained from $C_3(2, 1, 1)$ by attaching a P_2 to a pendant vertex whose support has degree 4.

2.2 Sets of graphs

We define the following sets of graphs.

$A_1 = \{C_3(3, 0, 0)\}$

$A_2 = \{C_4(P_3, P_1, P_1, P_1),\ C_4(1, 0, 0, 0)\}$

$A_3 = \{C_7\}$

$A_4 = \{G_1, C_3(P_3, P_3, P_2), C_3(P_3, P_2, P_2)\}$

$A_5 = \{G_2, G_3, C_5(P_3, P_2, P_1, P_1, P_1), C_5(P_3, P_1, P_2, P_1, P_1), C_5(P_2, P_2, P_1, P_1, P_1), C_5(P_2, P_1, P_2, P_1, P_1), C_3(P_5, P_2, P_1), C_3(P_4, P_2, P_1), C_3(P_3, P_3, P_1)\}$

$A_6 = \{G_4, G_5, C_5(P_4, P_1, P_1, P_1, P_1), C_5(P_3, P_1, P_1, P_1, P_1), C_3(P_5, P_1, P_1), C_3(P_4, P_1, P_1)\}$

$A_7 = \{C_5(2, 0, 0, 0, 0), C_3(2, 1, 0), C_3(2, 1, 1)$ and $G_i\ :\ 6 \le i \le 12\}$

$A_8 = \{C_4,\ C_3(2, 0, 0),\ C_3(1, 1, 1),\ C_3(1, 1, 0),\ C_3(P_1, P_2, P_3),\ C_3(P_1, P_1, P_3),$
$C_5(1, 0, 0, 0, 0)\}$

2.3 Family of graphs

Let $\mathscr{F}_1$ be the family of connected unicyclic graphs of order n with odd cycle $C = (v_1, v_2, \cdots, v_k, v_1)$ satisfy the following conditions: (i) $6 \le n \le 9$ (ii) $\Delta = 3$ (iii) $s = |\{v \in C : degv = \Delta\}|$.

Let $\mathscr{F}_2$ be the family of connected unicyclic graphs of order n with odd cycle $C = (v_1, v_2, \cdots, v_k, v_1)$ satisfy the following conditions (i) $6 \le n \le 9$ (ii) $\Delta = 4$.

We assume through out that the graph G has no isolated vertices.

3 Main Results

Theorem 3.1. Let G be a graph without isolated vertices. Then $\gamma_{nt}(G) + \chi(G) \leq n + 2$ and equality holds if and only if G is isomorphic to sK_2.

Proof: By Theorem 1.7, $\gamma_{nt}(G) \leq n - \Delta + 1$ and by Theorem 1.1, $\chi(G) \leq \Delta + 1$. Hence $\gamma_{nt}(G) + \chi(G) \leq n + 2$.

Let G be a graph with $\gamma_{nt}(G) + \chi(G) = n + 2$. Then $\gamma_{nt}(G) = n - \Delta + 1$ and $\chi(G) = \Delta + 1$. Suppose G is connected. Then G is either a complete graph or an odd cycle. If $G = K_n, n \geq 3$ or an odd cycle, then $\gamma_{nt}(G) \leq n - \Delta$. Hence G is isomorphic to K_2.

Suppose G is disconnected. We claim that $\Delta(G) = 1$. Suppose $\Delta(G) \geq 2$. Let G_1 be a component of G with $\Delta(G_1) = \Delta(G)$ and let $|V(G_1)| = n_1$. Since $\gamma_{nt}(G) = n - \Delta + 1$, it follows that $\gamma_{nt}(G_1) = n_1 - \Delta + 1$. If $\Delta(G_1) < n_1 - 1$ then $\gamma_{nt}(G_1) \leq n_1 - \Delta$ and hence $\gamma_{nt}(G) \leq n - \Delta$. If $\Delta(G_1) = n_1 - 1$, then $\gamma_{nt}(G_1) = 1$ or 2. If $\gamma_{nt}(G_1) = 1$, then $\gamma_{nt}(G) \leq n - \Delta$. If $\gamma_{nt}(G_1) = 2$ then G_1 contains a support vertex and hence $\chi \leq \Delta$, which is a contradiction. Thus $\Delta = 1$ and each component of G is isomorphic to K_2. ■

Theorem 3.2. Let G be a connected graph. Then $\gamma_{nt}(G) + \chi(G) = n + 1$ if and only if $G = C_5$ or $K_n; (n \geq 3)$ or $K_n - Y$ where Y is a set of edges incident with a vertex of K_n with $|Y| = n - 2$.

Proof: Let G be a connected graph with $\gamma_{nt}(G) + \chi(G) = n + 1$.

Case 1: $\Delta < n - 1$.

Then $\gamma_{nt} \leq n - \Delta$. Since $\gamma_{nt} + \chi = n + 1$, it follows that $\chi \geq \Delta + 1$ and hence $\chi = \Delta + 1$. Thus G is an odd cycle and $\gamma_{nt} = n - \Delta = n - 2$. Hence $G = C_5$.

Case 2: $\Delta = n - 1$.

Then $\gamma_{nt} \leq 2$. If $\gamma_{nt} = 1$ then $\chi = n$ and hence G is isomorphic to $K_n, n \geq 3$. Suppose $\gamma_{nt} = 2$. It follows from Theorem 1.3 that G has exactly one vertex v with $\deg v = n - 1$ and v is adjacent to a vertex of degree 1. Since $\chi(G) = n - 1$, it follows that $G - v$ has exactly two components H_1 and H_2 where $H_1 = K_1$ and $\langle V(H_2) \cup \{v\} \rangle = K_{n-1}$. Hence $G = K_n - Y$ where Y is a set of edges incident with a vertex of K_n with $|Y| = n - 2$. The converse is obvious. ■

Corollary 3.3. Let G be a graph with $\gamma_{nt}(G) + \chi(G) = n + 1$. Then G is isomorphic to $sK_2 \cup H$ where H is isomorphic to C_5 or K_{n-2s} $(n - 2s \neq 2)$ or $K_{n-2s} - Y$ where Y is the set of edges incident with a vertex of K_{n-2s} with $|Y| = n - 2s - 2$.

Theorem 3.4. Let T be a tree of order n. Then $\gamma_{nt} + \chi = n$ if and only if T is isomorphic to P_4 or P_5 or $K_{1,3}$.

Proof: Suppose $\gamma_{nt} + \chi = n$. Since $\chi = 2$ for any nontrivial tree, $\gamma_{nt} = n - 2$. If $\Delta = n - 1$, then $\gamma_{nt} = 2$ so that $n = 4$ and $T = K_{1,3}$. Suppose $\Delta < n - 1$. Then $\gamma_{nt} \leq n - \Delta$. Since $\gamma_{nt} = n - 2$, it follows that $\Delta \leq 2$, so that T is a path. Hence $\gamma_{nt} \leq \lceil \frac{n}{2} \rceil$ which implies $n \leq 5$. Since $\Delta < n - 1$, T is isomorphic to either P_4 or P_5. The converse is obvious. ■

Theorem 3.5. Let G be a connected unicyclic graph of order n with cycle $C = (v_1, v_2, \ldots, v_k, v_1)$. Then $\gamma_{nt} + \chi = n$ if and only if $G \in A_8$.

Proof: Suppose $\gamma_{nt} + \chi = n$.

Case 1: $\Delta = n - 1$.

Then $\gamma_{nt} \leq 2$. If $\gamma_{nt} = 1$ then G is K_3 and in this case $\gamma_{nt} + \chi = 4 > n$. Hence $\gamma_{nt} = 2$ and $\chi = n - 2$. If k is even then $\chi = 2$ and $n = 4$ and $k = 4$ so that $\Delta \neq n - 1$ which is a contradiction. Hence k is odd, $\chi = 3$, so that $n = 5$ and $\Delta = 4$. Hence $k = 3$ and G is isomorphic to $C_3(2, 0, 0)$.

Case 2. $\Delta < n - 1$.

If k is even, then $\chi = 2$ and $\gamma_{nt} = n - 2$. Since $\gamma_{nt} \leq \left\lceil \frac{n}{2} \right\rceil$, it follows that $n \leq 5$. If $n = 5$, then $k = 4$ and there exists a vertex not in C which is adjacent to a vertex in C. For this graph $\gamma_{nt} + \chi \neq n$. Hence $n = 4$ and G is isomorphic to C_4.

If k is odd, then $\chi = 3$ and $\gamma_{nt} = n - 3$. Since $\Delta < n - 1$ we have $\gamma_{nt} \leq n - \Delta$ and hence $\Delta \leq 3$. Also $\gamma_{nt} \leq \left\lceil \frac{n}{2} \right\rceil$, which gives $n \leq 7$.

If $\Delta = 2$, then G is isomorphic to C_3 or C_5 or C_7 and for these graphs $\gamma_{nt} + \chi \neq n$. Thus $\Delta = 3$.

Let X be the set of all pendent vertices of G. Since $\gamma_{nt} = n - 3$, $|X| \leq 3$ and C contains at most three vertices of degree 3. Suppose C contains three vertices of degree 3. Then $k = 3$. Let $C = (v_1, v_2, v_3, v_1)$ and $u_i v_i \in E, i = 1, 2, 3$. Since $\gamma_{nt} + \chi = n$, it follows that $deg\ u_i = 1$ for all i and G is isomorphic to $C_3(1, 1, 1)$.

Suppose C contains two vertices of degree 3. Then $k = 3$ or 5. If $k = 5$, then $n = 7$, and in this case $\gamma_{nt} + \chi \neq n$. Hence $k = 3$. Let $C = (v_1, v_2, v_3, v_1)$, $deg\ v_1 = deg\ v_2 = 3$ and $u_1 v_1, u_2 v_2 \in E$. If $deg\ u_1 = deg\ u_2 = 1$, then G is isomorphic to $C_3(1, 1, 0)$. If $deg\ u_1 = deg\ u_2 = 2$, then $\gamma_{nt} + \chi \neq n$. Hence we may assume that $deg\ u_1 = 1$ and $deg\ u_2 \geq 2$. Since $\gamma_{nt} + \chi = n$, it follows that $deg\ u_2 = 2$ and if $u_2 w \in E(G)$, then $deg\ w = 1$. Hence G is isomorphic to $C_3(P_1, P_2, P_3)$.

Suppose C contains exactly one vertex of degree 3. Then $k = 3$ or 5. If $k = 3$, and $u_1 v_1 \in E$, then $deg\ u_1 = 2$ and if $u_1 w \in E$, then $deg\ w = 1$. Hence G is isomorphic to $C_3(P_3, P_1, P_1)$.

If $k = 5$ and $u_1 v_1 \in E$, then $deg\ u_1 = 1$ and G is isomorphic to $C_5(1, 0, 0, 0, 0)$. The converse is obvious. ■

Remark 3.6. There is no cubic graph of order n with $\gamma_{nt} + \chi = n$.

Proof: Let G be a cubic graph with $\gamma_{nt} + \chi = n$. If G is a complete graph then $\gamma_{nt} + \chi = n + 1$, which is a contradiction. Hence $\chi \leq 3$. Then $\gamma_{nt} \geq n - 3$. It follows from Theorem 1.8, $\gamma_{nt} \leq n - 3$. Thus we have $\gamma_{nt} = n - 3$ and then $\chi = 3$. Theorem 1.6 gives $\gamma_{nt} \leq \lceil \frac{n}{2} \rceil$ which implies $n \leq 7$. Since G is not a complete graph, we have $n = 6$. Then $\gamma_{nt} = 3$ and $\chi = 3$. Since each vertex v of G dominates four vertices, and all the vertices have degree 3, two vertices are sufficient to dominate six vertices. Hence there does not exist a cubic graph with $\gamma_{nt} + \chi = n$. ■

Theorem 3.7. Let T be a tree of order n. Then $\gamma_{nt} + \chi = n - 1$ if and only if T is $K_{1,4}$ or P_6 or it is obtained from $K_{1,3}$ by subdividing at least one edge.

Proof: Let T be a tree with $\gamma_{nt} + \chi = n - 1$. Since $\chi = 2$, we have $\gamma_{nt} = n - 3$ and hence $n \geq 5$. If $\Delta = n - 1$ then $\gamma_{nt} = 2$ so that $n = 5$ and $T = K_{1,4}$. Suppose $\Delta < n - 1$. Then $\gamma_{nt} \leq n - \Delta$ and hence $\Delta \leq 3$. Now from Theorem 1.6, $\gamma_{nt} \leq \lceil \frac{n}{2} \rceil$. So that $n - 3 \leq \lceil \frac{n}{2} \rceil$ which gives $n \leq 7$. Hence $5 \leq n \leq 7$. If $\Delta = 2$ then T is a path and hence $T = P_6$. Let $\Delta = 3$. If $n = 5$ then T is obtained from $K_{1,3}$ by subdividing exactly one edge. Suppose $n = 6$. Let $v \in V$ such that $degv = \Delta$ and let $N(v) = \{v_1, v_2, v_3\}$. Let $u_1,\ u_2 \in V - N[v]$. If $u_1,\ u_2 \in N(v_i)$ for some $i = 1,\ 2,\ 3$ then $\gamma_{nc} = 2$ which is a contradiction. Hence u_1 is adjacent to v_i and u_2 is adjacent to $v_j,\ j \neq i$. Thus T is isomorphic to a graph obtained from $K_{1,3}$ by subdividing two edges once.

Suppose $n = 7$. Let v be a vertex of degree Δ and let $N(v) = \{v_1, v_2, v_3\}$. Let $u_1, u_2, u_3 \in V - N[v]$. Suppose $\deg v_1 = 1$. If $\deg v_2 = 2$ and $\deg v_3 = 3$ then $\gamma_{nt} = 3 \neq n - 3$ which is a contradiction. If $\deg v_2 = 1$ and $\deg v_3 = 3$ then $u_1\, v_3,\ u_2\, v_3 \in E(T)$. Since T is a tree without loss of generality we assume u_3 is adjacent to u_1 then $\gamma_{nt} = 3 \neq n - 3$ which is a contradiction. Let $\deg v_2 = 1$ and $\deg v_3 = 2$ and $u_1\, v_3 \in E(T)$. If $u_1\, u_2,\ u_1\, u_3 \in E(T)$ then $\gamma_{nt} = 3 \neq n - 3$ which is a contradiction. If $u_1\, u_2,\ u_2\, u_3 \in E(T)$ then $\gamma_{nt} \neq n - 3$ which is a contradiction. Hence $\deg v_i = 2,\ 1 \leq i \leq 3$. Thus T is isomorphic to a graph obtained from $K_{1,3}$ by subdividing all the edges once. The converse is obvious. ∎

Lemma 3.8. Let G be a connected unicyclic graph of order n and $\Delta = n - 1$. Then $\gamma_{nt} + \chi = n - 1$ if and only if $G \in A_1$.

Proof: Let G be a connected unicyclic graph with cycle $C = (v_1, v_2, \cdots, v_k, v_1)$, $\Delta = n - 1$ and let $\gamma_{nt} + \chi = n - 1$. Then $\gamma_{nt} \leq 2$. If $\gamma_{nt} = 1$ then G is K_3 and $\gamma_{nt} + \chi = n + 1$. Hence $\gamma_{nt} = 2$ and $\chi = n - 3$. If $k \geq 4$ then $\Delta < n - 1$. Hence $k = 3$. Thus $\chi = 3,\ n = 6$ and $\Delta = 5$. Hence G is isomorphic to $C_3(3, 0, 0)$. The converse is obvious. ∎

Lemma 3.9. Let G be a connected unicyclic graph with even cycle $C = (v_1, v_2, \cdots, v_k, v_1)$ and $\Delta < n - 1$. Then $\gamma_{nt} + \chi = n - 1$ if and only if $G \in A_2$.

Proof: Since k is even $\chi = 2$ and $\gamma_{nt} = n - 3$. It follows from Theorem 1.6 that $\gamma_{nt} \leq \lceil \frac{n}{2} \rceil$ and hence $n \leq 7$.

Case 1: $n = 7$.

Then $k = 4$ or 6. If $k = 4$ then $C = (v_1, v_2, v_3, v_4, v_1)$. Let u_1, u_2, u_3 be the vertices not on C. If $\deg u_i = 1$ for all $i = 1, 2, 3$ then G is isomorphic to $C_4(1, 1, 1, 0)$ or $C_4(2, 1, 0, 0)$ or $C_4(2, 0, 1, 0)$ or $C_4(3, 0, 0, 0)$. For this graphs $\gamma_{nt} \neq n - 3$. Let $\deg u_1 = \deg u_2 = 1$. If $\deg u_3 = 2$ then G is isomorphic to $C_4(P_3, P_2, P_1, P_1)$ or $C_4(P_3, P_1, P_2, P_1)$. But $\gamma_{nt} = 3 \neq n - 3$. If $\deg u_3 = 3$ then G is a graph obtained from $C_4(P_3, P_1, P_1, P_1)$ by attaching a P_2 to the vertex $u \notin V(C)$ of degree 2. For this graph $\gamma_{nt} = 3 \neq n - 3$. If $\deg u_1 = 1$ and $\deg u_i \neq 1$, $i = 2,\ 3$ then $\deg u_2 = \deg u_3 = 2$. Hence G is isomorphic to $C_4(P_4, P_1, P_1, P_1)$. But $\gamma_{nt} = 3 \neq n - 3$. If $k = 6$ then G is isomorphic to $C_6(1, 0, 0, 0, 0, 0)$. But $\gamma_{nt} = 3 \neq n - 3$.

Case 2: $n = 6$.

Then $k = 4$ or 6. If $k = 6$ then G is C_6.But $\gamma_{nt}(C_6) = 2$. Hence $k = 4$. Then G is any one of the following graph (i) $H_1 = C_4(1, 1, 0, 0)$ (ii) $H_2 = C_4(2, 0, 0, 0)$ (iii) $H_3 = C_4(P_3, P_1, P_1, P_1)$. But $\gamma_{nt}(H_1) = \gamma_{nt}(H_2) = 2$. Hence G is isomorphic to $C_4(P_3, P_1, P_1, P_1)$.

If $n = 5$ then $k = 4$ and G is isomorphic to $C_4(1, 0, 0, 0)$. If $n = 4$ then G is C_4 and $\gamma_{nt} \neq n - 3$. The converse is obvious. ■

Lemma 3.10. Let $G \in \mathscr{F}_1$ and $s = 3$. Then $\gamma_{nt} + \chi = n - 1$ if and only if $G \in A_4$.

Proof: Since k is odd $\chi = 3$ and $\gamma_{nt} = n - 4$, also $s = 3$ gives $k = 3$ or 5. If $k = 5$ then G is isomorphic to $C_5(P_3, P_2, P_2, P_1, P_1)$ or $C_5(P_2, P_2, P_2, P_1, P_1)$ or $C_5(P_2, P_2, P_1, P_2, P_1)$ or $C_5(P_3, P_1, P_2, P_2, P_1)$ or $C_5(P_3, P_2, P_1, P_1, P_2)$ or $C_5(P_3, P_2, P_1, P_2, P_1)$. For this graphs $\gamma_{nt} \neq n - 4$ which is a contradiction. Thus $k = 3$. Let $C = (v_1, v_2, v_3, v_1)$ and $u_i v_i \in E,\ 1 \leq i \leq 3$. If $\deg u_i = 1,\ 1 \leq i \leq 3$ then G is isomorphic to $C_3(1, 1, 1)$. But $\gamma_{nt}[C_3(1, 1, 1)] = 3 \neq n - 4$ which is a contradiction. If $\deg u_1 = 3$ then at most one of u_2 and u_3 has degree 2. Let $\deg u_2 = 2$. Then $\deg u_3 = 1$. For this graph $\gamma_{nt} = 4 \neq n - 4$ which is a contradiction. If $\deg u_2 = 1$ and $\deg u_3 = 1$ then the graph G is isomorphic to G_1. Suppose $\deg u_1 = 2$. Then ($\deg u_2 = 2$ and $\deg u_3 = 2$) or ($\deg u_2 = 2$ and $\deg u_3 = 1$) or ($\deg u_2 = 1$ and $\deg u_3 = 1$).

If $\deg u_1 = 2,\ \deg u_2 = 2$ and $\deg u_3 = 2$ then G is isomorphic to $C_3(P_3, P_3, P_3)$. But $\gamma_{nt}[C_3(P_3, P_3, P_3)] = 4 \neq n - 4$ which is a contradiction. If $\deg u_1 = 2,\ \deg u_2 = 2$ and $\deg u_3 = 1$ then G is isomorphic to $C_3(P_3, P_3, P_2)$. If $\deg u_1 = 2,\ \deg u_2 = 1$ and $\deg u_3 = 1$ then G is isomorphic to $C_3(P_3, P_2, P_2)$. The converse is obvious. ■

Lemma 3.11. Let $G \in \mathscr{F}_1$ and $s = 2$. Then $\gamma_{nt} + \chi = n - 1$ if and only if $G \in A_5$.

Proof: Since k is odd $\chi = 3$ and $\gamma_{nt} = n - 4$. Also $s = 2$ gives $k = 3$ or 5 or 7.

Case 1: $k = 7$.

Then G is isomorphic to the graph obtained from C_7 by attaching P_2 to any two vertices. But $\gamma_{nt} < n - 4$ which is a contradiction.

Case 2: $k = 5$.

Let $C = (v_1, v_2, v_3, v_4, v_5, v_1)$ and let x be a pendant vertex in G. It is clear that $d(x, C) \leq 3$. If $d(x, C) = 3$ then G is isomorphic to $C_5(P_4, P_2, P_1, P_1, P_1)$ or $C_5(P_4, P_1, P_2, P_1, P_1)$. For this graphs $\gamma_{nt} \neq n - 4$ which is a contradiction.

Suppose $d(x, C) = 2$. Then $n = 8$ or 9. Suppose $n = 8$. Let $\deg v_1 = 3$ and (v_1, x_1, x) be a path in G. Since $n = 8$ there is a vertex x_2 in $V(G)$ and x_2 is adjacent to v_2 or v_3 or v_4 or v_5. Hence G is isomorphic to $C_5(P_3, P_2, P_1, P_1, P_1)$ or $C_5(P_3, P_1, P_2, P_1, P_1)$. Suppose $n = 9$. Then there are two vertices $x_2,\ x_3 \in V(G)$. If $\deg x_2 = \deg x_3 = 1$ then $x_2 v_2$ or $x_2 v_3 \in E$ and $x_1 x_3 \in E$. For these graphs $\gamma_{nt} \neq n - 4$ which is a contradiction. If $\deg x_2 = 2$ then $\deg x_3 = 1$ and $x_2 x_3 \in E$. Hence G is isomorphic to $C_5(P_3, P_3, P_1, P_1, P_1)$ or $C_5(P_3, P_1, P_3, P_1, P_1)$. For this graphs $\gamma_{nt} \neq n - 4$.

If $d(x, C) = 1$ then G is isomorphic to $C_5(P_2, P_2, P_1, P_1, P_1)$ or $C_5(P_2, P_1, P_2, P_1, P_1)$.

Case 3: $k = 3$.

Let $C = (v_1, v_2, v_3, v_1)$ and let x be a pendant vertex in G. Then $d(x, C) \leq 5$. If $d(x, C) = 5$ then $n = 9$ and G is isomorphic to $C_3(P_6, P_2, P_1)$. But $\gamma_{nt} \neq n - 4$ which is a contradiction.

Sub Case 3.1: $d(x, C) = 4$.

Let (v_1, x_1, x_2, x_3, x) be the $v_1 - x$ path. Then $n = 8$ or 9. Suppose $n = 8$ then G is isomorphic to $C_3(P_5, P_2, P_1)$. If $n = 9$ there exist two vertices x_4 and x_5 such that $x_4 v_2 \in E$ and x_5 is adjacent to any one of x_1, x_2, x_3 and x_4. All these cases $\gamma_{nt} \neq n - 4$.

Sub Case 3.2: $d(x, C) = 3$.

Let (v_1, x_1, x_2, x) be the $v_1 - x$ path. Then $7 \leq n \leq 9$. If $n = 7$ then G is isomorphic to $C_3(P_4, P_2, P_1)$. Let $n = 8$ and $x_3 v_2 \in E$. Then there is a vertex x_4 which is adjacent to any one of x_1, x_2 and x_3. Hence G is isomorphic to $C_3(P_4, P_3, P_1)$ or G_2. But $\gamma_{nt}(C_3(P_4, P_3, P_1)) \neq n - 4$. Hence G is isomorphic to G_2.Let $n = 9$ and $x_3 v_2 \in E$. Then there are two vertices x_4 and x_5 with $x_2 x_4,\ x_3 x_5 \in E$ or $x_2 x_4,\ x_1 x_5 \in E$ or $x_1 x_5,\ x_3 x_4 \in E$. All these cases $\gamma_{nt} \neq n - 4$.

Sub Case 3.3: $d(x, C) = 2$.

Let (v_1, x_1, x) be the $v_1 - x$ path and let $x_2 v_2 \in E$. Then $6 \leq n \leq 9$. If $n = 6$ then G is isomorphic to $C_3(P_3, P_2, P_1)$. For this graph $\gamma_{nt} \neq n - 4$. If $n = 7$ then G is isomorphic to $C_3(P_3, P_3, P_1)$ or G_3. If $n = 8$ then G is a graph obtained from $C_3(P_3, P_3, P_1)$ by attaching a P_2 to the vertex $u \notin V(C)$ of degree 2. For this graph $\gamma_{nt} \neq n - 4$. If $n = 9$ then G is a graph obtained from $C_3(P_3, P_3, P_1)$ by attaching a pendant vertex to all the vertices of degree 2 which are not on C. For this graph $\gamma_{nt} \neq n - 4$.

If $d(x, C) = 1$ then G is isomorphic to $C_3(P_2, P_2, P_1)$. But $\gamma_{nt} \neq n - 4$ which is a contradiction. The converse is obvious. ∎

Lemma 3.12. Let $G \in \mathscr{F}_1$ and $s = 1$. Then $\gamma_{nt} + \chi = n - 1$ if and only if $G \in A_6$.

Proof: Since k is odd, $\chi = 3$ and $\gamma_{nt} = n - 4$. Also $s = 1$ gives $k = 3$ or 5 or 7.

Case 1: $k = 7$.

Then G is isomorphic to $C_7(P_3, P_1, P_1, P_1, P_1, P_1, P_1)$ or $C_7(P_2, P_1, P_1, P_1, P_1, P_1, P_1)$. For these graphs $\gamma_{nt} \neq n - 4$.

Case 2: $k = 5$.

Let $C = (v_1, v_2, v_3, v_4, v_5, v_1)$ and let x be a pendant vertex in G. Also let us assume $\deg v_1 = 3$. Then $d(x, C) \leq 4$. If $d(x, C) = 4$ then $n = 9$ and G is isomorphic to $C_5(P_5, P_1, P_1, P_1, P_1)$. But $\gamma_{nt} \neq n - 4$. Let $d(x, C) = 3$ and let (v_1, x_1, x_2, x) be the $v_1 - x$ path. If $\deg x_1 = 3$ or $\deg x_2 = 3$ then $\gamma_{nt} \neq n - 4$. Hence G is isomorphic to $C_5(P_4, P_1, P_1, P_1, P_1)$. Let $d(x, C) = 2$ and let (v_1, x_1, x) be the $v_1 - x$ path. Then $n = 7$ or 8. If $n = 7$ then G is isomorphic to $C_5(P_3, P_1, P_1, P_1, P_1)$. If $n = 8$ then there is a vertex x_2 such that $x_2 x_1 \in E$. For this graph $\gamma_{nt} \neq n - 4$. If $d(x, C) = 1$ then G is isomorphic to $C_5(P_2, P_1, P_1, P_1, P_1)$. But $\gamma_{nt} \neq n - 4$.

Case 3: $k = 3$.

Let $C = (v_1, v_2, v_3, v_1)$ and let x be a pendant vertex in G. Also let us assume $\deg v_1 = 3$. Then $d(x, C) \leq 6$. If $d(x, C) = 6$ then $n = 9$ and G is isomorphic to $C_3(P_7, P_1, P_1)$. But $\gamma_{nt} \neq n - 4$.

Let $d(x, C) = 5$ and let $(v_1, x_1, x_2, x_3, x_4, x)$ be the $v_1 - x$ path. If $\deg x_i = 2,\ 1 \leq i \leq 4$ then G is isomorphic to $C_3(P_6, P_1, P_1)$.But $\gamma_{nt}(C_3(P_6, P_1, P_1)) \neq n - 4$. If $\deg x_i = 3$ for some i, $1 \leq i \leq 4$ then $\gamma_{nt} \neq n - 4$.

Let $d(x, C) = 4$ and let (v_1, x_1, x_2, x_3, x) be the $v_1 - x$ path. If $\deg x_i = 2$, $1 \leq i \leq 3$ then G is isomorphic to $C_3(P_5, P_1, P_1)$. If $\deg x_i = 3$ for some i, $1 \leq i \leq 3$ then $\gamma_{nt} \neq n - 4$.

Let $d(x, C) = 3$ and let (v_1, x_1, x_2, x) be the $v_1 - x$ path. Then $6 \leq n \leq 9$. If $n = 6$ then G is isomorphic to $C_3(P_4, P_1, P_1)$. If $n = 7$ then $\deg x_1 = 3$ or $\deg x_2 = 3$. Hence G is isomorphic to G_4. If $n = 8$ then $\deg x_i = 3, 1 \leq i \leq 2$ or $\deg x_1 = 3$ and $\deg x_2 = 2$ and x_1 is not a support vertex. For these graphs $\gamma_{nt} \neq n - 4$. If $n = 9$ then $\deg x_i = 3, 1 \leq i \leq 2$ and x_1 is not a support vertex. For this graph $\gamma_{nt} \neq n - 4$.

Let $d(x, C) = 2$ and let (v_1, x_1, x) be the $v_1 - x$ path. If $\deg x_1 = 2$ then $n = 5$ which is a contradiction. Thus $\deg x_1 = 3$ and hence G is isomorphic to G_5.

If $d(x, C) = 1$ then $n = 4$ which is a contradiction. The converse is obvious. ■

Lemma 3.13. Let $G \in \mathscr{F}_2$. Then $\gamma_{nt} + \chi = n - 1$ if and only if $G \in A_7$.

Proof: Since k is odd $\chi = 3$ and $\gamma_{nt} = n - 4$.

Case 1: C contains a vertex of degree Δ.

Then $k = 3$ or 5 or 7. If $k = 7$ then G is isomorphic to $C_7(2, 0, 0, 0, 0, 0, 0)$ and $\gamma_{nt} \neq n - 4$.

Sub case 1.1: $k = 5$.

Let $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. If x is a pendant vertex then $d(x, C) \leq 3$. If $d(x, C) = 3$ then $\gamma_{nt} \neq n - 4$. If $d(x, C) = 2$ then by similar arguments given in Lemma 3.11 and 3.12 $\gamma_{nt} \neq n - 4$. If $d(x, C) = 1$ then $n = 7$ or 8 or 9. If $n = 8$ or 9 we have $\gamma_{nt} \neq n - 4$. Then G is isomorphic to $C_5(2, 0, 0, 0, 0)$.

Sub Case 1.2: $k = 3$.

Let $C = (v_1, v_2, v_3, v_1)$. If x is a pendant vertex then $d(x, C) \leq 5$. If $d(x, C) = 5$ or 4 then $\gamma_{nt} \neq n - 4$. If $d(x, C) = 3$ then $n = 7$ or 8 or 9. If $n = 8$ or 9 then by similar arguments given in Lemma 3.11 and Lemma 3.12 $\gamma_{nt} \neq n - 4$. If $n = 7$ then G is isomorphic to G_6. If $d(x, C) = 2$ then $6 \leq n \leq 9$. If $n = 6$ then G is isomorphic to G_7. If $n = 7$ then G is isomorphic to G_8 or G_9 or G_{10}. If $n = 8$ then G is isomorphic to G_{11} or G_{12}. If $n = 9$ then no graph exists. If $d(x, C) = 1$ then $6 \leq n \leq 9$. Then by similar arguments given in Lemma 3.11 and Lemma 3.12 there is no graph of order 8 and 9. If $n = 6$ then G is isomorphic to $C_3(2, 1, 0)$. If $n = 7$ then G is isomorphic to $C_3(2, 1, 1)$.

Case 2: C does not contains maximum degree vertex.

Then $k = 3$ or 5. If $k = 5$ then G is a graph obtained from $C_5(P_3, P_1, P_1, P_1, P_1)$ by attaching two

P_2 to the vertex $u \notin V(C)$ of degree 2. For this graph $\gamma_{nt} \neq n-4$. If $k=3$ then by similar arguments given in Lemma 3.11 and 3.12 there is no graph. The converse is obvious. ■

Theorem 3.14. Let G be a connected unicyclic graph of order n. Then $\gamma_{nt} + \chi = n-1$ if and only if $G \in \bigcup_{i=1}^{7} A_i$.

Proof: Let G be a unicyclic graph with cycle $C = (v_1, v_2, \cdots, v_k, v_1)$ and let $\gamma_{nt} + \chi = n-1$. If $\Delta = n-1$ or $\Delta < n-1$ with k is even then $G \in A_1 \cup A_2$.

Suppose $\Delta < n-1$ and k is odd.Then $\chi = 3$ and $\gamma_{nt} = n-4$. Also it follows from Theorem 1.6 that $\gamma_{nt} \leq \lceil \frac{n}{2} \rceil$ so that $6 \leq n \leq 9$. Further since $\Delta < n-1$ we have $\gamma_{nt} \leq n - \Delta$ and hence $\Delta \leq 4$. If $\Delta = 2$ then G is isomorphic to C_7 or C_9. But $\gamma_{nt}(C_9) = 3 \neq n-4$. Hence G is isomorphic to C_7. Thus $G \in A_3$. Since $\gamma_{nt} = n-4$, G contains at most four pendant vertices and hence C contains at most four vertices of degree 3. Then $s \leq 4$. If $s = 4$ then $k = 5$. Let $v \in V(C)$ of degree 2 and let $u_1,\ u_2,\ u_3,\ u_4 \in V(G-C)$. Then $\deg u_i = 1,\ 1 \leq i \leq 4$ and $S = V - \{v, u_1, u_2, u_3, u_4\}$ is a ntd set of cardinality $n-5$ which is a contradiction. Hence $s \leq 3$. Then $G \in \mathscr{F}_1$ or $\mathscr{F}_2$. Hence from Lemmas $3.10, 3.11, 3, 12, 3, 13$ the result follows. The converse is obvious. ■

Theorem 3.15. Let G be a connected cubic graph of order n. Then $\gamma_{nt} + \chi = n-1$ if and only if $G = C_3^{(2)}$.

Proof: Let G be a connected cubic graph with $\gamma_{nt} + \chi = n-1$. If G is a complete graph then $\gamma_{nt} + \chi = n+1$ which is a contradiction. Hence $\chi \leq 3$. Then $\gamma_{nt} \geq n-4$. It follows from Theorem 1.8, $\gamma_{nt} \leq n-3$. Thus we have $\gamma_{nt} = n-4$ or $n-3$.

Case 1: $\gamma_{nt} = n-3$.

Then $\chi = 2$. It follows from Theorem 1.6 that $\gamma_{nt} \leq \lceil \frac{n}{2} \rceil$ which gives $n \leq 7$. Since G is not a complete graph we have $n = 6$. Then $\gamma_{nt} = 3$ and $\chi = 2$. Since each vertex v of G dominates four vertices and all the vertices having degree three, two vertices are sufficient to dominate six vertices. Hence $\gamma_{nt} \leq 2$ which is a contradiction.

Case 2: $\gamma_{nt} = n-4$.

Then $\chi = 3$. It follows from Theorem 1.6 that $\gamma_{nt} \leq \lceil \frac{n}{2} \rceil$ which gives $n \leq 9$. Since G is not a complete graph we have $n = 6$ or 8.

Suppose $n = 6$. Then $\gamma_{nt} = 2$ and $\chi = 3$

Let $S = \{v_1, v_2\}$ be the γ_{nt}-set of G and let $V - S = \{u_1, u_2, u_3, u_4\}$

Sub case 2.1: $\langle S \rangle = \overline{K_2}$.

Let v_1 be adjacent to u_1, u_2 and u_3. If v_2 is also adjacent to u_1, u_2 and u_3 then u_4 is adjacent to $u_1, u_2,$ and u_3. For this graph $\chi = 2$ which is a contradiction. Suppose v_2 is adjacent to u_1, u_2 and u_4. If $u_1u_2 \in E$ then G is not a cubic graph. Hence u_1 is adjacent to u_3 or u_4. If $u_1u_3 \in E$ then

$u_2u_4,\ u_3u_4 \in E$. Then the graph G is isomorphic to the graph $C_3^{(2)}$. If $u_1u_4 \in E$ then $u_2u_3,\ u_3u_4 \in E$. Then the graph G is isomorphic to $C_3^{(2)}$.

Sub case 2.2: $\langle S \rangle = K_2$.

Let v_1 be adjacent to u_1 and u_2. If v_2 is also adjacent to u_1 and u_2 then G is not a cubic graph. Suppose v_2 is adjacent to u_1 and u_3. Then u_4 is adjacent to $u_1,\ u_2$ and u_3 and hence $u_2\,u_3 \in E$. Then G is isomorphic to the graph $C_3^{(2)}$. Suppose v_2 is adjacent to u_3 and u_4. If $u_1\,u_2 \in E$ then $u_2\,u_4,\ u_3\,u_4,\ u_1\,u_4 \in E$. Then G is isomorphic to $C_3^{(2)}$. If $u_1\,u_2 \notin E$ then $u_1\,u_3,\ u_1\,u_4,\ u_2\,u_3,\ u_2u_4 \in E(G)$. For this graph $\chi = 2$ which is a contradiction.

Suppose $n = 8$ then $\gamma_{nt} = 4$ and $\chi = 3$. Since each vertex v of G dominates four vertices and all the vertices having degree three maximum of three vertices are sufficient to dominate eight vertices. Hence $\gamma_{nt} \leq 3$ which is a contradiction. The converse is obvious. ■

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