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Z_{4p}- Magic labeling for some special graphs

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Abstract

For any non-trivial abelian group A under addition a graph G is said to be $A-$ magic if there exists a labeling f of the edges of G with non zero elements of A such that, the vertex labeling f^+ defined as $f^+(v) = \Sigma f(uv)$ taken over all edges uv incident at v is a constant [5]. A graph is said to be A-magic if it admits an A-magic labeling. In this paper we prove that splitting graph of a path, triangular snake and book graphs are Z_4-magic graphs. Also we generalize that they are all Z_{4p}-magic graphs for any positive integer p.

Keywords: A - magic labeling, Z_4 - magic labeling, Z_{4p} -magic labeling, Z_{4p}-magic graphs.
AMS Subject Classification(2010): 05C78.

1 Introduction

In this paper by a graph $G(V, E)$ we mean G is a finite, simple, undirected graph. The concept of magic labelings were introduced by Sedlacek in 1963. Kong, Lee and Sun [4] used the term magic labeling for the labeling of edges with non negative integers such that for each vertex v, the sum of the labels of all edges incident at v is same for all v.

For any non-trivial abelian group A under addition a graph G is said to be $A-$ magic if there exists a labeling f of the edges of G with non zero elements of A such that, the vertex labeling f^+ defined as $f^+(v) = \Sigma f(uv)$ taken over all edges uv incident at v is a constant. In this paper, we choose Z_4 which is additive modulo 4 as the abelian group and we prove the splitting graph of a path, triangular snake, book graph and $F_n^{(t)}$ are Z_4-magic graphs. We also prove that they are all Z_{4p}-magic graphs.

2 Definitions

Definition 2.1. [6] For each point v of a graph G take a new vertex v' and join v' to those points of G adjacent to v. The graph thus obtained is called the splitting graph of G and is denoted as $S'(G)$.

Definition 2.2. [2] The block - cutpoint graph of a graph G is a bipartite graph in which one partite set consists of the cut vertices of G and the other has a vertex b_i for each block B_i of G.

Definition 2.3. [2] A block of a graph is a maximal connected subgraph that has no cut-vertex.

Definition 2.4. [2] A triangular cactus is a connected graph all of whose blocks are triangles.

Definition 2.5. [2] A triangular snake is a triangular cactus whose block-cutpoint graph is a path.

Definition 2.6. [2] A book with n pages is defined as the Cartesian product of the complete bipartite graph $K_{1,n}$ and a path of length 1 and is denoted by B_n.

Definition 2.7. [2] The graph $P_n + K_1 \;\; n \geq 2$ is called a fan and it is denoted by F_n.

Definition 2.8. $F_n^{(t)}$ is the one-point union of t fans of length n.

3 Main Results

Theorem 3.1. $S'(P_n)$ is Z_4-magic for $n \geq 2$.

Proof: Let the vertex set $V(S'(P_n)) = \{v_i/1 \leq i \leq n\} \cup \{v_i'/1 \leq i \leq n\}$ and the edge set $E(S'(P_n)) = \{v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{v_iv_{i+1}'/1 \leq i \leq n-1\} \cup \{v_i'v_{i+1}/1 \leq i \leq n-1\}$, where $v_1', v_2', ...v_n'$ are the new vertices joined corresponding to $v_1, v_2, ...v_n$ of the path P_n.
Define $f : E(S'(P_n)) \to Z_4 - \{0\}$ as

$$\begin{aligned}
f(v_iv_{i+1}) &= \left\{ \begin{array}{ll} 1 & for\ i = 1, n-1 \\ 2 & 2 \leq i \leq n-2 \end{array} \right\} \\
f(v_2v_1') &= 2 = f(v_{n-1}v_n') \\
f(v_iv_{i+1}') &= 1,\ 1 \leq i \leq n-2 \\
\text{and} f(v_i'v_{i+1}) &= 1,\ 2 \leq i \leq n-1
\end{aligned}$$

Then the mapping $f^+ : V(S'(P_n)) \to Z_4$ is given by

$$\begin{aligned}
f^+(v_i) &= f(v_{i-1}'v_i) + f(v_iv_{i+1}') + f(v_iv_{i-1}) + f(v_iv_{i+1}),\ 2 \leq i \leq n-1 \\
f^+(v_1) &= f(v_2'v_1) + f(v_2v_1) \\
f^+(v_n) &= f(v_{n-1}v_n) + f(v_{n-1}'v_n) \\
f^+(v_i') &= f(v_i'v_{i-1}) + f(v_i'v_{i+1}),\ 2 \leq i \leq n-1 \\
f^+(v_1') &= f(v_2v_1') \\
f^+(v_n') &= f(v_{n-1}v_n') \\
\text{Clearly, } f^+(v_1) &= 2 \\
f^+(v_i) &= 2,\ 2 \leq i \leq n \\
f^+(v_i') &= 2,\ 1 \leq i \leq n
\end{aligned}$$

Thus $S'(P_n)$ admits Z_4 - magic labeling. Hence, $S'(P_n)$ is a Z_4-magic graph. ■

Example 3.2. Z_4-magic labelings of $S'(P_7)$ and $S'(P_6)$ are given below.

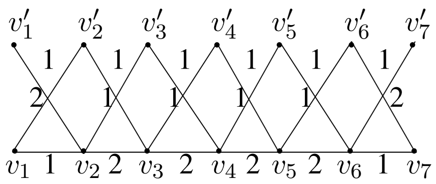

Figure 1: Z_4 - *magic labeling of* $S'(P_7)$.

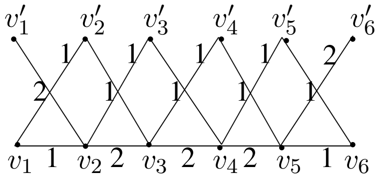

Figure 2: Z_4 - *magic labeling of* $S'(P_6)$.

Theorem 3.3. Triangular snake T_n is Z_4-magic, for $n \geq 2$.

Proof: Let $V(T_n) = \{v_i \ / \ 1 \leq i \leq n+1\} \cup \{v'_i \ / \ 1 \leq i \leq n\}$ and
$E(T_n) = \{v_i v_{i+1} \ / \ 1 \leq i \leq n\} \cup \{v'_i v_{i+1} \ / \ 1 \leq i \leq n\} \cup \{v'_i v_i / 1 \leq i \leq n\}$.
$|V(T_n)| = 2n+1$ and $|E(T_n)| = 3n$.
Case 1: n is odd.
Define $f : E(T_n) \to Z_4 - \{0\}$ as

$$\begin{aligned} f(v_{2i}v_{2i+1}) &= 3, \quad 1 \leq i \leq (n-1)/2 \\ f(v_{2i-1}v_{2i}) &= 1, \quad 1 < i < (n+1)/2 \\ f(v_j v'_j) &= f(v'_j v_{j+1}) = 1, \quad 1 \leq j \leq n \end{aligned}$$

Then $f^+ : V(T_n) \to Z_4$ is defined as

$$\begin{aligned} f^+(v_j) &= f(v_{j-1}v_j) + f(v_j v_{j+1}) + f(v_j v'_j) + f(v_j v'_{j-1}),\ 2 \leq j \leq n \\ f^+(v_1) &= f(v_1 v'_1) + f(v_1 v_2) \\ f^+(v_{n+1}) &= f(v_n v_{n+1}) + f(v_{n+1} v'_n) \\ f^+(v'_j) &= f(v_j v'_j) + f(v'_j v_{j+1}),\ \ 1 \leq j \leq n \end{aligned}$$

Then we have,

$$\begin{aligned} f^+(v_i) &= 2, \quad 1 \leq i \leq n+1 \\ f^+(v'_j) &= 2, \quad 1 \leq j \leq n. \end{aligned}$$

Hence, f^+ is a constant and it is equal to 2 for all $v \in \ V(T_n)$.
Case 2: n is even.
Define $f : E(T_n) \to Z_4 - 0$ by

$$\begin{aligned} f(v_{2i-1}v_{2i}) &= 3, \quad 1 \leq i \leq n/2 \\ f(v_{2i}v_{2i+1}) &= 1, \quad 1 \leq i \leq n/2 \\ f(v_j v'_j) &= 1, \quad 1 \leq j \leq n \\ f(v'_j v_{j+1}) &= 3, \quad 1 \leq j \leq n. \end{aligned}$$

Then $f^+ : V(T_n) \to Z_4$ is given by

$$\begin{aligned} f^+(v_j) &= f(v_{j-1}v_j) + f(v_j v_{j+1}) + f(v_j v'_j) + f(v_j v'_{j-1}); 2 \leq j \leq n \\ f^+(v_1) &= f(v_1 v_2) + f(v_1 v'_1); \\ f^+(v_{n+1}) &= f(v_n v_{n+1}) + f(v'_n v_{n+1}); \end{aligned}$$

$$f^+(v'_j) \;=\; f(v'_j v_{j+1}) + f(v_j v'_j) \;\; 1 \leq j \leq n.$$

Then we have,

$$\begin{aligned} f^+(v_i) &= 0, \;\; 1 \leq i \leq n+1 \\ \text{and } f^+(v'_j) &= 0, \;\; 1 \leq j \leq n. \end{aligned}$$

In both the cases T_n admits Z_4 - magic labeling. Hence, T_n is Z_4 - magic graph. ∎

Example 3.4. Z_4 - magic labelings of T_n for $n = 4$ and $n = 5$ are given below.

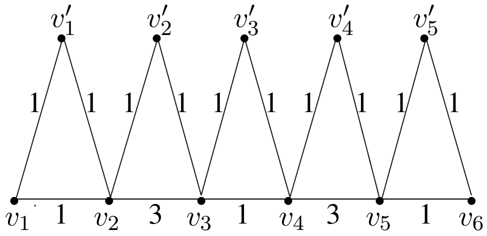

Figure 3: Z_4 - *magic labeling of* T_5.

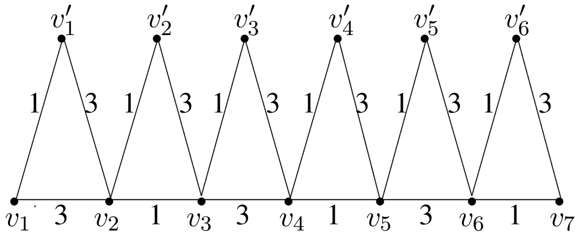

Figure 4: Z_4 - *magic labeling of* T_6.

Theorem 3.5. The graph B_n is Z_4 - magic for all $n \in N$.

Proof: Let $V(B_n) = \{u, v\} \cup \{u_i, v_i \,/\, 1 \leq i \leq n\}$ and
$E(B_n) = \{uv\} \cup \{uu_i \;/\; 1 \leq i \leq n\} \cup \{vv_i/1 \leq i \leq n\} \cup \{u_iv_i \,/\, 1 \leq i \leq n\}$.
Case 1: n is odd.
Define $f : E(B_n) \to Z_4 - \{0\}$ by

$$\begin{aligned} f(uv) &= 1 \\ f(u_iv_i) &= 1, \;\; 1 \leq i \leq n \\ f(uu_i) &= 2, \;\; 1 \leq i \leq n \\ f(vv_i) &= 2, \;\; 1 \leq i \leq n. \end{aligned}$$

Then $f^+ : V(B_n) \to Z_4$ is defined by

$$\begin{aligned} f^+(u) &= f(uv) + \Sigma_{i=1}^{n} f(uu_i) \\ f^+(v) &= f(uv) + \Sigma_{i=1}^{n} f(vv_i) \\ f^+(u_i) &= f(uu_i) + f(u_iv_i) \;\; 1 \leq i \leq n \\ f^+(v_i) &= f(vv_i) + f(u_iv_i) \;\; 1 \leq i \leq n. \end{aligned}$$

We have, $f^+(u) = 3,\ f^+(v) = 3,\ f^+(u_i) = 3,\ \ 1 \leq i \leq n and f^+(v_i) = 3.\ 1 \leq i \leq n.$

Case 2: n is even.

Define $f : E(B_n) \to Z_4 - \{0\}$ by

$$\begin{aligned} f(uv) &= 1 \\ f(u_iv_i) &= 3,\ 1 \leq i \leq n \\ f(uu_i) &= 2,\ 1 \leq i \leq n \\ f(vv_i) &= 2,\ 1 \leq i \leq n \end{aligned}$$

Then Clearly,

$$\begin{aligned} f^+(u) &= 1 = f^+(v) \\ f^+(u_i) &= 1 = f^+(v_i)\ \ 1 \leq i \leq n \end{aligned}$$

In both the cases B_n admits Z_4 - magic labeling. Hence, B_n is Z_4 - magic for all $n \in N$. ∎

Example 3.6. Z_4 - magic labelings of B_2 and B_3 are given in Figure 5 and Figure 6 respectively.

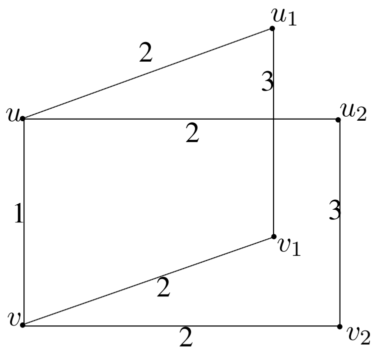

Figure 5: *Z_4 - magic labeling of B_2.*

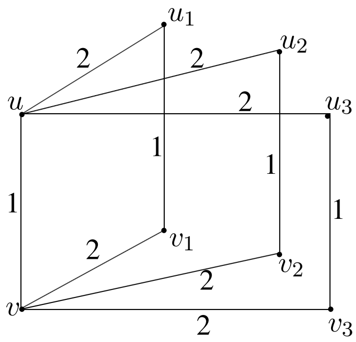

Figure 6: *Z_4 - magic labeling of B_3.*

Theorem 3.7. The graph $F_n^{(t)}$ is Z_4 - magic where t denotes the number of copies of the fan F_n.

Proof: Let the vertex set and the edge set be given by $V(F_n^{(t)}) = \{u, v_i^{(j)}/1 \leq i \leq n, 1 \leq j \leq t\}$ and $E(F_n^{(t)}) = \{uv_i^{(j)}\ /\ 1 \leq i \leq n, 1 \leq j \leq t\} \cup \{v_i^{(j)}v_{i+1}^{(j)}/1 \leq i \leq n-1, 1 \leq j \leq t\}$.

Case 1: Suppose $n = 4k - 1$ and $t \in N$.

Define $f : E(F_n^{(t)}) \to Z_4 - \{0\}$ by

$$\begin{aligned}
f(uv_1^{(i)}) &= 3, \quad 1 \le j \le t \\
f(uv_n^{(j)}) &= 3, \quad 1 \le j \le t \\
f\left(uv_{i+1}^{(j)}\right) &= 2, \quad 1 \le i \le n-2, \ 1 \le j \le t \\
f\left(v_i^{(j)} v_{i+1}^{(j)}\right) &= 1 \ for \ 1 \le i \le n-1, \ 1 \le j \le t.
\end{aligned}$$

Then $f^+ : V(F_n^{(t)}) \to Z_4$ is given by

$$\begin{aligned}
f^+(u) &= \Sigma_{j=1}^{t} \Sigma_{i=1}^{n} f\left(uv_i^{(j)}\right) \\
&\equiv (3 + 2 + 2 + ...(4k-3) \ times \ 2 + 3) \ mod \ 4 \times t \\
&= 0 \\
f^+(v_1^{(j)}) &= f(uv_1^{(j)}) + f(v_1^{(j)} v_2^{(j)}) \\
&\equiv (3+1) \ (mod \ 4) = 0, \ \ 1 \le j \le t \\
f^+(v_n^{(j)}) &= 0, \ 1 \le j \le t \\
f^+(v_i^{(j)}) &= f(uv_i^{(j)}) + f(v_i^{(j)} v_{i+1}^{(j)}) + f(v_{i-1}^{(j)} v_i^{(j)}) \\
&\equiv (2+1+1)(mod \ 4) = 0, \ \ 2 \le i \le n-1 \text{ and } 1 \le \text{j} \le \text{t}.
\end{aligned}$$

We get f^+ is constant and equals to 0 for all vertices of $F_n^{(t)}$.

Case 2: Suppose $n = 4k + 1$ and $t \in N$ where $k \in N$.

Let $f : E(F_n^{(t)}) \to Z_4 - \{0\}$ be defined as follows:

$$\begin{aligned}
f(uv_1^{(j)}) &= 2 \ \ 1 \le j \le t \\
f(uv_n^{(j)}) &= 3 \ \ 1 \le j \le t \\
f(uv_i^{(j)}) &= 1 \ \ 2 \le i \le n-1, \ 1 \le j \le t \\
f(v_{i-1}^{(j)} v_i^{(j)}) &= 2 \ \ 2 \le i \le n-1, \ 1 \le j \le t \\
f(v_i^{(j)} v_{i+1}^{(j)}) &= 1 \ \ 2 \le i \le n-1, \ 1 \le j \le t.
\end{aligned}$$

Then $f^+ : V(F_n^{(t)}) \to Z_4$ is given by

$$\begin{aligned}
f^+(u) &= \Sigma_{j=1}^{t} \Sigma_{i=1}^{n} f(uv_i^{(j)}) \\
&\equiv (2 + 1 + 1...(4k-1) times 1 + 3)t \ (mod \ 4) \\
&= 0 \\
f^+(v_1^{(j)}) &= f(uv_1^{(j)}) + f(v_1^{(j)} v_2^{(j)}) \\
&\equiv (2+2) \ (mod \ 4) \ = 0, \quad 1 \le j \le t \\
f^+(v_n^{(j)}) &\equiv (3+1) \ (mod \ 4) = 0, \quad 1 \le j \le t \\
f^+(v_i^{(j)}) &= f(uv_i^{(j)}) + f(v_i^{(j)} v_{i+1}^{(j)}) + f(v_{i-1}^{(j)} v_i^{(j)}) \\
&\equiv (1+1+2) \ (mod \ 4) = 0
\end{aligned}$$

Thus, $f^+(v_i^{(j)}) = 0$ for $2 \le i \le n-1, \ \ 1 \le j \le t$. Hence f^+ is a constant mapping and is equal to 0 for all vertices in $F_n^{(t)}$.

Case 3: Suppose $n = 4k$.

Sub case (i): $t \equiv 0\ (mod\ 2)$.

Define $f : E(F_n^{(t)}) \to Z_4 - \{0\}$ as

$$\begin{aligned} f(uv_1^{(j)}) &= 2 = f(uv_n^{(j)}) \ \ 1 \le j \le t \\ f(uv_i^{(j)}) &= 1, \ \ 2 \le i \le n-1, \ \ 1 \le j \le t \\ f(v_{2i-1}^{(j)} v_{2i}^{(j)}) &= 2, \ \ 1 \le i \le n/2, \ \ 1 \le j \le t \\ f(v_{2i}^{(j)} v_{2i+1}^{(j)}) &= 1, \ \ 1 \le i \le (n-2)/2, \ \ 1 \le j \le t. \end{aligned}$$

Then $f^+ : V(F_n^{(t)}) \to Z_4$ is given by

$$\begin{aligned} f^+(u) &= \Sigma_{j=1}^t \Sigma_{i=1}^n f(uv_i^{(j)}) \\ &= (2 + 1 + 1...(4k-2) times 1 + 2).t \\ &\equiv 0\ (mod\ 4) = 0 \\ f^+(v_1^{(j)}) &= f(uv_1^{(j)}) + f(v_1^{(j)} v_2^{(j)}) \ \ 1 \le j \le t \\ &= (2 + 2) \equiv 0\ (mod\ 4) = 0 \\ f^+(v_i^{(j)}) &= f(uv_i^{(j)}) + f(v_i^{(j)} v_{i+1}^{(j)}) + f(v_{i-1}^{(j)} v_i^{(j)}), \ \ 2 \le i \le n-1, \ 1 \le j \le t \\ f^+(v_i^{(j)}) &= (1 + 1 + 2) \equiv 0\ (mod\ 4), 2 \le i \le n-1, \ 1 \le j \le t \\ &= 0 \\ f^+(v_n^{(j)}) &= (2 + 2) \equiv 0\ (mod 4), \ 1 \le j \le t \end{aligned}$$

Hence f^+ is a constant mapping and is equal to 0 for all vertices in $F_n^{(t)}$.

Sub case (ii): $t \equiv 1\ (mod\ 2)$.

Let $f : E(F_n^{(t)}) \to Z_4 - \{0\}$ be defined as

$$\begin{aligned} f(uv_1^{(j)}) &= 3 = f(uv_n^{(j)}) \ 1 \le j \le t \\ f(uv_i^{(j)}) &= 2, \ \ 2 \le i \le n-1, \ 1 \le j \le t \\ f(v_{2i-1}^{(j)} v_{2i}^{(j)}) &= 3, \ \ 1 \le i \le n/2, \ 1 \le j \le t \\ f(v_{2i}^{(j)} v_{2i+1}^{(j)}) &= 1, \ \ 1 \le i \le (n-2)/2, \ 1 \le j \le t \end{aligned}$$

Then $f^+ : V(F_n^{(t)}) \to Z_4$ is given by

$$\begin{aligned} f^+(u) &= \Sigma_{j=1}^t \Sigma_{i=1}^n f(uv_i^{(j)}) \\ &\equiv (3 + 2 + 2 + ...(4k-2) \text{ times } + 3).t \\ &\equiv 2 (mod\ 4).t = 2 \\ f^+(v_i^{(j)}) &\equiv (3 + 1 + 2)\ (mod\ 4), \ 2 \le i \le n-1, 1 \le j \le t \\ &= 2 \\ f^+(v_1^{(j)}) &\equiv (3 + 3)\ (mod\ 4) = 2 = f^+(v_n^{(j)}) \equiv (3 + 3)\ (mod\ 4), 1 \le j \le t. \end{aligned}$$

Hence, f^+ is a constant mapping and is equal to 2 for all vertices in $F_n^{(t)}$.

Case 4: Suppose $n = 4k + 2$ and $t \in N$.
Let $f : E(F_n^{(t)}) \to Z_4 - \{0\}$ be defined as follows:

$$\begin{aligned} f(uv_1^{(j)}) &= 2 = f(uv_n^{(j)}) \;\; 1 \le j \le t \\ f(uv_i^{(j)}) &= 1, \;\; 2 \le i \le n-1, \; 1 \le j \le t \\ f(v_{2i-1}^{(j)} v_{2i}^{(j)}) &= 2, \; 1 \le i \le n/2, 1 \le j \le t \\ f(v_{2i}^{(j)} v_{2i+1}^{(j)}) &= 1, \; 1 \le i \le (n-2)/2, \; 1 \le j \le t \end{aligned}$$

Then $f^+ : V(F_n^{(t)}) \to Z_4$ is given by

$$\begin{aligned} f^+(u) &= \Sigma_{j=1}^{t} \Sigma_{i=1}^{n} f(uv_i^{(j)}) \\ f^+(u) &\equiv (2 + 1 + 1 ... 4k \text{ times } + 2).t(mod\, 4) \\ &\equiv 0 \; (mod\, 4) = 0 \\ f^+(v_1^{(j)}) &\equiv (2+2) \; (mod\, 4) = 0 = f^+(v_n^{(j)}) \equiv (2+2)(mod\, 4), 1 \le j \le t \\ f^+(v_i^{(j)}) &\equiv (2+1+1) \equiv 0 \; (mod\, 4) = 0, \; 2 \le i \le n-1, \; 1 \le j \le t. \end{aligned}$$

Hence f^+ is a constant mapping and is equal to 2 for all vertices in $F_n^{(t)}$.
In all the cases $F_n^{(t)}$ admits Z_4 -magic labeling. Hence, $F_n^{(t)}$ is Z_4 - magic. ∎

Example 3.8. Z_4-magic labelings of some one point union of fans are given in this example.

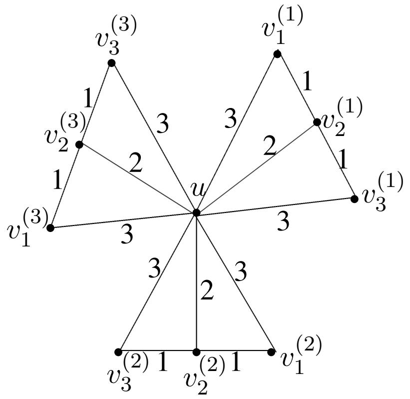

Figure 7: Z_4 - *magic labeling of* $F_3^{(3)}$.

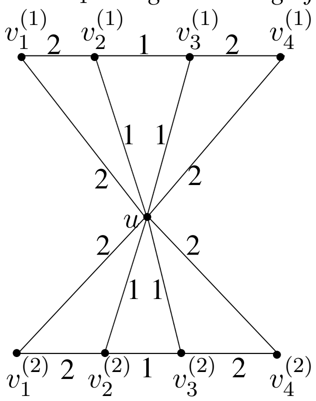

Figure 8: Z_4 - *magic labeling of* $F_4^{(2)}$.

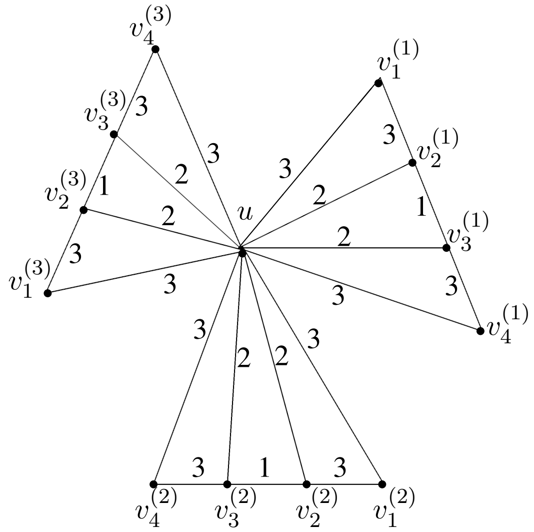

Figure 9: Z_4 - *magic labeling of* $F_4^{(3)}$.

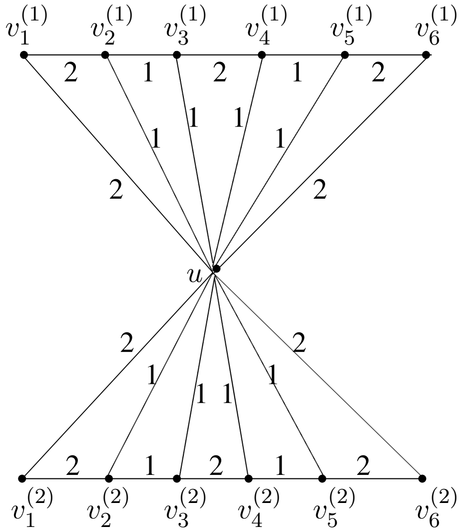

Figure 10: Z_4 - *magic labeling of* $F_6^{(2)}$.

Observation 3.9. In all the theorems, if we multiply the edge labeling by a positive integer p, the vertex labeling remains to be a constant and is equal to p times the constant value we obtained. Hence all the above graphs admit Z_{4p}-magic labeling. Hence, $S'(P_n),\ T_n,\ B_n$ and $F_n^{(t)}$ are all Z_{4p}-magic graphs.

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