International Journal of Mathematics and Soft Computing
Vol.5, No.1 (2015), 111 - 117.

ISSN Print : 2249 - 3328
ISSN Online: 2319 - 5215

Graceful labeling for grid related graphs

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Abstract

In this paper we present graceful labelings for $C(t \cdot P_n \times P_m)$, $(P_n \times P_m)^\star$ and path union of t copies of the grid graph $P_n \times P_m$.

Keywords: Graceful labeling, star of a graph, cycle of a graph, path union of a graph.
AMS Subject Classification(2010): 05C78.

1 Introduction

The graceful labeling was introduced by Rosa [8]. Golomb [4] named such labeling as graceful labeling, which was called earlier as $\beta-$valuation. Bu and Cau [2] have discussed gracefulness of complete bipartite graph and its union with path. Acharya and Gill [1] have investigated graceful labeling for the grid graph $(P_n \times P_m)$. Kaneria and Makadia [6] discussed gracefulness of $(P_n \times P_m) \cup (P_r \times P_s)$, $C_{2f+3} \cup (P_n \times P_m) \cup (P_r \times P_s)$, tensor product $P_2(T_p)P_n$ and star of cycle $C_n^\star$ ($n \equiv 0 \pmod 4$). For a dynamic survey on graph labeling we refer to Gallian [3].

We begin with a simple, undirected finite graph $G = (V, E)$ with $|V| = p$ and $|E| = q$. For all terminology and notations we follows Harary [5].

Definition 1.1. A function f is called *graceful labeling* of a graph $G = (V, E)$ if $f : V \longrightarrow \{0, 1, \ldots, q\}$ is injective and the induce function $f^\star : E \longrightarrow \{1, 2, \ldots, q\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E$. A graph G is called *graceful graph* if it admits a graceful labeling.

Definition 1.2. [9] Let G be a graph on n vertices. The graph obtained by replacing each vertex of the star $K_{1,n}$ by a copy of G is called a star of G and is denoted by $G^\star$.

Definition 1.3. [7] If each vertex of a cycle C_n is replaced by connected graphs $G_1, G_2, \dots, G_n$ then the resulting graph is known as *cycle of graphs* and is denoted by $C(G_1, G_2, \dots, G_n)$. If we replace each vertex by a graph G, that is, $G_1 = G$, $G_2 = G$, ..., $G_n = G$, then the cycle of the graph G is denoted by $C(n \cdot G)$.

Definition 1.4. Let G be a graph and $G_1, G_2, \dots, G_n$, $n \geq 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} $(1 \leq i \leq n-1)$ is called *path union* of G.

In this paper we establish graceful labelings of some grid related graphs.

2 Main Results

Theorem 2.1. Cycle of grid graph $C(t \cdot P_n \times P_m)$ ($t \equiv 0$ (mod 2)) is graceful, where $m, n \geq 2$.

Proof: Let $V(P_n \times P_m) = \{u_1 = v_{1,1}, u_2, \dots, u_n, u_{n+1}, \dots, u_{mn} = v_{m,n}\}$ be the vertices of grid graph $P_n \times P_m$. It is proved that $P_n \times P_m$ is graceful [1]. In the proof we can observe that the labeling are given diagonally by an increasing sequence and a decreasing sequence alternatively with $f(v_{1,1}) = q$ and $f(v_{n,m}) = \lfloor \frac{q}{2} \rfloor$, as shown in Figure 1.

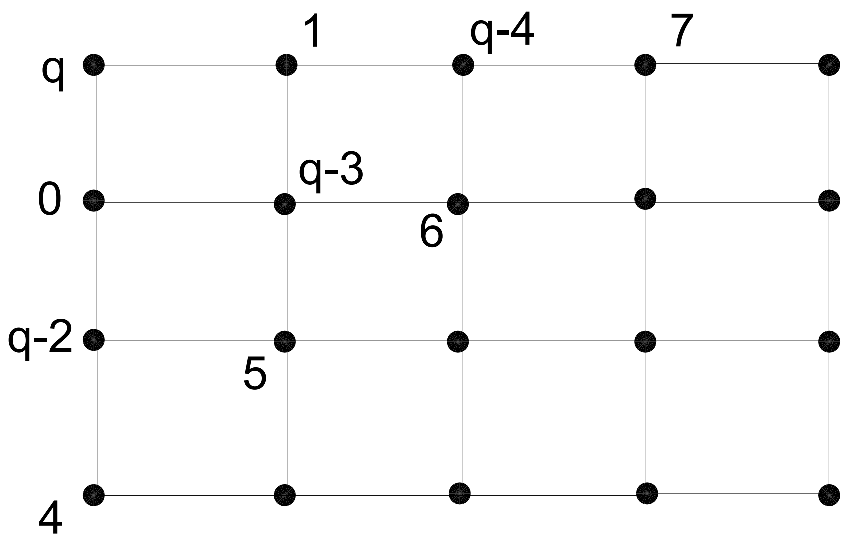

Figure 1: A grid graph on q edges with its graceful labeling.

Let G be a cycle of the graph $P_n \times P_m$ with t copies. We have $G = C(t \cdot P_n \times P_m)$ ($t \equiv 0$ (mod2)), with $|V(C(t \cdot P_n \times P_m))| = tmn$ vertices and $Q = |E(C(t \cdot P_n \times P_m))| = t(2mn - (m+n) + 1)$ edges. Let $u_{i,j}$ $(1 \leq j \leq mn)$ be the vertices of i^{th} copy of $P_n \times P_m$ in G, $\forall\, i = 1, 2, \dots, t$, where the vertex set of i^{th} copy of $P_n \times P_m$ is $p = mn$ and the edge set of i^{th} copy of $P_n \times P_m$ is $q = 2mn - (m+n)$.

To define a labeling function $g : V(C(t \cdot P_n \times P_m)) \longrightarrow \{0, 1, \dots, Q\}$, we consider the following two cases.

Case 1: q is odd.

$$\begin{aligned} g(u_{1,j}) &= f(u_j), && \text{if } f(u_j) < \tfrac{q}{2} \\ &= f(u_j) + Q - q, && \text{if } f(u_j) > \tfrac{q}{2}, \quad \forall\, j = 1, 2, \ldots, mn; \\ g(u_{i,j}) &= g(u_{i-1,j}) - (\tfrac{q+1}{2}), && \text{if } f(u_j) > \tfrac{q}{2} \\ &= g(u_{i-1,j}) + (\tfrac{q+1}{2}), && \text{if } f(u_j) < \tfrac{q}{2}, \\ & && \forall\, j = 1, 2, \ldots, mn, \forall\, i = 2, 3, \ldots, \tfrac{t}{2}; \\ g(u_{\frac{t}{2}+1,j}) &= g(u_{\frac{t}{2},j}) - (\tfrac{q+1}{2}), && \text{if } f(u_j) > \tfrac{q}{2} \\ &= g(u_{\frac{t}{2},j}) + (\tfrac{q+3}{2}), && \text{if } f(u_j) < \tfrac{q}{2}, \\ & && \forall\, j = 1, 2, \ldots, mn; \\ g(u_{i,j}) &= g(u_{i-1,j}) - (\tfrac{q+1}{2}), && \text{if } f(u_j) > \tfrac{q}{2} \\ &= g(u_{i-1,j}) + (\tfrac{q+1}{2}), && \text{if } f(u_j) < \tfrac{q}{2}, \\ & && \forall\, j = 1, 2, \ldots, mn, \forall\, i = \tfrac{t}{2} + 2, \tfrac{t}{2} + 3, \ldots, t. \end{aligned}$$

Case 2: q is even.

$$\begin{aligned} g(u_{1,j}) &= f(u_j), && \text{if } f(u_j) < \tfrac{q}{2} \\ &= f(u_j) + Q - q, && \text{if } f(u_j) \geq \tfrac{q}{2}, \\ & && \forall\, j = 1, 2, \ldots, mn; \\ g(u_{2,j}) &= g(u_{1,mn+1-j}) + Q - q, && \text{if } f(u_j) < \tfrac{q}{2} \\ &= g(u_{1,mn+1-j}) - Q + q, && \text{if } f(u_j) \geq \tfrac{q}{2}, \\ & && \forall\, j = 1, 2, \ldots, mn; \\ g(u_{i,j}) &= g(u_{i-2,j}) - (q+1), && \text{if } g(u_{i-2,j}) > \tfrac{Q}{2} \\ &= g(u_{i-2,j}) + (q+1), && \text{if } g(u_{i-2,j}) < \tfrac{Q}{2}, \\ & && \forall\, j = 1, 2, \ldots, mn, \forall\, i = 3, 4, \ldots, \tfrac{t}{2}; \\ g(u_{\frac{t}{2}+1,j}) &= g(u_{\frac{t}{2}-1,j}) + (q+1) + \tfrac{1}{2} + \tfrac{1}{2}(-1)^{\frac{t}{2}}, && \text{if } g(u_{\frac{t}{2}-1,j}) < \tfrac{Q}{2} \\ &= g(u_{\frac{t}{2}-1,j}) - (q+1) - \tfrac{1}{2} + \tfrac{1}{2}(-1)^{\frac{t}{2}}, && \text{if } g(u_{\frac{t}{2}-1,j}) > \tfrac{Q}{2}, \\ & && \forall\, j = 1, 2, \ldots, mn; \\ g(u_{\frac{t}{2}+2,j}) &= g(u_{\frac{t}{2},j}) + (q+1) + \tfrac{1}{2} + \tfrac{1}{2}(-1)^{\frac{t}{2}}, && \text{if } g(u_{\frac{t}{2},j}) < \tfrac{Q}{2} \\ &= g(u_{\frac{t}{2},j}) - (q+1) - \tfrac{1}{2} + \tfrac{1}{2}(-1)^{\frac{t}{2}}, && \text{if } g(u_{\frac{t}{2},j}) > \tfrac{Q}{2}, \\ & && \forall\, j = 1, 2, \ldots, mn; \\ g(u_{i,j}) &= g(u_{i-2,j}) - (q+1), && \text{if } g(u_{i-2,j}) > \tfrac{Q}{2} \\ &= g(u_{i-2,j}) + (q+1), && \text{if } g(u_{i-2,j}) < \tfrac{Q}{2}, \\ & && \forall\, j = 1, 2, \ldots, mn, \forall\, i = \tfrac{t}{2} + 3, \tfrac{t}{2} + 4, \ldots, t. \end{aligned}$$

We join the vertices $u_{i,mn}$ and $u_{i+1,1}$ $\forall\, i = 1, 2, \ldots, t-1$. Also join $u_{t,mn}$ and $u_{1,1}$ if either ($t \equiv 2 \pmod 4$ and q is odd or ($t \equiv 0 \pmod 4$)), otherwise join $u_{t,(m-1)n}$ and $u_{1,n+1}$.

The above labeling pattern gives rise a graceful labeling to the graph G. ■

Illustration 2.2. $C(6 \cdot P_3 \times P_3)$ and its graceful labeling shown in Figure 2.

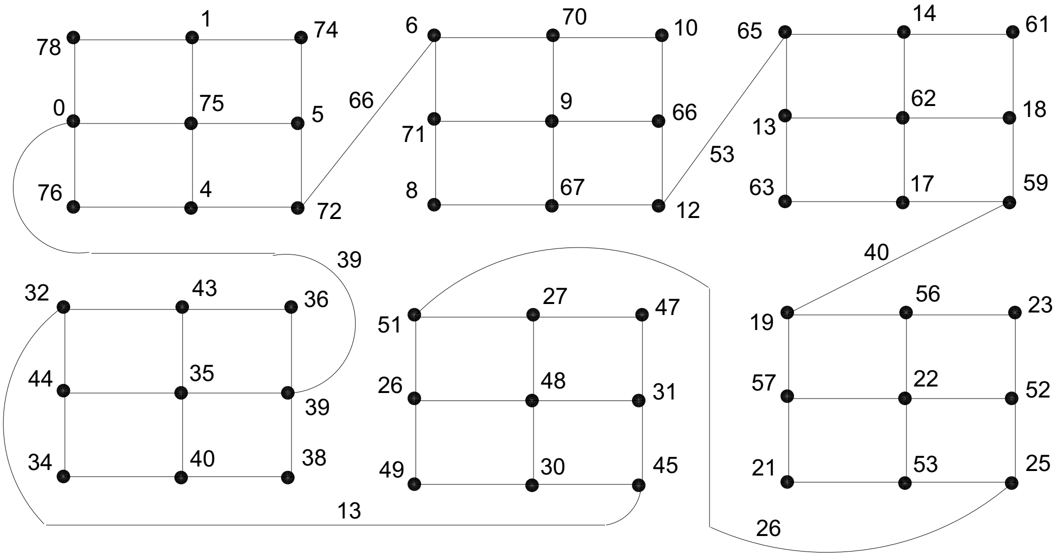

Figure 2: A cycle of the grid graph $P_3 \times P_3$ and its graceful labeling.

Theorem 2.3. Star of the grid graph $P_n \times P_m$ is graceful, $\forall\, m, n \geq 2$.

Proof: Let $G = (P_n \times P_m)^\star$ be the star of the grid graph $P_n \times P_m$. We know that the grid graph $P_n \times P_m$ is a graceful graph on $p = mn$ vertices and $q = 2mn - (m + n)$ edges. Let $f : V(P_n \times P_m) \longrightarrow \{0, 1, \ldots, q\}$ be the graceful labeling with two sequences of labels, one is increasing and another is decreasing which starts with $f(v_{1,1}) = q$ and ends with $f(v_{n,m}) = \lfloor \frac{q}{2} \rfloor$.

Let $V(P_n \times P_m) = \{u_1 = v_{1,1}, u_2, \ldots, u_n, u_{n+1}, \ldots, u_{mn} = v_{m,n}\}$. That is, we have $f(u_1 = q$ and $f(u_{mn}) = \lfloor \frac{q}{2} \rfloor$, as shown in Figure 1.

We have $G = (P_n \times P_m)^\star$, with $|V((P_n \times P_m)^\star)| = p(p + 1)$ vertices and the number of edges $Q = |E((P_n \times P_m)^\star)| = (p + 1)(q + 1) - 1$, where $p = mn$ and $q = 2mn - (m + n)$. Let $u_{0,j}$ $(1 \leq j \leq mn)$ be vertices of the central copy $P_n \times P_m$ in G and $u_{i,j}$ $(1 \leq j \leq mn)$ be vertices of i^{th} copy of $P_n \times P_m$ in G, $\forall\, i = 1, 2, \ldots, mn$. We define the labeling function $g : V((P_n \times P_m)^\star) \longrightarrow \{0, 1, \ldots, Q\}$ as follows:

$$
\begin{aligned}
g(u_{0,j}) &= f(u_j), && \text{if } f(u_j) < \tfrac{q}{2} \\
&= f(u_j) + Q - q, && \text{if } f(u_j) \geq \tfrac{q}{2}, \quad \forall\, j = 1, 2, \ldots, mn; \\
g(u_{1,j}) &= g(u_{0,j}) + p(q + 1), && \text{if } g(u_{0,j}) < \tfrac{Q}{2} \\
&= g(u_{0,j}) - p(q + 1), && \text{if } g(u_{0,j}) > \tfrac{Q}{2}, \\
& \forall\, j = 1, 2, \ldots, mn; \\
g(u_{l,j}) &= g(u_{l-2,j}) + (q + 1), && \text{if } g(u_{l-2,j}) < \tfrac{Q}{2} \\
&= g(u_{l-2,j}) - (q + 1), && \text{if } g(u_{l-2,j}) > \tfrac{Q}{2}, \\
& \forall\, j = 1, 2, \ldots, mn, \forall\, l = 2, 3, \ldots, mn.
\end{aligned}
$$

The above labeling function g gives rise to the edge labels $1, 2, \ldots, q, q + 2, q + 3, \ldots, 2q + 1, 2q +$

$3, \ldots, p(q+1)-1, p(q+1)+1, p(q+1)+2, \ldots, Q = (p+1)(q+1)-1$. In order to make $(P_n \times P_m)^{\star}$ as a graceful graph, we require edge labels $q+1, 2(q+1), \ldots, p(q+1)$.

We see that the difference of vertex labels for the central copy $(P_n \times P_m)^{(0)}$ of G and the other copies $(P_n \times P_m)^{(i)}$ $(1 \leq i \leq mn)$ are precisely $p(q+1), (q+1), (p-1)(q+1), 2(q+1), \ldots, \lfloor \frac{p}{2} \rfloor (q+1)$.

Using this sequence, we produce the required edge labels by joining the corresponding vertices of $(P_n \times P_m)^{(0)}$ with the other copies $(P_n \times P_m)^{(i)}$ $(1 \leq i \leq mn)$ in G. Thus, G admits graceful labeling. ∎

Illustration 2.4. The star of $P_3 \times P_3$ and its graceful labeling shown in Figure 3.

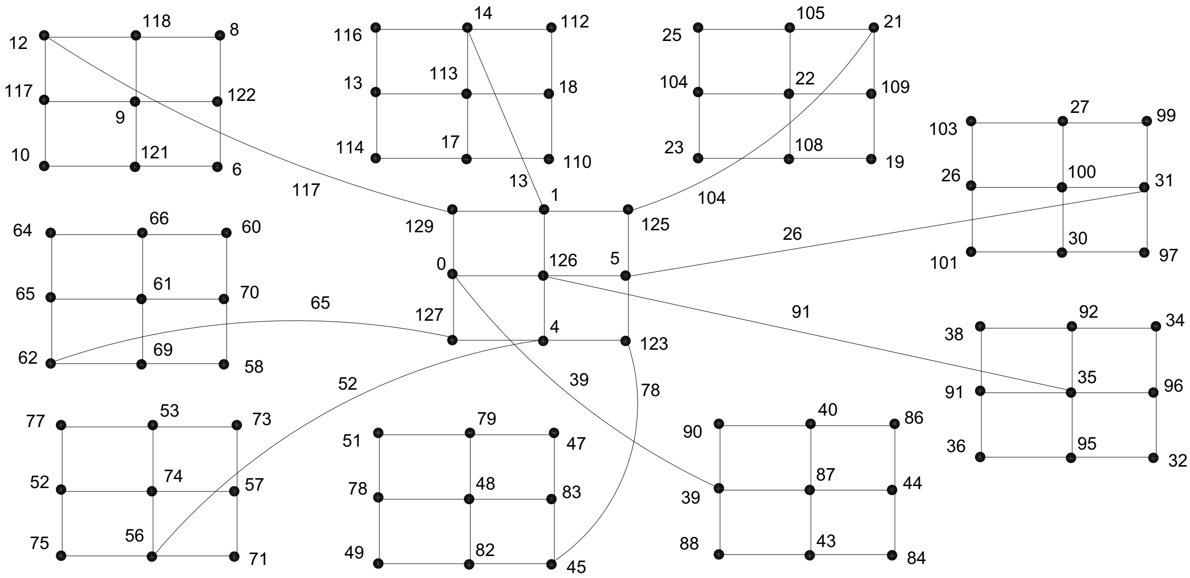

Figure 3: A star of the grid $P_3 \times P_3$ and its graceful labeling.

Theorem 2.5. The path union of finite copies of the grid graph $P_n \times P_m, \forall\, m, n \geq 2$ is graceful.

Proof: Let G be a path union of t copies of the grid graph $P_n \times P_m$, $\forall\, t, m, n \geq 2$. Let f be the graceful labeling of $P_n \times P_m$ as we mentioned in Theorem 2.1, where we have $V(P_n \times P_m) = \{u_1, u_2 \ldots, u_n, u_{n+1}, \ldots, u_{mn}\}$.

We see that $|V(G)| = tmn$ vertices and $Q = |E(G)| = t(2mn - (m+n) + 1) - 1$ edges in G. Let $u_{i,j}$ $(1 \leq j \leq mn)$ be the vertices of i^{th} copy of $P_n \times P_m, \forall\, i = 1, 2 \ldots, t$.

To define labeling $g : V(G) \longrightarrow \{0, 1, \ldots, Q\}$, we consider the following two cases.

Case 1: q is odd.

$$g(u_{1,j}) = f(u_j), \qquad \text{if } f(u_j) < \tfrac{q}{2}$$
$$= f(u_j) + Q - q, \qquad \text{if } f(u_j) > \tfrac{q}{2}$$
$$\forall\, j = 1, 2, \ldots, mn;$$

$$\begin{aligned} g(u_{i,j}) &= g(u_{i-1,j}) - (\tfrac{q+1}{2}), && \text{if } g(u_{i-1,j}) > \tfrac{Q}{2} \\ &= g(u_{i-1,j}) + (\tfrac{q+1}{2}), && \text{if } g(u_{i-1,j}) < \tfrac{Q}{2} \\ &\forall\, j = 1, 2, \ldots, mn, \forall\, i = 2, 3, \ldots, t. \end{aligned}$$

Case 2: q is even.

$$\begin{aligned} g(u_{1,j}) &= f(u_j), && \text{if } f(u_j) < \tfrac{q}{2} \\ &= f(u_j) + Q - q, && \text{if } f(u_j) \geq \tfrac{q}{2} \\ &\forall\, j = 1, 2, \ldots, mn; \\ g(u_{2,j}) &= g(u_{1,mn+1-j}) + Q - q, && \text{if } g(u_{1,mn+1-j}) < \tfrac{Q}{2} \\ &= g(u_{1,mn+1-j}) - Q + q, && \text{if } g(u_{1,mn+1-j}) > \tfrac{Q}{2} \\ &\forall\, j = 1, 2, \ldots, mn; \\ g(u_{i,j}) &= g(u_{i-2,j}) + (q+1), && \text{if } g(u_{i-2,j}) < \tfrac{Q}{2} \\ &= g(u_{i-1,j}) - (q+1), && \text{if } g(u_{i-2,j}) > \tfrac{Q}{2} \\ &\forall\, j = 1, 2, \ldots, mn, \forall\, i = 3, 4, \ldots, t. \end{aligned}$$

We join these consecutive copies of the grid graph $P_n \times P_m$ by an edge. Also join $u_{i,mn}$ with $u_{i+1,1}$, $\forall\, i = 1, 2, \ldots, t-1$ by an edge to form the path union of t copies of the grid graph $P_n \times P_m$. Above labeling pattern gives rise a graceful labeling to the given graph G. ■

Illustration 2.6. The path union of 3 copies of $P_3 \times P_5$ (it is related with case 2) and its graceful labeling is shown in Figure 4.

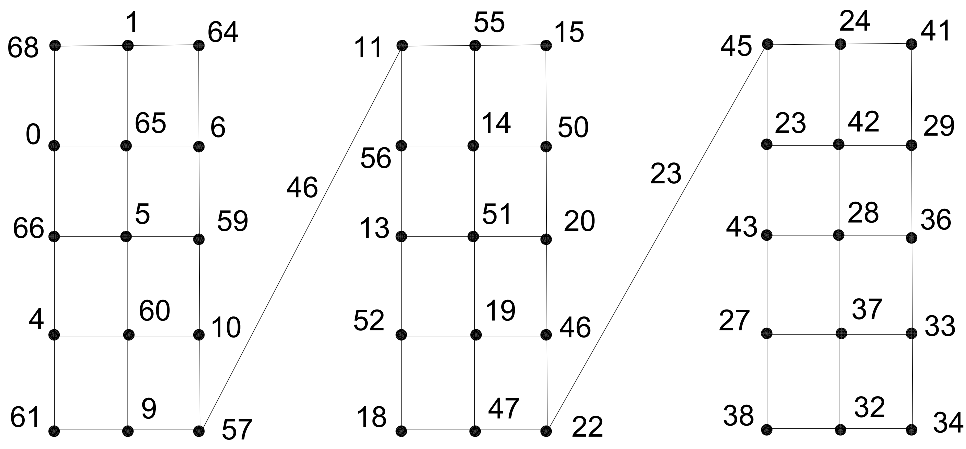

Figure 4: The path union of 3 copies of $P_3 \times P_5$ and its graceful labeling.

Acknowledgement: The authors of this paper would like to thank the reviewers for their valuable suggestions.

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