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Odd and even distance graphs

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Abstract

Let $G = (V, E)$ be a (p, q) - graph. A shortest path P is called an odd path if $l(P)$ is an odd positive integer. Odd distance graph $OD(G)$ of a graph G is defined as the graph with vertex set $V(G)$ and two vertices are adjacent if the distance between them is odd. In this paper, we deal with the odd distance graph of standard graphs such as path, cycle, star, wheel, complete graph, complete bipartite graph and corona of a complete graph etc. Also we give some properties and characterizations for odd distance graphs.

Keywords: Odd path, odd distance graph , even path, even distance graph, bi-regular.
AMS Subject Classification(2010): 05C12, 05C76.

1 Introduction

By a graph, we mean a finite undirected graph without loops and multiple edges. For terms not defined here, we refer to Harary[7].

Distance between the two vertices in a graph is the length of the shortest path between them. In this paper, we find the odd and even distance graphs of some standard graphs such as path, cycle, wheel, star, complete graph, complete bipartite graph, corona of a complete graph etc. It is also found that there are some properties and characterizations for odd and even distance graphs.

Definition 1.1. Let a, b be positive integers, $1 \leq a < b$. A graph G is said to be (a, b)- bi- regular if its vertices have degree either a or b.

2 Main Results

Definition 2.1. Let G= (V, E) be a (p, q)-graph. A shortest path P is called odd path if $l(P)$ is an odd positive integer.

Definition 2.2. Let G= (V, E) be a (p, q)-graph. Odd distance graph $OD(G)$ of a graph G is defined as the graph with vertex set $V(G)$ and two vertices are adjacent if the distance between them is odd.

Example 2.3. A graph G and its odd distance graph $OD(G)$are shown in Figure 1.

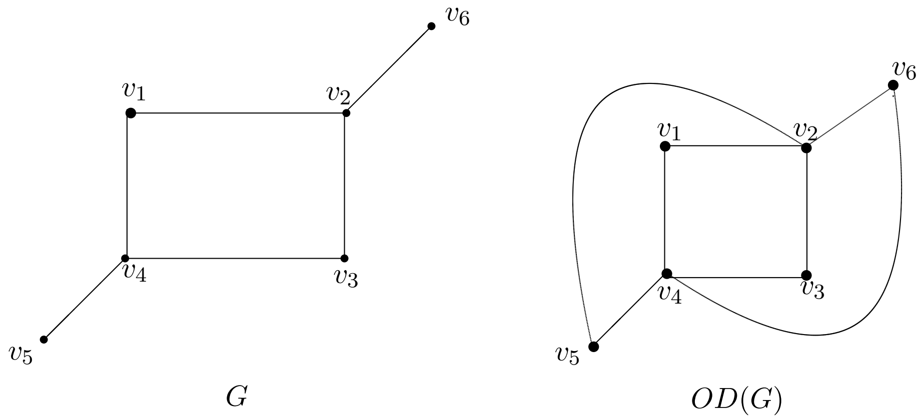
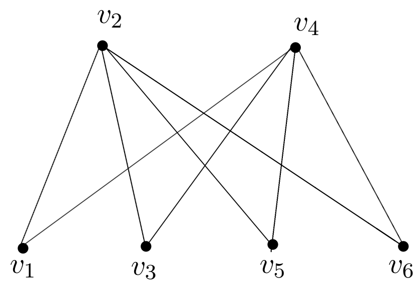


Figure 1: A graph G and its odd distance graph $OD(G)$.

Now, the graph $OD(G)$ is redrawn below. Note that $OD(G) \cong K_{2,4}$.



$OD(G)$

Figure 2

Observation 2.4. Since every edge is an odd path, G is isomorphic to a spanning subgraph of $OD(G)$ for any graph G.

Result 2.5. If G is complete, then $OD(G) \cong G$.

Proof: Since G is complete, $d(v_i, v_j) = 1$, for all $i \neq j$. We observe that the only odd paths in G are edges of G and hence $OD(G) \cong G$. ■

First, we see that the odd distance graph of a path is either regular or bi-regular graph.

Theorem 2.6. For a path P_p, $OD(P_p)$ is either $\frac{p}{2}$- regular or $(\frac{p-1}{2}, \frac{p+1}{2})$- regular when p is even or odd respectively.

Proof: Suppose p is even.

Then in P_p, it is clear that the number of vertices having odd distance from each v_i is $\frac{p}{2}$. So, in $OD(P_p)$, degree of each vertex is $\frac{p}{2}$. Hence, $OD(P_p)$ is $\frac{p}{2}$- regular.

Suppose p is odd.

Let $v_1v_2v_3 \dots v_{p-1}v_p$ be a path. We note that in P_p, the number of vertices having odd suffix is $\frac{p+1}{2}$ and that of even suffix is $\frac{p-1}{2}$. Since the distance between the odd suffix vertices and even suffix vertices is odd, in $OD(P_p)$ we join each odd suffix vertex to every even suffix vertices. So, in $OD(P_p)$, degree of each odd suffix vertex is $\frac{p-1}{2}$ and that of even suffix vertex is $\frac{p+1}{2}$. Thus, $OD(P_p)$ is $(\frac{p-1}{2}, \frac{p+1}{2})$-regular. ∎

Example 2.7. The paths P_8, P_9 and their odd distance graphs $OD(P_8)$ and $OD(P_9)$are given in Figure 3.

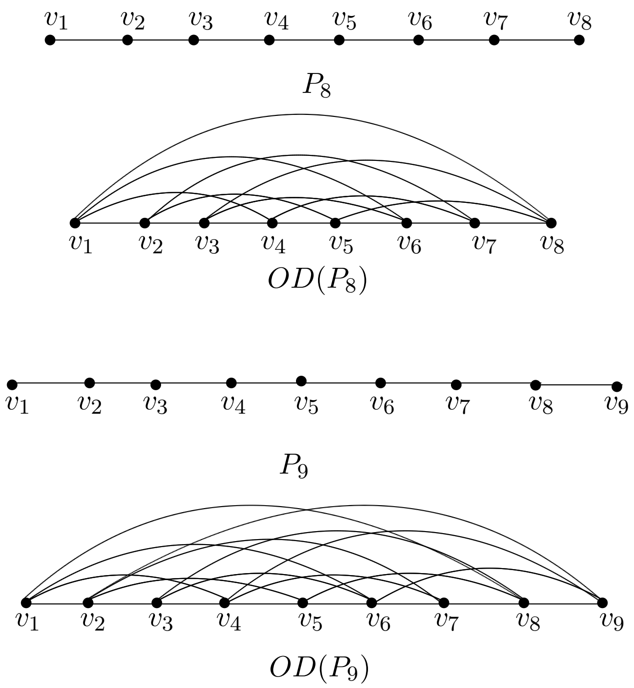


Figure 3: The paths P_8, P_9 and their odd distance graphs $OD(P_9)$ and $OD(P_9)$.

Now the graphs $OD(P_8)$ and $OD(P_8)$ are redrawn as below.

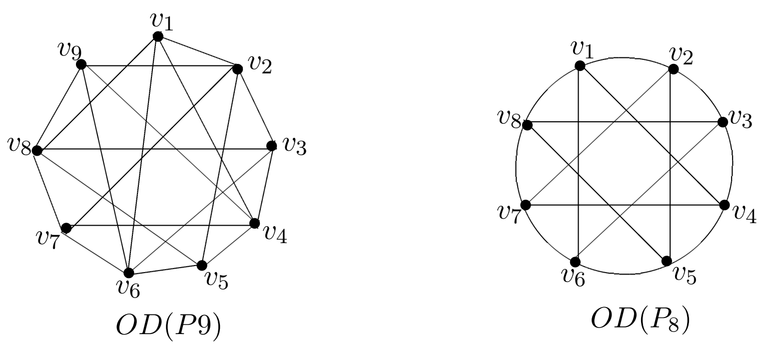


Figure 4

Note that $OD(P_8)$ is 4- regular and $OD(P_9)$ is $(4,5)$- regular.

Next, we shall find the formula for the number of edges in odd distance graph of a path.

Corollary 2.8. For a path P_p,
$$q(OD(P_p)) = \begin{cases} \frac{p^2}{4} & \text{if } p \text{ is even.} \\ \frac{p^2-1}{4} & \text{if } p \text{ is odd.} \end{cases}$$

Proof: Suppose p is even. Since $OD(P_p)$ is $\frac{p}{2}$- regular, we see that $2q(OD(P_p)) = \sum d_i = p(\frac{p}{2})$ and so, $q(OD(P_p)) = \frac{p^2}{4}$.

Suppose p is odd. From Theorem 2.6, $OD(P_p)$ is $(\frac{p-1}{2}, \frac{p+1}{2})$- regular and $\frac{p+1}{2}$ vertices of degree $\frac{p-1}{2}$ and $\frac{p-1}{2}$ vertices of degree $\frac{p+1}{2}$. Then, we see that $2q(OD(P_p)) = \sum d_i = (\frac{p+1}{2})(\frac{p-1}{2})+(\frac{p-1}{2})(\frac{p+1}{2}) = 2(\frac{p^2-1}{4})$. Hence, $q(OD(P_p)) = \frac{p^2-1}{4}$. ■

Remark 2.9. From Example 2.7, it can be noted that $q(OD(P_8)) = \frac{8^2}{4} = 16$ and $q(OD(P_9)) = \frac{9^2-1}{4} = 20$.

Next, we prove some results regarding the regularity of the odd distance graph of a cycle.

Theorem 2.10. For a cycle C_p, $p > 3$, $OD(C_p)$ is $\frac{p}{2}$- regular, if p is even and $\frac{p-1}{2}$ or $\frac{p+1}{2}$- regular, if p is odd.

Proof: Let $v_1, v_2, \cdots, v_p$ be the vertices of C_p.

Case (i): Suppose p is even.

Then, $diamC_p$ is $\frac{p}{2}$.

Since p is even, there are two paths between v_i and $v_{\frac{p}{2}+i}$. Consider the vertex v_i in C_p and the two paths of length $\frac{p}{2}$ from v_i namely $v_i, v_{i+1} \dots v_{\frac{p}{2}+i}$ and $v_{\frac{p}{2}+i}, v_{\frac{p}{2}+i+1} \dots v_i$ for all i= 1, 2, ..., p.

Note that the end vertices of these paths are the same and internal vertices of these paths are distinct.

Subcase (i): Suppose $\frac{p}{2}$ is odd.

Then, the above paths have $\frac{p}{2}+1$ vertices and that is even. By Theorem 2.6, the degree of v_i in $OD(P_{\frac{p}{2}+1})$ is $\frac{\frac{p}{2}+1}{2} = \frac{p+2}{4}$.

Since $\frac{p}{2}$ is odd, v_i is adjacent to $v_{\frac{p}{2}+i}$ in $OD(P_{\frac{p}{2}+1})$. But we have two paths between v_i and $v_{\frac{p}{2}+i}$ and in $OD(P_{\frac{p}{2}+1})$, these paths form multiple edges from v_i to $v_{\frac{p}{2}+i}$. Since $OD(P_{\frac{p}{2}+1})$ is a simple graph, the degree of v_i is $2(\frac{p+2}{4}) - 1 = \frac{p}{2}$. Thus, the degree of each v_i in $OD(C_p)$ is $\frac{p}{2}$.

Subcase (ii): Suppose $\frac{p}{2}$ is even.

Then, $\frac{p}{2}+1$ is odd. Hence by Theorem 2.6, the degree of v_i in $OD(P_{\frac{p}{2}+1})$ is $\frac{\frac{p}{2}+1-1}{2}$= $\frac{p}{4}$. Since $\frac{p}{2}$ is even, v_i is not adjacent to $v_{\frac{p}{2}+i}$ in $OD(P_{\frac{p}{2}+1})$. Hence, degree of v_i is $2(\frac{p}{4})$= $\frac{p}{2}$. Thus, the degree of each v_i in $OD(C_p)$, is $\frac{p}{2}$ and so, $OD(C_p)$ is $\frac{p}{2}$- regular.

Case (ii): Suppose p is odd.

Then, $diamC_p$ is $\frac{p-1}{2}$. In C_p, we observe that for each v_i there exist exactly two vertices v_j and v_k, for all $i \neq j$ and $i \neq k$, such that $d(v_i, v_j) = \frac{p-1}{2}$ and $d(v_i, v_k) = \frac{p-1}{2}$. Also the paths $v_i - v_j$ and $v_i - v_k$ are distinct and each path have $\frac{p+1}{2}$ vertices.

Subcase (i): Suppose $\frac{p-1}{2}$ is odd.

Then by Theorem 2.6, the degree of each vertex in $OD(P_{\frac{p+1}{2}})$ is $\frac{p+1}{4}$. Here we have two paths, so the degree of each vertex in $OD(C_p)$ is $\frac{p+1}{2}$. Thus, in this case $OD(C_p)$ is $\frac{p+1}{2}$- regular.

Subcase (ii): Suppose $\frac{p-1}{2}$ is even.

Then by Theorem 2.6, the degree of each vertex in $OD(P_{\frac{p+1}{2}})$ is $\frac{\frac{p+1}{2}-1}{2}= \frac{p-1}{4}$. Here we have two paths, so the degree of each vertex in $OD(C_p)$ is $\frac{p-1}{2}$.

Thus, in this case $OD(C_p)$ is $\frac{p-1}{2}$- regular.

Hence, $OD(C_p)$ is either $\frac{p-1}{2}$ or $\frac{p+1}{2}$- regular if p is odd. ∎

Example 2.11. The cycle C_{10} and its odd distance graph are given in Figure 5.

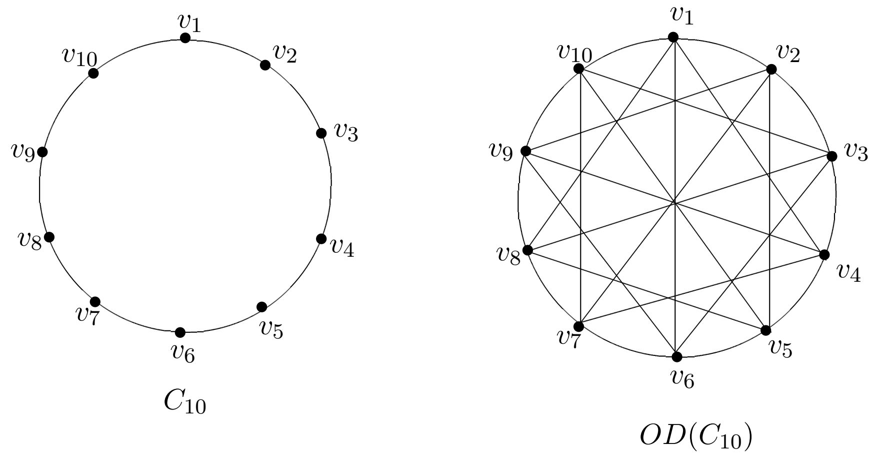


Figure 5: The cycle C_{10} and its odd distance graph $OD(C_{10})$.

Note that $OD(C_{10})$ is 5- regular.

Example 2.12. The cycle C_8 and its odd distance graph are given in Figure 6.

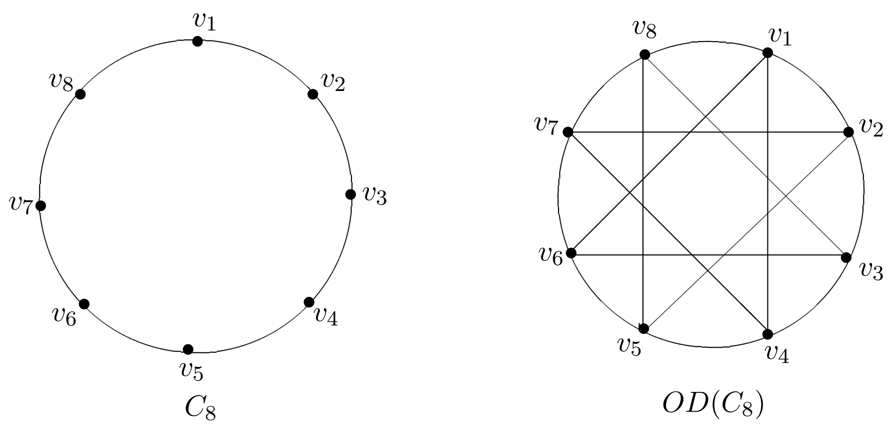


Figure 6: The cycle C_8 and its odd distance graph $OD(C_8)$.

Example 2.13. The cycle C_{11} and its odd distance graph are given in Figure 7.

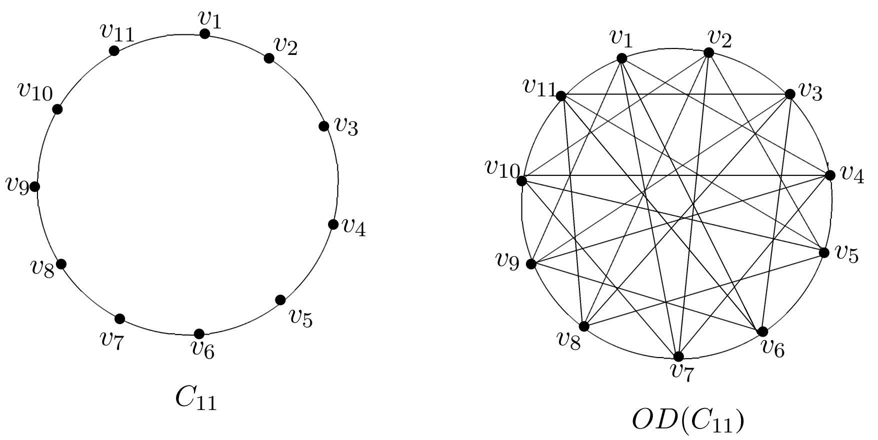


Figure 7: The cycle C_{11} and its odd distance graph $OD(C_{11})$.

Note that $OD(C_{11}$ is 6- regular.

Example 2.14. The cycle C_9 and its odd distance graph are given in Figure 8.

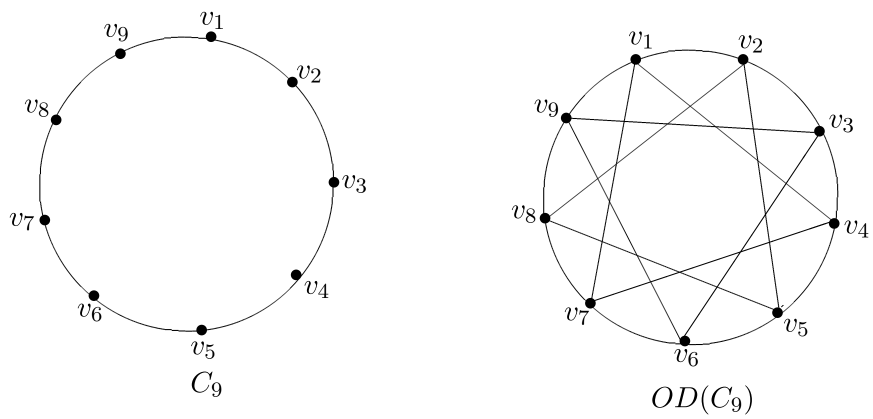


Figure 8: The cycle C_9 and its odd distance graph $OD(C_9)$.

Note that $OD(C_9)$ is 4- regular. Next, we find the formula for the number of edges of the odd distance graph of cycle.

Corollary 2.15. For a cycle C_p, $p > 3$, $q(OD(C_p)) = \begin{cases} \frac{p^2}{4} \text{ if } p \text{ is even} \\ \frac{p(p-1)}{4} or \frac{p(p+1)}{4} \text{ if } p \text{ is odd.} \end{cases}$

Proof: Suppose p is even. From Theorem 2.6, $OD(C_p)$ is $\frac{p}{2}$- regular, we see that $2q(OD(C_p)) = \sum d_i = p(\frac{p}{2})$ and so, $q(OD(C_p)) = \frac{p^2}{4}$.

Suppose p is odd. From Theorem 2.6, $OD(C_p)$ is either $\frac{p-1}{2}$ or $\frac{p+1}{2}$- regular .

Suppose $OD(C_p)$ $\frac{p-1}{2}$- regular, we see that $2q(OD(C_p)) = \sum d_i = \frac{p(p-1)}{2}$ and so $q(OD(C_p)) = \frac{p(p-1)}{4}$.

Suppose $OD(C_p)$ $\frac{p+1}{2}$- regular, we see that $2q(OD(C_p)) = \sum d_i = \frac{p(p+1)}{2}$ and so $q(OD(C_p)) = \frac{p(p+1)}{4}$. ∎

Remark 2.16. From Examples 2.12, 2.14 and 2.13, it can be noted that, $q(OD(C_8)) = \frac{8^2}{4} = 16$, $q(OD(C_9)) = \frac{9(9-1)}{4} = 18$ and $q(OD(C_{11})) = \frac{11(11+1)}{4} = 33$.

Next, we shall find the necessary and sufficient condition for a graph is isomorphic to its odd distance graph.

Theorem 2.17. For any graph G, $OD(G) \cong G$ if and only if diam G is at most 2.

Proof: Suppose $OD(G) \cong G$.

Then, we prove that $diamG$ is atmost 2.

Suppose $diamG$ is greater than 2. Then in G, we have paths of length greater than or equal to 3. Take one path of length 3. In $OD(G)$, the end vertices of this path must be adjacent and so, $OD(G) \ncong G$.

This is a contradiction. Thus, $diamG$ is at most 2.

Conversely, Suppose $diamG$ is atmost 2.

Then the maximum length of any path in G is 2 and the end vertices of this path are not adjacent in $OD(G)$. Thus, edges are only odd paths in G.

Hence, $OD(G) \cong G$. ∎

Remark 2.18. Since the graphs $K_{m,n}$, W_p and $K_{1,p}$ have diameter 2, they are isomorphic to their respective odd distance graphs.

Next, we shall find the necessary and sufficient condition for odd distance graph of a graph is isomorphic to a cycle.

Theorem 2.19. Let G be a (p, q)- graph. Then $OD(G) \cong Cp$ if and only if G is a cycle of diameter atmost 2 or $G \cong P_4$.

Proof: Suppose G is a cycle of diameter atmost 2.
Then, by Theorem 2.17, $OD(G) \cong C_p$. Suppose $G \cong P_4$.
Then, $OD(G) \cong C_4$.
Thus, in both the cases, $OD(G) \cong C_p$.
Conversely, suppose $OD(G) \cong C_p$.

Then, from the Observation 2.4 , G is isomorphic to a connected spanning subgraph of C_p. But, P_p and C_p are only connected spanning subgraphs of C_p. Hence G is isomorphic to either P_p or C_p

Case (i): Suppose G is isomorphic to P_p. Suppose $p \neq 4$, then $OD(G)$ is a path if $p \leq 3$ and $OD(G)$ has atleast one vertex of degree atleast 3 which is a contradiction. Hence, G isomorphic to P_4.

Case (ii): Suppose G is isomorphic to C_p.

Suppose diam(C_p) ≥ 3, then $OD(G)$ has atleast one vertex of degree 3 and hence $OD(G)$ is not isomorphic to C_p . ∎

Next, we shall find the necessary and sufficient condition for odd distance graph of a graph is isomorphic to a path.

Theorem 2.20. Let G be a (p, q)- graph. Then $OD(G) \cong p$ if and only if G is a path of length atmost two.

Proof: Suppose G is a path of length atmost 2.

Then by Theorem 2.17, $OD(G) \cong P_p$.

Conversely, assume that $OD(G) \cong P_p$.

From the Observation 2.4, G is a spanning subgraph of P_p. Also we observe that, the spanning subgraph of P_p is P_p only.

We claim that G is a path of length atmost 2.

Suppose G is a path of length greater than 2. Then, $OD(G)$ is either a cycle or has atleast one vertex of degree atleast 3. Thus, $OD(G) \ncong P_p$. Hence, G must be a path of length atmost two. ■

Next, we shall find the necessary and sufficient condition for odd distance graph of a graph is isomorphic to a complete graph.

Theorem 2.21. Let G be a (p, q)- graph. Then $OD(G) \cong K_p$ if and only if $G \cong K_p$.

Proof: Suppose $G \cong K_p$. Then $diam K_p = 1$.

Then by Theorem 2.17, $OD(G) \cong K_p$.

Conversely, Suppose $OD(G) \cong K_p$.

Suppose $G \ncong K_p$. Then, G contains a path of length atleast two. Thus, in $OD(G)$, we have atleast two vertices which are non adjacent. Thus, $OD(G) \ncong K_p$.

Hence, $G \ncong K_p$. ■

Theorem 2.22. For a tree T , then $OD(T)$ is either isomorphic to T or contains a cycle.

Proof: Suppose edges of G are the only odd paths of G, then it is clear that $OD(G) \cong G$. Suppose G has odd paths other than edges, then in $OD(G)$, we join the end vertices of that odd path which produces a cycle in $OD(G)$. ■

Next, we shall find the odd distance graph of corona of a complete graph.

Theorem 2.23. For a graph $K_p^+, p \geq 3, OD(K_p^+) \cong K_p \square K_p$

Proof: Let $u_1, u_2, \ldots, u_p$ be the vertices of K_p and $v_1, v_2, \ldots, v_p$ be the pendent vertices attached to $u_1, u_2, \ldots, u_p$ respectively. We note that K_p^+ has $2p$ vertices and $\frac{p(p+1)}{2}$ edges.
In K_p^+, we observe that
(i) $d(u_i, u_j) = 1$, for all $i \neq j$.

(ii) $d(v_i, v_j) = 3$, for all $i \neq j$.

(iii) $d(u_i, v_j)$= either 1 or 2, for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, p$.

The following table gives the number of vertices having the distance 1 or 2 or 3 from the vertices of K_p^+.

Vertex v	Distance D	Number of vertices having distance d from v
u_i	1	p
	2	$p-1$
	3	0
v_i	1	1
	2	$p-1$
	3	$p-1$

From the table, it is found that in K_p^+, the number of vertices having distance 1 from u_i is p and that of distance 2 from u_i is $p-1$ and that of distance 3 from u_i is 0 .

The number of vertices having distance 1 from v_i is 1 and that of distance 2 from v_i is $p-1$ and that of distance 3 from v_i is $p-1$.

From (i), (ii) and (iii) we see that in $OD(K_p^+)$ each u_i is adjacent to u_j for all $i \neq j$ and each v_i is adjacent to v_j for all $i \neq j$. Then, in $OD(K_p^+)$, the vertices $u_1, u_2, ...u_p$ induces a complete subgraph of order p and so the vertices $v_1, v_2, ...v_p$. By the construction of K_p^+, $d(u_i, v_i) = 1$ for $i = 1, 2, 3, ...p$. So, in $OD(K_p^+)$ each u_i is adjacent to v_i for $i = 1, 2, 3, ...p$. Then, in $OD(K_p^+)$ the corresponding vertices of these two complete subgraphs are adjacent.

Thus, $OD(K_p^+) \cong K_p \square K_p$. ∎

Example 2.24. The graph K_4^+ and its its odd distance graph are given in Figure 9.

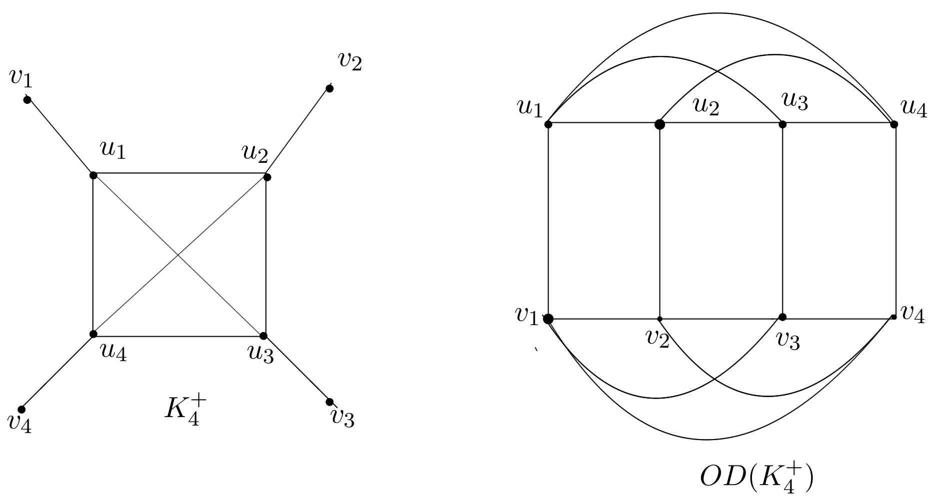


Figure 9: The graph K_4^+ and its odd distance graph $OD(K_4^+)$.

Definition 2.25. Let G= (V, E) be a (p, q)-graph. The even distance graph $ED(G)$ of a graph G has the vertex set $V = V(G)$ and two vertices in $ED(G)$ are adjacent if the distance between them is even in G.

Example 2.26. The graph G and its even distance graph $ED(G)$ are shown in Figure 10.

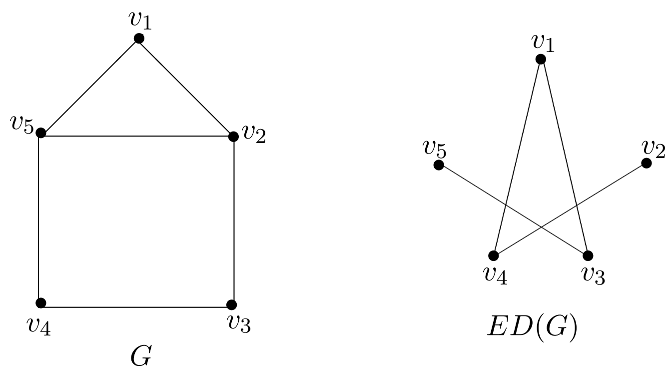


Figure 10: A graph G and its even distance graph $ED(G)$.

Now, we redraw the above $ED(G)$ as follows.

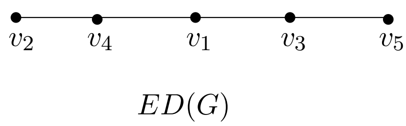


Figure 11

Observation 2.27. Since every edge is not a even distance path, for any graph G, $ED(G)$ is isomorphic to a subgraph of $\overline{G}$.

Theorem 2.28. For any graph G, $ED(G) \cong \overline{OD(G)}$.

Proof: Two vertices in $ED(G)$ are adjacent if they have an even distance path between them and two vertices in $\overline{OD(G)}$ are adjacent if they do not have an odd distance path between them. Thus, two vertices in $\overline{OD(G)}$ are adjacent if they have an even distance path between them.

Thus, $ED(G) \cong \overline{OD(G)}$. ∎

Note 2.29. From the above theorem, we observe that many of the results related to even distance graphs came from odd distance graphs. We state some results without proof.

Result 2.30. If G is complete then $ED(G) \cong \overline{G}$.

Result 2.31. For a path P_p,

$$ED(P_p) \cong \begin{cases} 2K_{\frac{p}{2}} & \text{if } p \text{ is even} \\ K_{\frac{p+1}{2}} \cup K_{\frac{p-1}{2}} & \text{if } p \text{ is odd.} \end{cases}$$

Result 2.32. For a cycle C_p, $p > 3$, $ED(C_p) \cong 2K_{\frac{p}{2}}$, if p is even and $\frac{p-1}{2}$ or $\frac{p-3}{2}$- regular, if p is odd.

Result 2.33. For a graph K_p^+, $p \geq 3$, $ED(K_p^+)$ is a bi-partite graph.

References

[1] T.M. Apostol, *Introduction to Analytic Number Theory*, Third edition, Narosa edition, 1991.

[2] R. Aravamudhan and B. Rajendran, *Graph Equations involving antipodal graphs,* Presented at the Seminar on Combinatorics and applications.ISI, Calcutta, 1984, 40-43.

[3] R. Aravamudhan and B. Rajendran. *On antipodal graphs*, Discrete Math.49, 1984, 193 - 195.

[4] J.A. Bondy and U.S.R. Murty, *Graph theory with Applications*, North Holland, 1983 Fifth Printing.

[5] F. Buckley and F. Harary, *Distance in graphs*, Addison-wesely, Reading, 1990.

[6] David M. Burton, *Elementary Number Theory,* Tata McGraw-Hill Publishing company Limited, 2006.

[7] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, Reading, Mass, 1972.

[8] K.M. Kathiresan and G. Marimuthu, *A Study on radial graphs*, Ars Combinatoria, 2010, 96, 353-360.

[9] S. Saravanakumar and K.Nagarajan, *Distant Divisor Graphs*, International Journal of Mathematical Sciences and Engineering Applications, 2013, Vol.7 No.IV, 83-97.

[10] S. Saravanakumar and K. Nagarajan, *Distant Divisor Graph of Subdivision of a Graph*, International Journal of Mathematics and Soft Computing , Vol.4, No.1(2014), 51-60.