International Journal of Mathematics and Soft Computing
Vol.3, No.3 (2013), 7 - 13.

ISSN Print : 2249 - 3328
ISSN Online: 2319 - 5215

Some new perspectives on distance two labeling

S K Vaidya
Saurashtra University, Rajkot - 360005,
Gujarat, INDIA.
E-mail: samirkvaidya@yahoo.co.in

D D Bantva
L. E. College, Morvi-363 642
Gujarat, INDIA.
E-mail: devsi.bantva@gmail.com

Abstract

An $L(2,1)$-labeling (or distance two labeling) of a graph G is a function f from the vertex set $V(G)$ to the set of nonnegative integers such that $|f(u) - f(v)| \geq 2$ if $d(u,v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d(u,v) = 2$. The $L(2,1)$-labeling number $\lambda(G)$ of G is the smallest number k such that G has an $L(2,1)$-labeling with max$\{f(v) : v \in V(G)\} = k$. In this paper we find λ-number for some cacti.

Keywords: Interference, channel assignment, distance two labeling, λ-number, cactus.
AMS Subject Classification(2010): 05C78.

1 Introduction

The assignment of channels to the transmitters is one of the fundamental problems for any network which is widely known as channel assignment problem introduced by Hale [3]. The interference between two transmitters plays a vital role in the assignment of channels to transmitters in the network. If we divide interference in two categories - avoidable and unavoidable, then as suggested by Roberts [4], the transmitters having unavoidable interference must receive channels that are at least two apart and the transmitters having avoidable interference must receive different channels.

The channel assignment problem can be linked with distance two labeling of graphs in which transmitters are represented by vertices of graph and interference by edges with definite rule - transmitters having unavoidable interference are joined by direct edge and transmitters having avoidable interference are put distance two apart while interference free transmitters are at distance three or more than three. Motivated through this problem Griggs and Yeh [2] introduced $L(2,1)$-labeling which is defined as follows:

Definition 1.1. A distance two labeling (or $L(2,1)$-labeling) of a graph $G = (V(G), E(G))$ is a function f from vertex set $V(G)$ to the set of nonnegative integers such that the following conditions are satisfied:
(1) $|f(u) - f(v)| \geq 2$ if $d(u,v) = 1$
(2) $|f(u) - f(v)| \geq 1$ if $d(u,v) = 2$

The *span of* f is defined as $\max\{|f(u)-f(v)| : u, v \in V(G)\}$. The λ-number for a graph G, denoted by $\lambda(G)$, is the minimum span of a distance two labeling for G. The $L(2,1)$-labeling is explored in the past two decades and received the focus of many researchers like Chang and Kuo [1], Sakai [5], Yeh [13], Vaidya *et al.* [6] and Vaidya and Bantva [7, 8, 9, 10, 11].

Proposition 1.2. [2] The λ-number of a star $K_{1,\Delta}$ is $\Delta + 1$, where Δ is the maximum degree.

Proposition 1.3. [2] The λ-number of a complete graph K_n is $2n - 2$.

Proposition 1.4. [1] $\lambda(H) \leq \lambda(G)$, for any subgraph H of a graph G.

Proposition 1.5. [2] Let G be a graph with maximum degree $\Delta \geq 2$. If G contains three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta + 2$.

2 Main Results

We begin with a finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. A complete graph K_n is a simple graph in which each pair of distinct vertices is joined by an edge. A vertex v of a graph G is called a cut vertex if its deletion leaves a graph disconnected. A block of a graph G is a maximal connected subgraph of G that has no cut-vertex. If H is a block of graph G then H has no cut vertex but H may contain vertices that are cut vertices of G and two blocks in a graph share at most one vertex. The block cutpoint graph G is a bipartite graph H in which one partite set consists of the cut vertices of G and the other has a vertex b_i for each block B_i of G. We include vb_i as an edge of H if and only if $v \in B_i$. A spider is a tree that has at most one vertex (called the center) of degree greater than 2. We denote a spider by $S_{n_1,n_2,\ldots,n_k}$ with $n_1 \geq n_2 \geq \ldots \geq n_k$, $k \geq 3$, where $n_i \in Z^+$ is the length of the ith leg. Hence, $|V(S_{n_1,n_2,\ldots,n_k})| = n_1 + n_2 + \ldots + n_k + 1$. The vertex set of spider is denoted by $V(S_{n_1,n_2,\ldots,n_k}) = V_1 \cup V_2 \cup \ldots \cup V_k$, where each V_i is the vertex set of the ith leg; that is assuming $v_{i,0} = v_{0,0}$, $V_i = \{v_{i,j} : 0 \leq j \leq n_k\}$, where $v_{i,j}v_{i,j+1} \in E(S_{n_1,n_2,\ldots,n_k})$, $0 \leq j \leq n_k - 1$. A lobster is a tree with the property that the removal of the endpoints leaves a caterpillar. A caterpillar is a tree with the property that the removal of its endpoints leaves a path. For terms not defined in this paper, readers shall refer West [12].

Definition 2.1. A linear cactus $P_m(K_n)$ is a connected graph in which all the blocks are isomorphic to a complete graph K_n and block-cutpoint graph is a path P_{2m-1}.

Definition 2.2. A spider cactus $S_{n_1,n_2,\ldots,n_k}(K_n)$ is a connected graph in which all the blocks are isomorphic to complete graph K_n and block-cutpoint graph is a spider $S_{2n_1,2n_2,\ldots,2n_k}$.

Definition 2.3. A caterpillar cactus is a cactus obtained by replacing each edge of a caterpillar by a complete graph K_n.

Definition 2.4. A lobster cactus is a cactus obtained by replacing each edge of a lobster by a complete graph K_n.

Theorem 2.5. For $P_m(K_n)$,

$$\lambda(P_m(K_n)) = \begin{cases} \Delta + 1; & \text{if} \quad m = 3 \\ \Delta + 2; & \text{if} \quad m \geq 4 \end{cases}$$

Proof: Let $P_m(K_n)$ be the linear cacti whose vertex set is $V(P_m(K_n))$ = $\{v_i^j, v_{m+1}^1 : 1 \leq i \leq m, 1 \leq j \leq n-1\}$ and $E(P_m(K_n))$ = $\{v_i^j v_{i+1}^1, v_i^j v_i^k : 1 \leq i \leq m, 1 \leq j,k \leq n-1, j \neq k\}$. The graph $K_{1,\Delta}$ is a subgraph of $P_m(K_n)$ and hence by Proposition 1.2 and Proposition 1.4, it follows that $\lambda(P_m(K_n))$ $\geq \Delta + 1$. For m = 3, define f by $f(v_1^1)$ = 3, $f(v_1^l)$ = $2l$+1, $f(v_2^1)$ = 0, $f(v_2^l)$ = $2l$-2, $f(v_3^1)$ = $\Delta + 1$, $f(v_3^l)$ = $2l$-3, $f(v_4^1)$ = $2n-3$, where $2 \leq l \leq n-1$ which is an $L(2,1)$-labeling of $P_3(K_n)$ and hence $\lambda(P_3(K_n))$ = $\Delta + 1$. For $m \geq 4$, in the graph $P_m(K_n)$, the close neighborhood of each v_i^1 where i = 3, ..., $m-1$ contains three vertices with degree Δ and hence by Proposition 1.5, $\lambda(P_m(K_n)) \geq \Delta + 2$. Now for each i = 1, 2, ... ,$m+1$ and j = 1, 2, ... ,$n-1$ define $f : V(P_m(K_n)) \rightarrow$ {0, 1, 2,..., $\Delta + 2$} as follows:

$f(v_i^1)$ = 0 if $i \equiv 1 \pmod 4$
$f(v_i^1)$ = $\Delta + 1$ if $i \equiv 2 \pmod 4$
$f(v_i^1)$ = 1 if $i \equiv 3 \pmod 4$
$f(v_i^1)$ = $\Delta + 2$ if $i \equiv 0 \pmod 4$
$f(v_i^j)$ = $2j-2$ if $i \equiv 1 \pmod 2$
$f(v_i^j)$ = $2j-1$ if $i \equiv 0 \pmod 2$

In above defined function, redefine f at $f(v_i^2)$, where $i \equiv 3 \pmod 4$ as $f(v_i^2)$ = $\Delta - 2$ then f is an $L(2,1)$-labeling for $P_m(K_n)$ and from the definition of f it is clear that $\lambda(P_m(K_n)) \leq \Delta + 2$, for $m \geq$ 4.

Thus we have,

$$\lambda(P_m(K_n)) = \begin{cases} \Delta + 1; & \text{if} \quad m = 3 \\ \Delta + 2; & \text{if} \quad m \geq 4 \end{cases}$$

■

Example 2.6. In Figure 1, an optimal $L(2,1)$-labeling of linear cactus $P_7(K_4)$ is shown for which $\lambda(P_7(K_4))$ = $\Delta + 2$ = 6 + 2 = 8.

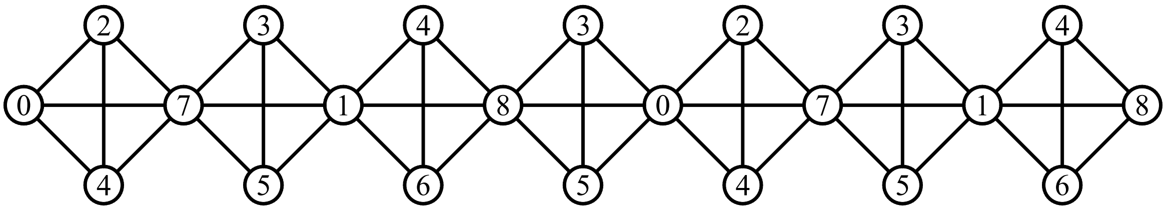

Figure 1: *An optimal $L(2,1)$-labeling of linear cactus $P_7(K_4)$ with $\lambda(P_7(K_4))$ = 8.*

Theorem 2.7. $\lambda(S_{n_1,n_2,\ldots,n_k}(K_n))$ = $\Delta + 1$.

Proof: Let $S_{n_1,n_2,\ldots,n_k}(K_n)$ be the spider cactus whose vertex set is $V(S_{n_1,n_2,\ldots,n_k}(K_n))$ = $\{v_{i,j}^l, v_{n_l+1,1}^l : 1 \leq l \leq k, 1 \leq i \leq n_l, 1 \leq j \leq n-1$ where $v_{1,1}^1 = v_{1,1}^2 = \ldots = v_{1,1}^k = v_0\}$ and $E(S_{n_1,n_2,\ldots,n_k}(K_n))$ = $\{v_{i,j}^l v_{i+1,1}^l, v_{i,j}^l v_{i,m}^l : 1 \leq l \leq k, 1 \leq i \leq n_l, 1 \leq j,m \leq n-1, j \neq m\}$. The graph $K_{1,\Delta}$ is a subgraph of $S_{n_1,n_2,\ldots,n_k}(K_n)$ and hence by Proposition 1.2 and Proposition 1.4, it follows that $\lambda(S_{n_1,n_2,\ldots,n_k}(K_n)) \geq \Delta + 1$. Now define $f : V(S_{n_1,n_2,\ldots,n_k}(K_n)) \rightarrow$ {0, 1, 2,..., $\Delta + 1$} as follows:

$f(v_0) = \Delta + 1$
$f(v_{1,2}^1) = 0$
$f(v_{1,2}^2) = 1$
... ...
$f(v_{1,2}^k) = k - 1$
$f(v_{1,3}^1) = f(v_{1,2}^k) + 1$
$f(v_{1,3}^2) = f(v_{1,3}^1) + 1$
... ...
$f(v_{1,3}^k) = f(v_{1,3}^{k-1}) + 1$
... ...
$f(v_{1,n-1}^1) = f(v_{1,n-2}^k) + 1$
$f(v_{1,n-1}^2) = f(v_{1,n-1}^1) + 1$
... ...
$f(v_{1,n-1}^k) = f(v_{1,n-1}^{k-1}) + 1$
$f(v_{i,j}^l) = (f(v_{i-1,j}^l) + 2(n-1))(\text{mod } (\Delta + 1))$

The above defined function f is an $L(2,1)$-labeling of $S_{n_1,n_2,\ldots,n_k}(K_n)$ and it is clear that $\lambda(S_{n_1,n_2,\ldots,n_k}(K_n)) \leq \Delta + 1$.

Thus, we have $\lambda(S_{n_1,n_2,\ldots,n_k}(K_n)) = \Delta + 1$. ■

Example 2.8. In Figure 2, an optimal $L(2,1)$-labeling of spider cactus $S_{4,3,2,1}(K_4)$ is shown for which $\lambda(G) = \Delta + 1 = 12 + 1 = 13$.

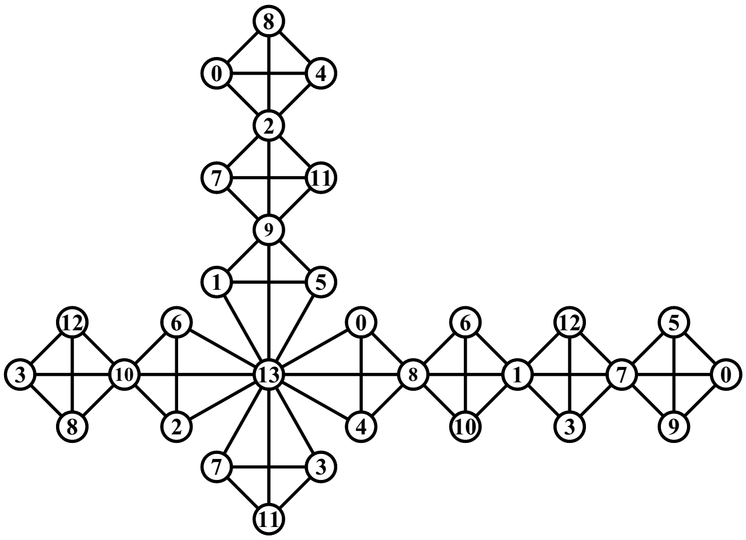

Figure 2: *An optimal $L(2,1)$-labeling of spider cactus $S_{4,3,2,1}(K_4)$.*

Corollary 2.9. For K_n^t(one point union of t complete graph K_n), $\lambda(K_n^t) = \Delta + 1$.

Example 2.10. In Figure 3, an optimal $L(2,1)$-labeling of K_4^3 is shown for which $\lambda(K_4^3) = \Delta + 1 = 9 + 1 = 10$.

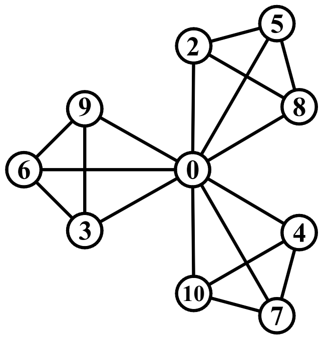

Figure 3: *An optimal $L(2,1)$-labeling of K_4^3.*

Theorem 2.11. Let G be a lobster cactus then $\lambda(G)$ = $\Delta+1$ or $\Delta+2$, where Δ is the maximum degree of the vertex.

Proof: Let G be a lobster cactus having a vertex with maximum degree Δ. The graph $K_{1,\Delta}$ is a subgraph of G and hence by Proposition 1.2 and Proposition 1.4, it follows that $\lambda(G) \geq \Delta + 1$.

Let v_0 be a vertex of degree Δ. Define $f(v_0) = 0$ and let S = $\{v_0\}$.
Let N(v_0) = $\{v_1$, v_2 ,..., $v_\Delta\}$. By definition of G, N(v_0) can be partitioned into sets V_1, V_2, ... ,V_k such that for each i = 1,2,...,k the graph induced by $V_i \cup \{v_0\}$ forms a complete subgraph of G. In addition, the partitions have the characteristic that for $u \in V_i$ and $v \in V_j$ with $i \neq j$, $d(u,v)$ = 2.

Choose a vertex $v_1 \in N(v_0)$ and define $f(v_1)$ = 2. Find a vertex $v_2 \in N(v_0)$ such that $d(v_1, v_2)$ = 2 and define $f(v_2)$ = 3. Continue this process till all the vertices of $N(v_0)$ are labeled. Take S^1 = $\{v_0\} \cup \{v \in V(G)/f(v)$ is a label of $v\}$.

For a labeled vertex $f(w)$ = i, find $N(w)$ and define $f(v)$ = the smallest number from the set {0,1,2,...}$-\{i-1, i, i+1\}$, where $v \in N(w) - S$ such that $|f(u) - f(v)| \geq 2$ if $d(u,v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d(u,v) = 2$ for any $u \in S$. Denote $S \cup \{v \in V(G)/f(v)$ is a label of $v\}$ = S^2.

Continuing this process we get S^n = V(G), where S^n = $S^{n-1} \cup \{v \in V(G)/f(v)$ is a label of $v\}$ for some n. Now all the vertices are labeled and we get max$\{f(v)/v \in V(G)\}$ = $\Delta+2$. Hence, $\lambda(G)$ $\leq \Delta + 2$.

Thus, $\lambda(G)$ is either $\Delta + 1$ or $\Delta + 2$. ■

Example 2.12. In Figure 4, an optimal $L(2,1)$-labeling of lobster cactus is shown for which $\lambda(G)$ = $\Delta + 1$ = 12 + 1 = 13.

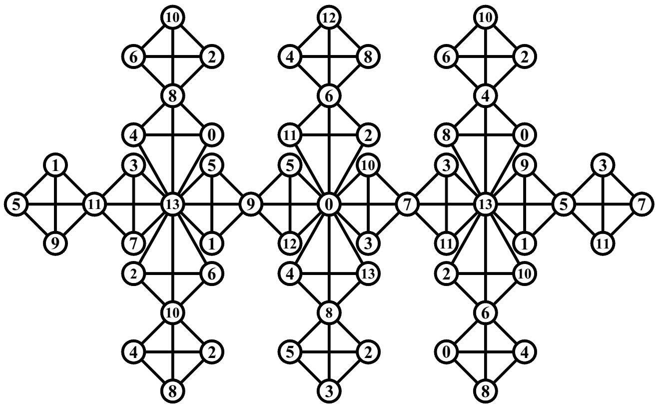

Figure 4: *An optimal $L(2,1)$-labeling of a lobster cactus G with $\lambda(G)$ = 13.*

Corollary 2.13. Let G be a caterpillar cactus with maximum degree of vertex Δ then $\lambda(G) = \Delta + 1$ or $\Delta + 2$.

Example 2.14. In Figure 5, an optimal $L(2,1)$-labeling of a caterpillar graph is shown for which $\lambda(G)$ = $\Delta + 1$ = 12 + 1 = 13.

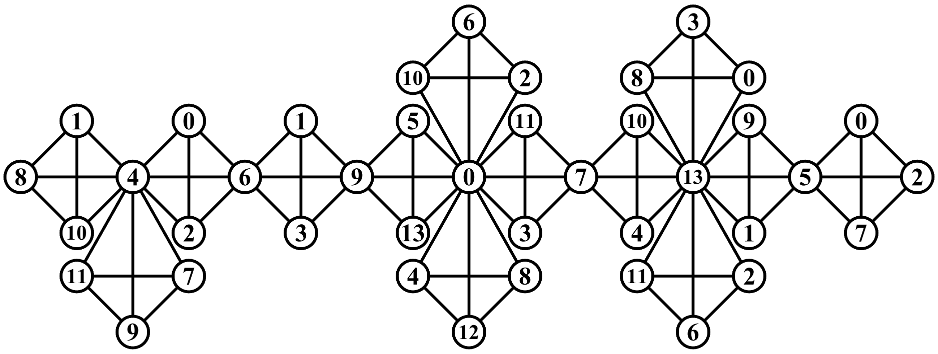

Figure 5: *An optimal $L(2,1)$-labeling of a caterpillar graph G with $\lambda(G)$ = 13.*

3 Concluding Remarks

The expansion of transmitter network requires more channels for broadcasting which is free of interference. Keeping this in mind, we introduce some new classes of cacti and investigate bounds on λ-number for the same. This work has a potential to serve better in the transmitter network.

Acknowledgement

The authors are highly thankful to the anonymous referee for valuable suggestions and comments.

References

[1] G.J. Chang and D. Kuo, *The L(2,1)-labeling problem on graphs*, SIAM J. Discrete Math., 9(2)(1996), 309-316.

[2] J.R. Griggs and R. K. Yeh, *Labeling graphs with a condition at diameter 2*, SIAM J. Discrete Math., 5(1992), 586-595.

[3] W.K. Hale, *Frequency assignment:Theory and applications*, Proc. IEEE, 68(12)(1980), 1497-1514.

[4] F.S. Roberts, *T-coloring of graphs:recent results and open problems*, Discrete Math., 93(1991), 229-245.

[5] D. Sakai, *Labeling chordal graphs: Distance two condition*, SIAM J. Discrete Math., 7(1)(1994), 133-140.

[6] S.K. Vaidya, P. L. Vihol, N. A. Dani and D. D. Bantva, $L(2,1)$*-labeling in the context of some graph operations*, Journal of Mathematics Research, 2(3)(2010), 109-119.

[7] S.K. Vaidya and D. D. Bantva, *The* $L(2,1)$*-labeling of some middle graphs*, Journal of Applied Computer Science and Mathematics, 9(4)(2010), 104-107.

[8] S.K. Vaidya and D. D. Bantva, *Distance two labeling of some total graphs*, General Mathematics Notes, 3(1)(2011), 100-107.

[9] S.K. Vaidya and D. D. Bantva, *Labeling cacti with a condition at distance two*, Le Mathematiche, 66(2011), 29-36.

[10] S.K. Vaidya and D. D. Bantva, *Distance two labeling of generalized cacti*, International J. Math. Combin., 2(2012), 103-109.

[11] S.K. Vaidya and D. D. Bantva, $L(2,1)$*-labeling for square graph of some trees*, Advances and Applications in Discrete Mathematics, 10(1) (2012), 39-47.

[12] D.B. West, Introduction to Graph theory, Prentice -Hall of India, 2001.

[13] R.K. Yeh, *A survey on labeling graphs with a condition at distance two*, Discrete Math., 306(2006), 1217-1231.