International Journal of Mathematics and Soft Computing
Vol.5, No.1. (2015), 23 - 29.

ISSN Print : 2249 – 3328
ISSN Online: 2319 – 5215

Even-Odd Harmonious graphs

N. Adalin Beatress[1], P. B. Sarasija[2]
[1] Department of Mathematics,
All Saint College of Education
Kaliyakkavilai, Tamil Nadu.
adalinkamal@yahoo.in

[2] Department of Mathematics
Noorul Islam Centre for Higher Education
Kumaracoil,Tamil Nadu.
sijavk@gmail.com

Abstract

A graph $G(V, E)$ with n vertices and m edges is said to be even-odd harmonious if there exists an injection $f : V(G) \rightarrow \{1, 3, 5, \ldots, 2n\text{-}1\}$ such that the induced mapping $f^*:E(G) \rightarrow \{0,2,4,\ldots,2(m\text{-}1)\}$ defined by $f^*(uv) = [f(u) + f(v)] \pmod{2m}$ is a bijection. The function f is called even-odd harmonious labeling of G. In this paper, we prove that the bistar graph $B_{m,n}$, cycle with one pendent edge, crown graph, the graph $K_{1,m,n}$. the prism graph C_3Y_n and the graph nP_2 are even-odd harmonious graphs.

Keywords: Harmonious labeling, bistar, complete bipartite graph.

AMS Subject Classification (2010): 05C78.

1 Introduction

In this paper, we consider finite, undirected, simple connected graphs. For notations and terminology we follow Bondy and Murthy [1].

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Graph labeling was introduced in the late 1960s. For a detailed survey on graph labeling we refer to Gallian [2]. Harmonious graphs naturally arose in the study by Graham and Sloane [3] of modular versions of additive base problems. Odd and even harmonious graphs were introduced in [4,6].

The even - odd harmonious labeling of a graph G with n vertices and m edges is an injection $f:V(G) \rightarrow \{1,3,5,\ldots,(2n\text{-}1)\}$ such that the induced mapping $f^*:E(G) \rightarrow \{0,2,4,\ldots,2(m\text{-}1)\}$ defined by $f^*(uv) = [f(u) + f(v)] \pmod{2m}$ is a bijection. The function f is called even-odd harmonious labeling of G.A graph which admits an even-odd harmonious labeling is called an even-odd harmonious graph.

In this paper we prove that the path P_n, cycle C_n, complete bipartite graph $K_{m,n}$, star graph $K_{1,n}$, bistar graph $B_{m,n}$, cycle with one pendant edge, crown graph, comb, the graph $K_{1,m,n}$. the prism graph C_3Y_n and the graph nP_2 are even-odd harmonious graphs.

Definition 1.1. The graph obtained by joining a pendant edge at each vertex of a path P_n is called a comb and is denoted by $P_n \odot K_1$. The graph obtained by joining a pendant edge at each vertex of a cycle C_n is called crown and is denoted by $C_n \odot K_1$.

Definition 1.2. The graph obtained by joining a single pendant edge at one vertex of a cycle C_n is denoted by $C_n @ K_1$.

Definition 1.3. The graph obtained by joining n pendant edge at one vertex of the cycle C_3 is denoted by $C_3 \odot nK_1$.

Definition 1.4. The bistar graph $B_{m,n}$ is the graph obtained from K_2 by joining m pendant edges to one end of K_2 and n pendant edges to the other end of K_2.

Definition 1.5. P_n^k, the k^{th} power of P_n, is the graph obtained from the path P_n by adding the edges that join all the vertices u and v with distance $d(u,v) = k$.

2 Main Results

Theorem 2.1. Every path P_n is an even-odd harmonious graph.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n. We consider two cases.

Case (i): n is even.

Define $f : V(P_n) \to \{1,3,5,\ldots,(2n-1)\}$ by $f(v_i) = 2i - 1$, $1 \le i \le n$.

Case (ii): n is odd.

Define $f : V(P_n) \to \{1,3,5,\ldots,(2n-1)\}$ as follows:

$$f(v_{2i-1}) = 2i - 1,\ 1 \le i \le \lceil \tfrac{n}{2} \rceil \text{ and } f(v_{2i}) = n + 2i,\ 1 \le i \le \lfloor n/2 \rfloor.$$

In both the cases, the edge labels are distinct. Hence, P_n is an even –odd harmonious graph. ■

Theorem 2.2. Any cycle of odd length is an even-odd harmonious graph.

Proof: Let C_{2n-1} ($n \ge 2$) be the cycle of odd length with vertices $v_1, v_2, \ldots, v_{2n-1}$. Since the odd cycle has $(2n - 1)$ edges, the modulo taken is $2(2n-1)$.

Define $f : V(C_{2n-1}) \to \{1,3,5,\ldots,4n-3\}$ by $f(v_i) = 2i - 1$, $1 \le i \le 2n - 1$. The edge labels are distinct. Then f is an even –odd harmonious labeling of odd cycles. ■

Theorem 2.3. The comb $P_n \odot K_1$ is an even-odd harmonious graph.

Proof: Let $P_n \odot K_1$ be the comb graph with $2n$ vertices. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n and $v_1, v_2, \ldots, v_n$ be the vertices adjacent to each vertex of the path P_n. The edge set of the graph is given by $E(P_n \odot K_1) = \{u_i u_{i+1} : 1 \le i \le n-1;\ u_i v_i : 1 \le i \le n\}$. $|V| = 2n$ and $|E| = 2n - 1 = m$.

Define $f : V(P_n \odot K_1) \to \{1,3,5,\ldots,4n-1\}$ by

$f(u_{2i-1}) = 2n + 4i - 3$, $1 \le i \le \lceil \frac{n}{2} \rceil$ and $f(u_{2i}) = 4i - 1$, $1 \le i \le \lfloor n/2 \rfloor$.

$f(v_{2i-1}) = 4i - 3$, $1 \le i \le \lceil \frac{n}{2} \rceil$ and $f(v_{2i}) = 2n + 4i - 1$, $1 \le i \le \lfloor n/2 \rfloor$.

Then f induces a bijection $f^* : E(P_n \odot K_1) \to \{0,2,4,\ldots,4n-4\}$. Hence the comb $P_n \odot K_1$ is an even - odd harmonious graph. ■

The following example shows that the graph $P_5 \odot K_1$ is an even-odd harmonious graph.

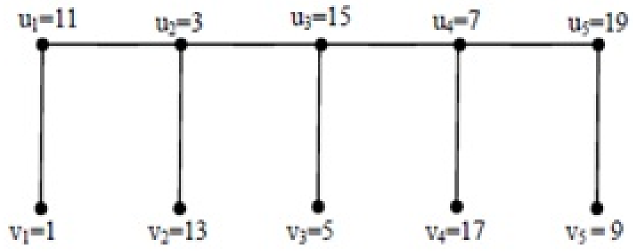

Figure 1: An even-odd harmonious labeling of $P_5 \odot K_1$.

Theorem 2.4. The complete bipartite graph $K_{m.n}$ is an even-odd harmonious graph for all m and n.
Proof: Let $u_1,u_2,\ldots,u_m$ and $v_1,v_2,\ldots,v_n$ be the vertices of the graph $K_{m,n}$.

Define $f : V(K_{m.n}) \to \{ 1,3,5,\ldots,2mn + 1\}$ by
$f(u_i) = 2i - 1,\ 1 \le i \le m$ and $f(v_j) = 2mj + 1,\ 1 \le j \le n$.

Then f induces a bijection $f^* : E(K_{m.n}) \to \{ 0, 2, 4,\ldots, 2mn - 2\}$
Then the complete bipartite graph is an even –odd harmonious graph. ■

Corollary 2.5. The star graph $K_{1,n}$ is an even –odd harmonious graph.
Proof: Replace m =1 in Theorem 2.5.
Let u be the vertex of degree n and $v_1, v_2, \ldots, v_n$ be the pendant vertices of $K_{1,n}$.Total number of vertices is (n+1). Define $f : V(K_{1,n}) \to \{ 1,3,5,\ldots,2n +1\}$ by $f(u) = 1$ and $f(v_i) = 2i + 1,\ 1 \le i \le n$. Hence, f is an even - odd harmonious labeling of the star graph. ■

Theorem 2.6. An odd cycle with a single pendant edge attached, C_{2n-1} @ K_1, ($n \ge 2$) is an even-odd harmonious graph.
Proof: Let $u_1,u_2,\ldots,u_{2n-1}, u_{2n}$ be the vertices of the graph C_{2n-1} @ K_1. We label the vertices of C_{2n-1} @ K_1 so that the pendant vertex is adjacent to the vertex u_{2n-1} as shown in Figure 2.

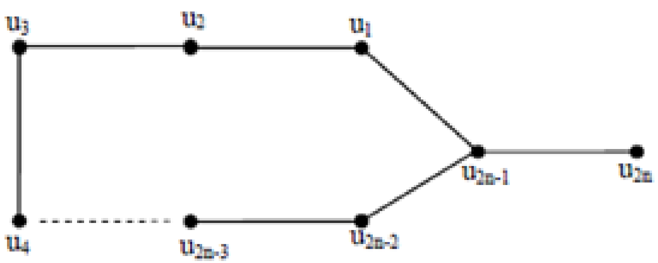

Figure 2: C_{2n-1} @ K_1.

Here, $| V | = | E | = 2n$.

Define $f : V(C_{2n-1}$ @ $K_1) \to \{ 1,3,5,\ldots,4n -1\}$ by $f(u_{2i-1}) = 2i - 1$ for $1 \le i \le n$ and $f(u_{2i}) = 2(n+i) - 1$ for $1 \le i \le n$.
Then f induces a bijection $f^* : E(C_{2n-1}$ @ $K_1) \to \{ 0, 2, 4,\ldots, 4n - 2\}$. Hence the graph is an even-odd harmonious graph. ■

Theorem 2.7. The crown graph $C_{2n-1} \odot K_1$, ($n \geq 2$) is an even-odd harmonious graph.

Proof : Let $u_1, u_2, \ldots, u_{2n-1}$ be the vertices of the cycle C_{2n-1} and $v_1, v_2, \ldots, v_{2n-1}$ be the vertices adjacent to each vertex of the cycle. We label the vertices of $C_{2n-1} \odot K_1$, ($n \geq 2$) as shown in Figure 3. Here, $|V| = |E| = 4n - 2$.

Define $f : V(C_{2n-1} \odot K_1) \rightarrow \{1,3,5,\ldots,8n-5\}$ by

$f(u_i) = 2i - 1$, $1 \leq i \leq 2n - 1$ and $f(v_i) = 2(2n+i) - 3$, $1 \leq i \leq 2n-1$. Then f is even-odd harmonious and hence, $C_{2n-1} \odot K_1$ is an even-odd harmonious graph. ∎

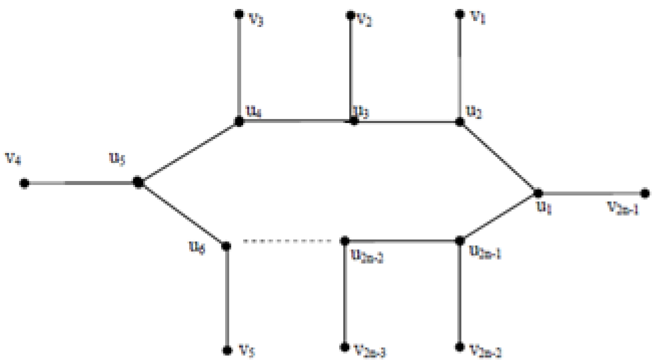

Figure 3: $C_{2n-1} \odot K_1$.

Theorem 2.8. The graph $C_3 \odot nK_1$ is an even-odd harmonious graph.

Proof: Let u, v, w be the vertices of the cycle C_3 and $w_1, w_2, \ldots, w_n$ be the vertices of the n pendant edges. We denote the vertices of $C_3 \odot nK_1$ as shown in Figure 4. $|V| = |E| = n+3$.

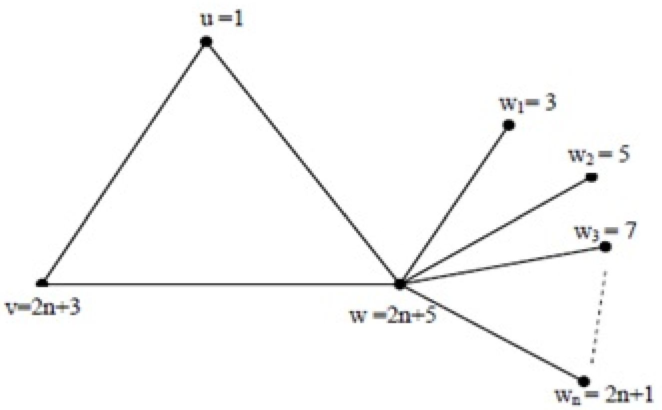

Figure 4: $C_3 \odot nK_1$.

Define $f : V(C_3 \odot nK_1) \rightarrow \{1,3,5,\ldots,2(n+3) - 1\}$ by

$f(u) = 1$, $f(v) = 2n+3$, $f(w) = 2n + 5$ and $f(w_i) = 2i + 1$ $1 \leq i \leq n$. Then f is even-odd harmonious and hence, $C_3 \odot nK_1$ is an even - odd harmonious graph. ∎

Theorem 2.9. The bistar graph $B_{m,n}$ is an even - odd harmonious.

Proof: Let u, v be the vertices of K_2 in $B_{m,n}$ and $U = \{u_j, 1 \leq j \leq m\}$, $V = \{v_i, 1 \leq i \leq n\}$ be the vertices adjacent to u and v respectively.

Define $f : V(B_{m,n}) \rightarrow \{ 1,3,5,\ldots,2(m+n)+3\}$ by

$f(u) = 1$, $f(v) = 2(m+n)+3$, $f(u_j) = 2n+2j+1$, $1 \leq j \leq m$ and $f(v_i) = 2i+1$, $1 \leq i \leq n$. Here we take the modulo $2(m+n+1)$. Then f induces a bijection $f^* : E(C_{2n-1} @ K_1) \rightarrow \{ 0, 2, 4,\ldots, 2(m+n)\}$. Hence, f is an even- odd harmonious labeling of the bistar $B_{m,n}$. ■

An even-odd harmonious labeling of the bistar $B_{5,6}$ is shown in Figure 5.

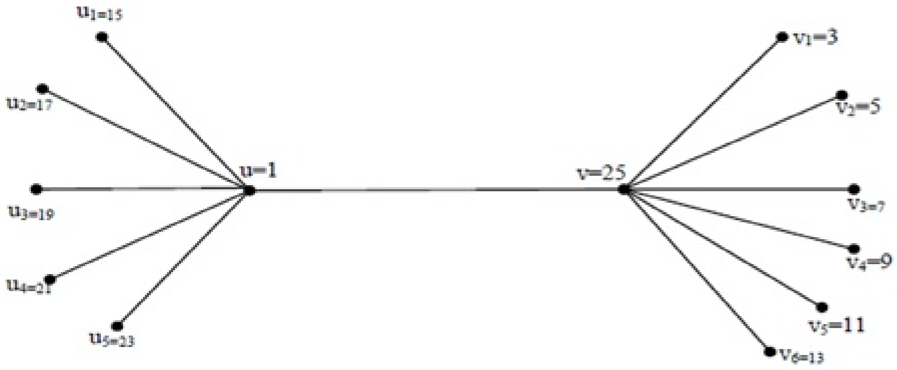

Figure 5: An even-odd harmonious labeling of the bistar $B_{5,6}$.

Theorem 2.10. The graph $K_{1,m,n}$ is an even -odd harmonious graph.

Proof: Let $V(c) = \{ u, u_j, v_i, j = 1,2,\ldots,m$ and $i = 1, 2,\ldots,n \}$

and $E(C_3Y_n) = \{ uv_i, uu_j, v_iu_j, j = 1, 2,\ldots,m$ and $i = 1,2,\ldots,n\}$

Also the number of vertices and edges of the graph is $(n+m+1)$ and $((m+n+mn)$ respectively.

Define $f : V(K_{1,m,n}) \rightarrow \{ 1,3,5,\ldots,2(n+mn)+1\}$ by

$f(u) = 1$, $f(u_j) = 2j+1$, $1 \leq j \leq m$ and $f(v_i) = 2(m+1)(n-i+1)+1$, $1 \leq i \leq n$.

Then f induces a bijection $f^* : E(K_{1,m,n}) \rightarrow \{ 0, 2, 4,\ldots, 2(m+n+mn)-2\}$.

Therefore, $K_{1,m\ n}$ admits an even -odd harmonious labeling. Hence this graph is an even - odd harmonious graph. ■

The following example shows that the graph $K_{1,3,4}$ is an even odd harmonious graph.

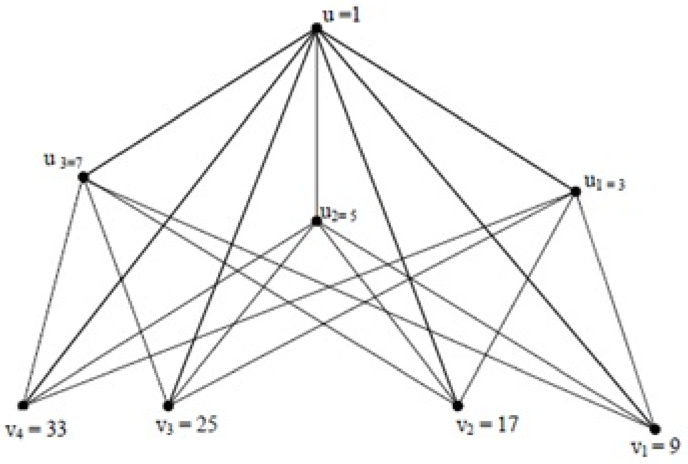

Figure 6: An even-odd harmonious labeling of $K_{1,3,4}$.

Theorem 2.11. The graph P_n^2 $(n \geq 4)$ is an even- odd harmonious graph.

Proof: Let $u_1, u_2, \ldots, u_n$ be the vertices of P_n^2. We arrange the vertices of the graph P_n^2 as shown in Figure 7.

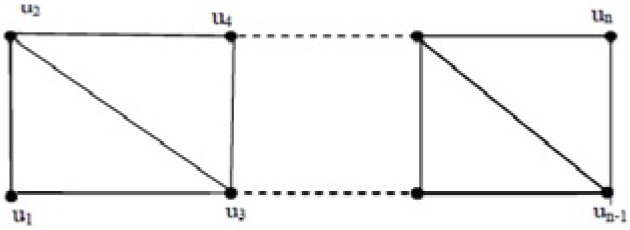

Figure 7: P_n^2.

Define $f(u_i) = 2i - 1$, $1 \leq i \leq n$. Since the graph P_n^2 has $2n - 3$ edges, the modulo is $2(2n-3)$. Then, f is an even-odd harmonious labeling. ■

Theorem 2.12. The prism graph C_3Y_n is an even - odd harmonious graph.

Proof: We denote the vertices of the graph C_3Y_n as shown in Figure 8.

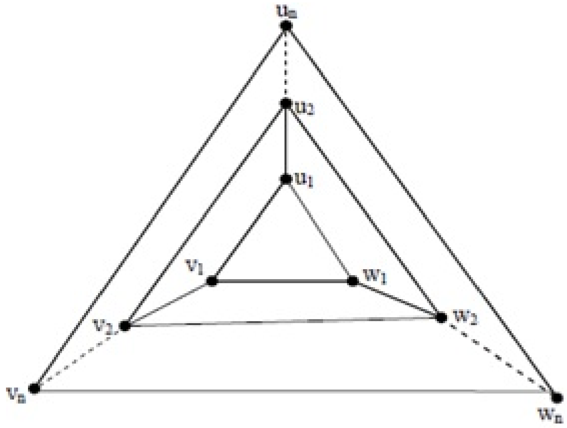

Figure 8: C_3Y_n.

Let the vertices of the prism graph C_3Y_n be $\{ u_i, v_i, w_i, 1 \leq i \leq n \}$. The number of vertices and edges of the graph are $3n$ and $6n - 3$ respectively.

Define $f : V(C_3Y_n) \rightarrow \{ 1,3,5,\ldots,6n - 1\}$ by

$f(u_{2i-1}) = 12i - 11, 1 \leq i \leq \lceil \frac{n}{2} \rceil$, $f(u_{2i}) = 12i - 1, 1 \leq i \leq \lfloor n/2 \rfloor$

$f(v_{2j-1}) = 12j - 9, 1 \leq j \leq \lceil \frac{n}{2} \rceil$, $f(v_{2j}) = 12j - 5, 1 \leq j \leq \lfloor n/2 \rfloor$

$f(w_{2k-1}) = 12k - 7, 1 \leq k \leq \lceil \frac{n}{2} \rceil$ and $f(w_{2k}) = 12k - 3, 1 \leq k \leq \lfloor n/2 \rfloor$.

It can be verified that f is an even - odd harmonious labeling.

Therefore, C_3Y_n admits an even - odd harmonious labeling. Hence, C_3Y_n is an even - odd harmonious graph. ■

Theorem 2.13. The graph nP_2 is an even odd harmonious graph.

Proof: Let u_kv_k be the k^{th} copy of the graph nP_2.

Define $f : V(nP_2) \rightarrow \{ 1,3,5,\ldots,4n - 1\}$ by $f(u_k) = 2k - 1, 1 \leq k \leq n$; $f(v_k) = 2k - 1, n+1 \leq k \leq 2n$.

f is an even - odd harmonious labeling. Hence, nP_2 is an even - odd harmonious graph. ∎

An even odd harmonious labeling of nP_2 is given in Figure 9.

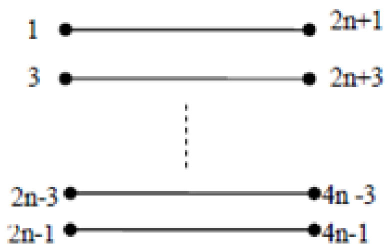

Figure 9: An even-odd harmonious labeling of nP_2.

Acknowledgement: The authors are thankful to the reviewers for their valuable comments and suggestions.

References

[1] J. A. Bondy and U.S.R. Murthy, *Graph Theory with Applications*, Macmillan, London, 1976.

[2] J.A. Gallian, *A dynamic survey of graph labeling*, The Electronics J. of Combinatorics, 16, 2012.

[3] R.L. Graham and N.J.A. Sloane, *On additive bases and harmonious graphs*, SIAM J. Alg. Discrete Meth., 1(1980), 382-404.

[4] Z.H. Liang and Z.L. Bai, On *the odd harmonious graphs with applications*, J. Appl. Math. Comput., 29(2009), 105-116.

[5] B. Liu and X. Zhang, *On harmonious labeling of graphs*, Ars. Combin., 36 (1993), 315 – 326.

[6] P. B. Sarasija and R. Binthiya, *Even harmonious graphs with applications*, International Journal of Computer Science and Information Security, Vol.9, No.7 (2011), 161-163.

[7] T. Tharmaraj and P.B. Sarasija, *Square graceful graphs*, International Journal of Mathematics and Soft Computing, Vol.4 No.1.(2014), 129-137.