CO-IDEALS AND CO-FILTERS
IN ORDERED SET UNDER CO-QUASIORDER

Daniel Abraham Romano

ABSTRACT. In this paper, basing our consideration on the sets with the apartness relation, we analyze characteristics of some special relations to these sets such as co-order and co-quasiorder and coequality relations. In addition, we analyze two special classes of subsets, co-filters and co-ideals, of ordered set under a co-quasiorder relation. This investigation is into the Bishop's Constructive mathematics.

1. INTRODUCTION

In this paper, we present a small portion of the theory of sets with apartness from a constructive point of view. The focus is on Bishops approach to constructive mathematics. Since the appearance of Bishops monograph [2] in 1967, there have been significant developments in Bishop-style analysis (see, for example [3, 4, 10]). The main goal of this paper is to provide a constructive definition of coquasiorder and other relations (co-order and coequality relations) concerning to this coquasiorder for an arbitrary set with apartness.

Our setting is Bishop's constructive mathematics [2, 3, 11, 19], mathematics developed with Constructive logic (or Intuitionistic logic [19]) - logic without the Law of the Excluded Middle $P \vee \neg P$. We have to note that 'the crazy axiom' $\neg P \implies (P \implies Q)$ is included in Constructive logic. Precisely in Constructive logic, the 'Double Negation Law' $P \iff \neg\neg P$ does not hold, but the following implication $P \implies \neg\neg P$ holds even in Minimal logic. In Constructive logic, the Weak Law of the Excluded Middle $\neg P \vee \neg\neg P$ does not hold. It is interesting, in Constructive logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but this

---

is impossible to prove without 'the crazy axiom'. One advantage of working in this manner is that proofs and results have more interpretations. On the one hand, Bishop's constructive mathematics is consistent with traditional mathematics. On the other hand, the results can be interpreted recursively or intuitionistically. If we are working constructively, the first problem is to obtain appropriate substitutes for the classical definitions. The classical theory of partially ordered sets is based on the negative concept of partial order. Unlike the classical case, an affirmative concept, introduced in the author's papers [**8, 9, 13, 14, 15, 16**] and similar to von Plato's [**20**] and Baroni's [**1**] excess relation, will be used as a primary relation.

Let $(S, =, \neq)$ be a constructive set in the sense of Mines et all. [**10**] and Troelstra and van Dalen [**19**]. The relation $\neq$ is a binary relation on S with the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq z \Longrightarrow x \neq y \vee y \neq z,$$
$$x \neq y \wedge y = z \Longrightarrow x \neq z.$$

It is called *apartness* (Heyting). Let S and T be two sets with apartness, then the relation $\neq$ on $S \times T$ is defined by

$$(x, y) \neq (u, v) \Longleftrightarrow (x \neq u \vee y \neq v)$$

for any $x, u \in S$ and any $y, v \in T$.

Let Y be a subset of S. Following [**8**], [**9**] we define a subset

$$Y^{\triangleright} = \{x \in S : x \triangleright Y\}$$

of S called the *complement of Y in S*, where $\triangleright$ is a relation bettween an element $x \in S$ and subset Y

$$x \triangleright Y \quad \Longleftrightarrow \quad (\forall y \in Y)(\ x \neq y).$$

(In all our earlier texts we used notation '$\bowtie$' instead notation '$\triangleright$'.)

Co-notions of relations are frequent in Brouwer's and Heyting's work on intuitionistic mathematics, to some extent occur in Bishop's writings, and have reappeared in and around formal topology, especially in the theory of apartness relations studied at length by Bridges, Vita ([**6**]) and others. Co-notions of predicates, subsets and substructures have proved useful for doing intuitionistic algebra.

As we have already pointed out, our primary objective is to show some characteristics of coequality, co-order and coquasiorder relations on set with apartness. Our work is based on applications of ideas and notions coming from [**11, 12, 13, 14, 15, 16**]. The paper is organized in the following way. A set with apartness together with a co-order and a coquasiorder relations, an equivalence and its dual coequivalence is the subject of section 2. The main results of this section are Theorem 2.1, Theorem 2.2 Theorem 2.3 and Theorem 2.4 and several corollaries in which we show that families of all coquasiorders and all coequalences on set with apaerness are lattices. Two important subsets, co-filters and co-ideals, of ordered set S with apartness under a co-quasiorder are studied in Section 3. The main results are given in Theorem 3.3 and Corollary 3.1 in which we prove that family of all co-filters (co-ideals) forms a lattice.

For undefined notions and notation, cf. [**8, 9, 11, 12, 13, 14, 15, 16, 17**]. Other general references for constructive mathematics are in books [**2, 3, 4, 6, 10, 19**].

Why study sets with apartness, relations and substructures on such sets? One can give an answer similar to the one given in [**4**]: ...doing constructive mathematics, in this case, sets with apartness and corresponding characteristics relations and subsets, is intellectually interesting and challenging...

Our next research should be on lattices mentioned relations and their interrelationships.

2. SOME IMPORTANT RELATIONS

2.1. Co-order relations. We will briefly recall the constructive definition of linear order and we will use a generalization of von Plato's [**20**] and Baroni's [**1**] excess relation for the definition of a partially ordered set. Let S be a nonempty set. A binary relation $<$ (less than) on S is called a *linear order* if the following axioms are satisfied for all elements x and y:

$$\neg(x < y \ \wedge \ y < x),$$
$$x < y \implies (\forall z \in S)(x < z \ \vee \ z < y).$$

An example is the standard strict order relation $<$ on **R**, as described in [**2**]. For an axiomatic definition of the real number line as a constructive ordered field, the reader is referred to [**2, 3, 5, 7**]. A detailed investigation of linear orders in lattices can be found in [**20**]. The binary relation $\nless$ on S is called an *excess relation* if it satisfies the following axioms:

$$\neg(x \nless x),$$
$$x \nless y \implies (\forall z \in S)(x \nless z \ \vee \ z \nless y).$$

We say that x exceeds y whenever $x \nless y$. Clearly, each linear order is an excess relation. As shown in [**16**], we obtain an apartness relation $\neq$ and a partial order $\leqslant$ on X by the following definitions:

$$x \neq y \iff (x \nless y \ \vee \ y \nless x) ,$$
$$x \leqslant y \iff \neg(x \nless y).$$

Note that the statement $\neg(x \leqslant y) \implies x \nless y$ does not hold in general.

As in [**13, 14, 15, 16**], we define our notion of co-order (in our earlier texts [**13, 14, 15, 16**] we used term 'anti-order'): a relation α on a set $(S, =, \neq)$ is a *co-order* on S if and only if

$$\alpha \subseteq \neq, \ \ \alpha \subseteq \alpha * \alpha, \ \ \alpha \cup \alpha^{-1} = \neq.$$

Here, $*$ is the filed product between relations defined by the following way: If α and β are relations on set S, then filed product $\beta * \alpha$ of relation α and β is the relation given by $\{(x, z) \in X \times X : (\forall y \in X)((x, y) \in \alpha \ \vee \ (y, z) \in \beta)\}$.

Our first proposition gives us an explanation of what kind of relation is a complement of a co-order relation.

LEMMA 2.1. *Let α be a co-order relation on the set $(S, =, \neq)$. Then the relation $\alpha^{\triangleright}$ is a partial order relation on $(S, \neg \neq, \neq)$. If the apartness $\neq$ is tight, then $\alpha^{\triangleright}$ is a partial order relation on the set S.*

PROOF. (1) Let (u, v) be an arbitrary element of α and let x be an element of S. Then, from $(u, x) \in \alpha \vee (x, v) \in \alpha$ it follows that $u \neq x \vee x \neq v$, i.e., $(u, v) \neq (x, x)$. So, the relation $\alpha^{\triangleright}$ is reflexive.

(2) Let $(x, y) \in \alpha^{\triangleright}$ and $(y, x) \in \alpha^{\triangleright}$ and suppose that $x \neq y$. Then by linearity of α, we have $(x, y) \in \alpha$ or $(y, x) \in \alpha$, which is impossible. So, we must have $\neg(x \neq y)$ and $x = y$ if the relation $\neq$ is tight.

(3) Now, let suppose that $(x, y) \in \alpha^{\triangleright}$ and $(y, z) \in \alpha^{\triangleright}$ and let (u, v) be an arbitrary element of α . Then, by cotransitivity of α, from $(u, x) \in \alpha$ or $(x, y) \in \alpha$ or $(y, z) \in \alpha$ or $(z, v) \in \alpha$ we have $(u, x) \in \alpha$ or $(z, v) \in \alpha$ because $(x, y) \in \alpha^{\triangleright}$ and $(y, z) \in \alpha^{\triangleright}$. Therefore, $u \neq x$ or $z \neq v$. So, $(x, z) \neq (u, v) \in \alpha$.   □

The reader can see some examples of this relation in our articles [**8, 9, 14, 15, 16**].

2.2. Co-quasiorder relations. Let S be a set. A relation ρ on S is a *quasi-order* if

$$= \; \subseteq \; \rho, \; \rho \circ \rho \subseteq \rho.$$

where he notation '$\circ$' is the standard mark for product of relations α and β: For elements x and z of set S holds

$$(x, z) \in \beta \circ \alpha \iff (\exists y \in S)((x, y) \in \alpha \wedge (y, z) \in \beta).$$

Then the relation e on S, defined by $e = \rho \cap \rho^{-1}$, is an equivalence on S. The constructive notion of a co-quasiorder relation is the parallel notion to the classical notion of a quasi-order relation. Let $(S, =, \neq)$ be a set with apartness. A consistent and cotransitive relation τ defined on S is called a *coquasiorder*:

$$\tau \subseteq \neq, \;\; \tau \subseteq \tau * \tau.$$

(In our texts [**13, 14, 15, 16**] we used term 'quasi-antiorder'.) We accept that the empty set $\emptyset$ is also a co-quasiorder relation on set S. In this paper and some other papers (for example, in [**8, 9, 13, 14, 15, 16, 17**]) we try to research the properties of co-quasiorders.

The strong complement $\sigma^{\triangleright}$ of a quasi-antiorder σ has the well known property.

LEMMA 2.2. *If σ is a co-quasiorder on S, then the relation $\sigma^{\triangleright} = \{(x, y) \in S \times S : (x, y) \triangleright \sigma\}$ is a quasi-order on S.*

PROOF. It is clear that $\sigma^{\triangleright}$ is a reflexive relation.

Let $(x, y) \in \sigma^{\triangleright}$ and $(y, z) \in \sigma^{\triangleright}$ and let (u, v) be an arbitrary element of σ. Then $(u, x) \in \sigma \vee (x, y) \in \sigma \vee (y, z) \in \sigma \vee (z, v) \in \sigma$. Hence, $u \neq x \vee z \neq v$, i.e., $(u, v) \neq (x, z)$. So, $(x, z) \in \sigma^{\triangleright}$.

Therefore, $\sigma^{\triangleright}$ is a coquasiorder relation on S.   □

THEOREM 2.1. *If $\{\sigma_k\}_{k \in J}$ is a family of co-quasiorders on a set $(S, =, \neq)$ all included in a relation R, then $\bigcup_{k \in J} \sigma_k$ is a co-quasiorder in S included in R. There exists the maximal co-quasiorder relation τ_{max} such that $\tau_{max} \subseteq \bigcap_{k \in J} \sigma_k$.*

PROOF. It is easy to check that $\bigcup_{k \in J} \sigma_k$ is a consistent relation in S.

Let (x, z) be an arbitrary elements of $X \times X$ such that $(x, z) \in \bigcup_{k \in J} \sigma_k$. then there exists k in J such that $(x, z) \in \sigma_k$. Hence, for every $y \in X$ we have $(x, y) \in \sigma_k \vee (y, z) \in \sigma_k$. So, $(x, y) \in \bigcup_{k \in J} \sigma_k$ or $(y, z) \in \bigcup_{k \in J} \sigma_k \sigma_k$. So, the relation $\bigcup_{k \in J} \sigma_k$ is cotransitive.

At the other side, for every k in J holds $\sigma_k \subseteq R$. From this we conclude $\bigcup_{k \in J} \sigma_k \subseteq R$.

By the first part of this theorem, there exists the biggest co-quasiorder relation on S included in $\bigcap_{k \in J}$. $\qquad\square$

COROLLARY 2.1. *The family of all co-quasiorders on a set $(S, =, \neq)$ is a completely lattice. The biggest element in this lattice is the apartness relation and last one is $\emptyset$.*

The reader can see some examples of this relation in our articles [**8, 9, 14, 15, 16**].

2.3. Coequality relations. A relation q on S is a *coequality relation* on S if and only if it is consistent with the apetness, symmetric and cotransitive

$$q \subseteq \neq, \ q^{-1} \subseteq q \ \text{ and } \ q \subseteq q * q.$$

If q is a coequivality relation defined on S, then a relation $q^{\triangleright}$ is an equivality relation on S and associated with q. (For two relations, α and β defined on S we say that α is *associated* with β if $\beta \circ \alpha \subseteq \alpha$.)

LEMMA 2.3. *Let q be an coequality relation on a set S with apartness. Then the relation $q^{\triangleright}$ is an equivalence on S associated with q.*

PROOF. It is true that $= \subseteq q^{\triangleright}$ and that $q^{\triangleright}$ is symmetric.

We need to prove that is $q^{\triangleright}$ transitive. Let $(x, y) \triangleright q$ and $(y, z) \triangleright q$ and let (u, v) be an arbitrary element of q. Then $(u, x) \in q \vee (x, y) \in q (y, z) \in q \vee (z, v) \in q$. Here follows $u \neq x \vee z \neq v$. So, $(u, v) \neq (x, z)$ and $(x, z) \triangleright q$.

Let us show that the relations of $q^{\triangleright}$ and q are associated. In order to show, take $(x, z) \in q^{\triangleright} \circ q$. By definition, there exists an element $y \in S$ such that $(x, y) \in q$ and $(y, z) \triangleright q$. Thus, by cotransitivity of q, we have $(x, z) \in q$ or $(z, y) \in q = q^{-1}$. Finally, we have $(x, z) \in q$ because $(y, z) \triangleright q$. $\qquad\square$

In some next sentences we will try to make clearer the notion of coequality relation to the reader. Let q be a coequality on S. As we saw, the relation $q^{\triangleright}$ - the strong complement of q - is an equality on S associate with q and we can construct the factor-set $S/(q^{\triangleright}, q) = \{aq^{\triangleright} : a \in S\}$, where $aq^{\triangleright} = \{u \in S : (a, u) \triangleright q\}$, with

$$aq^{\triangleright} = bq^{\triangleright} \Longleftrightarrow (a, b) \triangleright q, \ aq^{\triangleright} \neq bq^{\triangleright} \Longleftrightarrow (a, b) \in q.$$

and the family $S/q = \{aq : a \in S\}$, where $aq = \{u \in S : (a, u) \in q\}$ is the class of q generated by the element a, with

$$aq = bq \Longleftrightarrow (a, b) \triangleright q, \ aq \neq bq \Longleftrightarrow (a, b) \in q.$$

LEMMA 2.4. *If σ is a co-quasiorder on S, then the relation $q = \sigma \cup \sigma^{-1}$ is an coequality relation on S.*

PROOF.

$$\sigma \subseteq \neq \Longrightarrow \sigma^{-1} \subseteq \neq$$
$$\Longrightarrow \sigma \cup \sigma^{-1} \subseteq \neq;$$
$$q^{-1} = (\sigma \cup \sigma^{-1})^{-1} = \sigma^{-1} \cup \sigma = q.$$

Let x, y and z be arbitrary elements of set S. Then

$$(x,z) \in q = \sigma \cup \sigma^{-1} \Longleftrightarrow (x,z) \in \sigma \vee (z,x) \in \sigma$$
$$\Longrightarrow ((x,y) \in \sigma \vee (y,z) \in \sigma) \vee ((z,y) \in \sigma \vee (y,x) \in \sigma)$$
$$\Longleftrightarrow ((x,y) \in \sigma \vee (x,y) \in \sigma^{-1}) \vee ((y,z) \in \sigma \vee (y,z) \in \sigma^{-1})$$
$$\Longleftrightarrow (x,y) \in \sigma \cup \sigma^{-1} \vee (y,z) \in \sigma \cup \sigma^{-1}$$
$$\Longleftrightarrow (x,z) \in \sigma \cup \sigma^{-1} * \sigma \cup \sigma^{-1}.$$

Therefore, the relation q is a coequality relation on S. $\square$

THEOREM 2.2. *If $\{q_k\}_{k \in J}$ is a family of coequality relations on a set $(S,=,\neq)$ all included in a relation R, then $\bigcup_{k \in J} q_k$ is a coequality relation in S included in R.*

There exists the maximal coequality relation q_{max} such that $q_{max} \subseteq \bigcap_{k \in J} q_k$.

PROOF. Since a coequality relation q_k is a symmetric co-quasiorder relation in S for the completeness of proof the first statement we need the following

$$(\bigcup_{k \in J} q_k)^{-1} = \bigcup_{k \in J} (q_k)^{-1} = \bigcup_{k \in J} q_k.$$

So, the relation $\bigcup_{k \in J} q_k$ is a coequality relation on S. If q_k is included in a relation R, then $\bigcup_{k \in J} q_k$ is included in R also.

By second part of the Theorem 2.1, there exists the biggest co-quasiorder τ_{max} included in the intersection $\bigcap_{k \in J} q_k$. Thus, by Lemma 2.4, relation $\tau_{max} \cup (\tau_{max})^{-1}$ is a coequality relation included in the intersection $\bigcap_{k \in J} q_k$. $\square$

COROLLARY 2.2. *The family of all coequality relations in set $(S,=,\neq)$ is a completely lattice. The biggest element in this lattice is the apartness relation in S and last one is $\emptyset$.*

The reader can see some examples of this relation in our articles [**11, 12, 13**]

2.4. Functions. For a function $f : (S,=,\neq) \longrightarrow (T,=,\neq)$ between two sets we say that it is:
− a *strongly extensional* if $(\forall a,b \in S)(f(a) \neq f(b) \Longrightarrow a \neq b)$;
− an *embedding* if $(\forall a,b \in S)(a \neq b \Longrightarrow f(a) \neq f(b))$.

Without difficulties we can construct proof for statements in the following lemma.

LEMMA 2.5. *Let q be a coequality relation on a set S. Then:*
(1) The function $\pi_S : S \longrightarrow S/(q^{\triangleright},q)$, determined by $\pi_S(a) = aq^{\triangleright}$, is a strongly extensional surjective mapping.
(2) The function $\vartheta_S : S \longrightarrow S/q$, determined by $\vartheta_S(a) = aq$, is a strongly extensional surjective mapping.
(3) There exist the strongly extensional injective, surjective and embedding mapping between $S/(q^{\triangleright},q)$ and S/q.

PROOF. It is clear that the connections π_S and ϑ_S are functions. From determinations of $aq^\triangleright \neq bq^\triangleright \iff (a,b) \in q \subseteq \neq$ and $aq \neq bq \iff (a,b) \in q \subseteq \neq$ we conclude that π_S and ϑ_S are strongly extensional and surjective mappings.

It is easy to check that the connection $\psi : S/(q^\triangleright, q) \longrightarrow S/q$, defined by $\psi(aq^\triangleright) = aq$ $(a \in S)$, is a strongly extensional, injective, embedding and surjective mapping. $\square$

For a function $f : (S, =, \neq, \alpha) \longrightarrow (T, =, \neq, \beta)$ between two relational systems we say that it is:
— *isotone* if $(\forall a, b \in S)((a,b) \in \alpha \implies (f(a), f(b)) \in \beta)$;
— *reverse isotone* if $(\forall a, b \in S)((f(a), f(b)) \in \beta \implies (a,b) \in \alpha)$.

THEOREM 2.3. *Let σ be a co-quasiorder relation on a set T. If $f : S \longrightarrow T$ be a stronaly extensional reverse isotone mapping, then the relation $f^{-1}(\sigma) = \{(a,b) \in S \times S : (f(a), f(b)) \in \sigma\}$ is a co-quasiorder on S. In addition, there exists a strongly extensional mapping $\psi : S/q_S \longrightarrow T/q_T$ such that $\psi \circ \vartheta_S = \vartheta_T \circ f$, where $q_S = f^{-1}(\sigma) \cup (f^{-1}(\sigma))^{-1}$ and $q_T = \sigma \cup \sigma^{-1}$.*

PROOF. Thus, from $(f(a), f(b)) \in \sigma \subseteq \neq$ follows $f(a) \neq f(b)$ and $a \neq b$ since the mapping f is a reverse isotone. So, the relation $f^{-1}(\sigma)$ is a consistent relation. Let (a,b) be s an arbitrary element of $f^{-1}(\sigma)$ and let b be an arbitrary element of set S. Thus $(f(a), f(c)) \in \sigma$. For an element $b \in S$, since σ is a cotrantitive relation on T, we have $(f(a), f(b)) \in \sigma$ or $(f(b), f(c)) \in \sigma$. It is means $(a,b) \in f^{-1}(\sigma)$ or $(b,c) \in f^{-1}(\sigma)$. Therefore, the relation $f^{-1}(\sigma)$ is a cotransitive relation on S. So, the relation $f^{-1}(\sigma)$ is a co-quasiorder on S.

If the mapping $\psi : S/q_S \longrightarrow T/q_T$ determine by the following way $\psi(aq_S) = f(a)q_T$ we immediately give seeing equality. $\square$

In this part we will give our main results. Let $(S, =, \neq, \sigma)$ be a ordered set under a co-quasiorder σ. In the following theorem we will give the solution of the problem of existence of a co-order relation on the set S/q.

THEOREM 2.4. *Let σ be a co-quasiorder relation on a set S, $q = \sigma \cup \sigma^{-1}$. Then there exists a relation θ on S/q, defined by $(aq, bq) \in \theta \iff (a,b) \in \sigma$, such that it is a consistent, cotransitive and linear relation on S/q.*

PROOF. Let $(aq, bq) \in \theta$. That $(a,b) \in \sigma$. According to $\sigma \subseteq q$, we have $(a,b) \in q$. So, $aq \neq bq$.

Let $(aq, cq) \in \theta$. Then $(a,c) \in \sigma$. Thus, $(a,b) \in \sigma$ or $(b,c) \in \sigma$ for arbitrary element $b \in S$. Finally, we have $(aq, bq) \in \theta$ or $(bq, cq) \in \theta$. It is means that θ is a cotransitive relation.

Let $aq \neq bq$. Thus $(a,b) \in q = \sigma \cup \sigma^{-1}$. Then $(a,b) \in \sigma$ or $(b,a) \in \sigma$, i.e., then $(aq, bq) \in \theta$ or $(bq, aq) \in \theta$. Hence θ is a linear relation on S/q. $\square$

Some more information about function the reader can see in our articles [**17, 18**].

3. SOME IMPORTANT SUBSETS
IN ORDERED SET UNDER CO-QUASIORDER

We will start this section with the following statement.

PROPOSITION 3.1. *Let τ be a co-quasiorder on a set S. Then classes $a\tau$ and τb are detachable subsets of S such that $a \rhd a\tau$ and $b \rhd \tau b$, for any $a, b \in S$. Moreover, the following implications hold:*
(1) $y \in a\tau \wedge x \in S \Longrightarrow x \in a\tau \vee (x, y) \in \tau$;
(2) $y \in \tau b \wedge x \in S \Longrightarrow x \in \tau b \vee (y, x) \in \tau$.
(2) $(a, b) \in \tau \Longrightarrow a\tau \cup \tau b = S$.

PROOF. Let $x \in S$ and $y \in a\tau$. Then, by cotransitivity of τ, we have $(a, x) \in \tau$ or $(x, y) \in \tau$. So, $x \in a\tau$ or, by consistency of τ, $x \neq y$. Thus, $a\tau$ is a detachable subset of S, and $a \rhd a\tau$ holds.

$$y \in a\tau \wedge x \in S \Longleftrightarrow (a, y) \in \tau \wedge x \in S$$
$$\Longrightarrow (a, x) \in \tau \vee (x, y) \in \tau$$
$$\Longrightarrow x \in a\tau \vee (x, y) \in \tau.$$

In a similar manner we can prove that τb is a detachable subset of S with $b \rhd \tau b$ and that the implication $y \in \tau b \wedge x \in S \Longrightarrow x \in \tau b \vee (y, x) \in \tau$ holds.

Let $a, b \in S$ such that $(a, b) \in \tau$ and let $x \in S$. This means that $(a, x) \in \tau \vee (x, b) \in \tau$, i.e., $x \in a\tau$ or $x \in \tau b$, or, equivalently, $x \in a\tau \cup \tau b$. Therefore, $S \subseteq a\tau \cup \tau b$, which implies equality. □

Generalizing the example (1) and (2) in Proposition 3.1, we can introduce the concept of a special subsets in co-quasiordered set.

3.1. Co-ideals and co-filters.

DEFINITION 3.1. Let S be ordered set under co-quasiorder τ. For detachable subset cF of S we say that it is a *co-filter* in S if

$$y \in cF \wedge x \in S \Longrightarrow x \in cF \vee (x, y) \in \tau.$$

So, the subset $a\tau$ is a *principal co-filter* of S generated by the element a. In addition, the sets $\emptyset$ and S are trivial co-filters of S.

DEFINITION 3.2. For detachable subset cJ of ordered subset S under a co-quasiorder τ we say that it is a *co-ideal* in S if

$$y \in cJ \wedge x \in S \Longrightarrow x \in cJ \vee (y, x) \in \tau.$$

So, the subset τb is a *principal co-ideal* of S generated by the element b. In addition, the sets $\emptyset$ and S are trivial co-ideals of S.

THEOREM 3.1. *If cF is a co-filter of ordered set S under co-quasiorder τ, then $cF^{\rhd}$ is a filter in ordered set S under quasiorder $\tau^{\rhd}$.*

If cJ is a co-ideal of ordered set S under co-quasiorder τ, then $cJ^{\rhd}$ is an ideal in ordered set S under quasiorder $\tau^{\rhd}$.

PROOF. Let cF be a co-filter of ordered set S under co-quasiorder τ. Then $\tau^{\triangleright}$ is an quasiorder on set S. Suppose that $x \in cF^{\triangleleft}$ and $(x,y) \triangleright \tau$. Let u be an arbitrary element of cF. Thus, from the implication $u \in cF \implies x \in cF \vee (x,u) \in \tau$ follows $(x,u) \in \tau$ because $x \triangleright cF$. Further on, by cotransitivity of τ, we have $(x,y) \in \tau \vee (y,u) \in \tau \subseteq \neq$. Hence, we conclude $y \neq u \in cF$ because $(x,y) \triangleright \tau$. Finally, $y \in cF^{\triangleright}$. So, the subset $cF^{\triangleright}$ is a filter in S.

The second statement of this theorem we can prove by analogy to previous statement. $\qquad\square$

THEOREM 3.2. *If $\{K_j\}_{j \in J}$ be a family of co-filters (co-ideals) in ordered set S under co-quasiorder τ, then $\bigcup_{j \in J} K_j$ is a co-filter (co-ideal respectively) too.*

If G_1 and G_2 are co-filters (co-ideals), then the intersection $G_1 \cap G_2$ is also co-filter (co-ideal respectively) in S.

PROOF. Let $y \in \bigcup_{j \in J} K_j$ and $x \in S$. Thus, there exists an index $j \in J$ such that $y \in K_j$. Hence, by definition of co-filter, we have $x \in K_j$ or $(x,y) \in \tau$. Finally, we conclude $x \in \bigcup_{j \in J} K_j$ or $x,y) \in \tau$. therefore, the union $\bigcup_{j \in J} K_j$ is a co-filter in S too.

The second statement we derive by the following implications:
$y \in G_1 \cap G_2 \wedge x \in S \implies$
$(x \in G_1 \vee (x,y) \in \tau) \wedge (x \in G_2 \vee (x,y) \in \tau) \implies$
$x \in G_1 \cap G_2 \vee (x,y) \in \tau$.

Corresponding statements for co-ideals we can prove by analogous to proofs of previous statements. $\qquad\square$

COROLLARY 3.1. *Let S is a ordered set with apartness under co-quasiorder τ. Then the family $\mathfrak{cF}$ of all co-filters (the family $\mathfrak{cJ}$ of all co-ideals) in S forms a lattice. The greatest element in this lattice is S.*

COROLLARY 3.2. *Let S is a ordered set with apartness under co-quasiorder τ. Then the family $\mathfrak{cF} \cup \{\emptyset\}$ (the family $\mathfrak{cJ} \cup \{\emptyset\}$) in S forms an Alexandrov topology on S.*

THEOREM 3.3. *Let $f : (S, \sigma) \longrightarrow (T, \tau)$ be a reverse isotone mapping between two co-quasiordered sets. If K is a co-filter (co-ideal) in T, then the set $f^{-1}(K) = \{a \in S : f(a) \in K\}$ is a co-filter (co-ideal, respectively) in S.*

PROOF. Let $y \in f^{-1}(K)$ amd $x \in S$ be arbitrary elements. Thus, $f(y) \in K$ and $f(x) \in T$. Hence $f(x) \in K$ or $(f(x), f(y)) \in \tau$. Therefore, we have $x \in f^{-1}(K)$ or $(x,y) \in \sigma$ because f is a reverse isotone mapping.

The proof for the statement for co-ideal we can construct by analogy to previous proof. $\qquad\square$

This subsection we finish with following observation. Let S be a set with apartness ordered under a co-quasiorder τ. Mappings $i_L : S \ni s \longmapsto s\tau \in \mathfrak{cJ}$ and $i_R : S \ni s \longmapsto \tau s \in \mathfrak{cF}$ are strongly extensional functions.

3.2. Ordered co-ideals and co-filters.

DEFINITION 3.3. Let S be a set with apartness ordered under a co-quasiorder τ. Subsets of the form

$$A^L = \{z \in S : (\exists a \in A)((z,a) \in \tau)\} \quad (A \subseteq S)$$

is *ordered co-ideals* in S, while subsets of the form

$$A^R = \{z \in S : (\exists a \in A)((a,z) \in \tau)\} \quad (A \subseteq S)$$

is *ordered co-filter* in S.

Particulary, for each element $a \in S$ the set $\{a\}^L$ is the principal ordered co-ideal generated by a, and $\{a\}^R$ is the principal ordered co-filter generated by a and in addition $\{a\}^L = \tau a$ and $\{a\}^R = a\tau$ hold. Besides, $A^L = \bigcup_{a \in A} \tau a$ and $A^R = \bigcup_{a \in A} a\tau$ hold also. Therefore, by Theorem 3.2, ordered co-ideal (co-filter) is a co-ideal (co-filter).

THEOREM 3.4. *If $\{K_j\}_{j \in J}$ be a family of ordered co-filters (ordered co-ideals) in ordered set S under co-quasiorder τ, then $\bigcup_{j \in J} K_j$ is an ordered co-filter (order co-ideal respectively) too.*

PROOF. Let $\{K_j\}_{j \in J}$ be a family of ordered co-filters in order set S under co-quasiorder τ. Then for any $j \in J$ there exists a subset A_j of S such that $K_j = A_j^R$. Assertions of this theorem we get by standards way of direct checking since $(\bigcup_{j \in J} A_j)^R = \bigcup_{j \in J} A_j^R$ holds. $\square$

REMARK 3.1. Let us note that if K_1 and K_2 be two order co-filters (order co-ideals) than the intersection $K_1 \cap K_2$ is not a ordered co-filter (ordered co-ideal) in general case. For example, the intersection of two ordered co-filters $K_1 = A^R$ and $K_2 = B^R$ is a ordered co-filter if the following holds

$$(\forall a \in A)(\forall b \in B)(\exists c \in A \cap B)((c,a) \rhd \tau \wedge (c,b) \rhd \tau).$$

Indeed. For arbitrary elements $y \in A^R \cap B^R$ and $x \in S$ there exist elements $a \in A$ and $b \in B$ such that $(y,a) \in \tau$ and $(y,b) \in \tau$. There exists an element $c \in A \cap B$ such that $(c,a) \rhd \tau$ and $(c,b) \rhd \tau$ by hypothesis. Thus, we have $(y,c) \in \tau$. Further, from this follows $(y,x) \in \tau$ or $(x,c) \in \tau$ and finally we have $(y,x) \in \tau$ or $x \in (A \cap B)^R$.

COROLLARY 3.3. *The family $\mathfrak{OcF}$ (the family $\mathfrak{OcJ}$) of all ordered co-filters (ordered co-ideals) form meet semi-lattice.*

3.3. Normal co-ideals and co-filters.

DEFINITION 3.4. Let S be a set with apartness ordered under a co-quasiorder τ. Subsets of the form

$$A_L = \{z \in S : (\forall a \in A)((z,a) \in \tau)\} \quad (A \subseteq S)$$

is *normal co-ideals* in S, while subsets of the form

$$A_R = \{z \in S : (\forall a \in A)((a,z) \in \tau)\} \quad (A \subseteq S)$$

is *normal co-filter* in S.

Particulary, for each element $a \in S$ the set $\{a\}_L$ is the principal normal co-ideal generated by a, and $\{a\}_R$ is the principal normal co-filter generated by a and in addition $\{a\}_L = \tau a = \{a\}^L$ and $\{a\}_R = a\tau = \{a\}^R$ hold. Besides, $A_L = \bigcap_{a \in A} \tau a$ and $A_R = \bigcap_{a \in A} a\tau$ hold also. In addition, a normal co-ideal A_L of S is a co-ideal of S. Indeed. Let $y \in A_L$ be an arbitrary element and $x \in S$. Then from $(\forall a \in A)((z, a) \in \tau)$ we have $(\forall a \in A)((y, a) \in \tau \vee (x, a) \in \tau)$ and $(z, x) \in \tau \vee (\forall a \in A)((x, a) \in \tau)$. Therefore, the following implication $y \in A_L \wedge x \in S \Longrightarrow (z, x) \in \tau \vee x \in A_L$ is valid. We can demonstrate analogous analysis for normal co-filters, too.

Let K be a normal co-ideal (normal co-filter) of set S ordered under co-quasiorder τ. Then there exists a subset A of S such that $K = A_L$ ($K = A_R$). If z is an arbitrary element of K, we have $(\forall a \in A)((z, a) \in \tau)$ $((\forall a \in A)((a, z) \in \tau))$ and $z \rhd A$ because τ is a consistent relation. So, we have $(\forall z \in K)(z \rhd A)$.

THEOREM 3.5. *If $\{K_j\}_{j \in J}$ be a family of normal co-ideals (normal co-filters) in ordered set S under co-quasiorder τ, then $\bigcap_{j \in J} K_j$ is a normal co-ideal (normal co-filter respectively) too.*

PROOF. Let $\{K_j\}_{j \in J}$ be a family of normal co-ideals in ordered set S under co-quasiorder τ. Then for each $j \in J$ there exists a subset A_j of S such that $K_j = (A_j)_L$. Since $\bigcap_{j \in J}(A_j)_L = (\bigcup_{j \in J} A_j)_L$ holds, we conclude that the intersection $\bigcap_{j \in J} K_j$ is a normal co-ideal of S.

Proof that the intersection $\bigcap_{j \in J} K_j$ of the family $\{K_j\}_{j \in J}$ of normal co-filters is a normal co-filter we can got by analogy to previous proof.   □

REMARK 3.2. Let K_1 and K_2 be two normal co-ideals of ordered set S under co-quasiorder τ. Then there exist subsets A and B of S such that $K_1 = A_L$ and $K_2 = B_L$. Since $A_L \cup B_L \subseteq (A \cap B)_L$ is valid, we conclude that the union of two normal co-ideals is not always normal co-ideal but the union $K_1 \cup K_2$ is included in a normal co-ideal $(A \cap B)_L$. The reverse inclusion is not true because the formula

$$(\forall a \in A)((y, x) \in \tau \vee (x, a) \in \tau) \Longrightarrow (y, x) \in \tau \vee (\forall a \in A)((x, a) \in \tau)$$

can not be proven in the general case. We can demonstrate analogous analysis for normal co-filters, too.

COROLLARY 3.4. *The family $\mathfrak{NcJ}$ (the family $\mathfrak{NcF}$) of all normal co-ideals (normal co-filters) form joint semi-lattice.*

References

[1] M. Baroni, *Constructive order completeness*, UCDMS 2004/12, University of Canterbury, Christchurch, New Zealand, 2004.

[2] E. Bishop, *Foundations of Constructive Analysis*; McGraw-Hill, New York, 1967.

[3] E. Bishop and D. Bridges, *Constructive Analysis*; Grundlehren der mathematischen Wissenschaften 279, Springer, Berlin, 1985.

[4] D. Bridges and F. Richman, *Varieties of Constructive Mathematics*; London Mathematical Society Lecture Notes 97, Cambridge Univ. Press, 1987.

[5] D. S. Bridges: Constructive mathematics: A foundation for computable analysis, *Theoretical Computer Science*, **219**(1999), 95-109.

[6] D. S. Bridges and L. S. Vita, *Apartness and Uniformity: A Constructive Development*, CiE series - Theory and Applications of Computability, Springer Verlag, Berlin Heidelberg, 2011.

[7] N. Greenleaf, Linear order in lattices: A constructive study; in G.-C. Rota (ed.), *Advances in Mathematics Supplementary Studies 1* (pp. 11-30); Academic Press, New York, 1978.

[8] S. Crvenković, M. Mitrović and D. A. Romano, Semigroups with Apartness; *Math. Logic Quart.*, **59**(6)(2013), 407-414.

[9] S. Crvenković, M. Mitrović and D. A. Romano, Basic Notions of (Constructive) Semigroups with Apartness; *Semigroup Forum*, **92**(3)(2016), 659-674.

[10] R. Mines, F. Richman and W.Ruitenburg, *A Course of Constructive Algebra*; Springer, New York 1988.

[11] D. A. Romano, Rings and fields, a constructive view; *Math. Logic Quart.*, **34**(1)(1988), 25-40.

[12] D. A. Romano, Equality and coequality relations on the Cartesian product of sets; *Math. Logic Quart.*, **34**(5)(1988), 471-480.

[13] D. A. Romano, Coequality relations, a survey; *Bull.Soc.Math.Banja Luka*, **3**(1996), 1-36.

[14] D. A. Romano, A Note on Quasi-antiorder in Semigroup; *Novi Sad J. Math.*, **37**(1)(2007), 3-8.

[15] D. A. Romano, An Isomorphism Theorem for Anti-ordered Sets; *Filomat*, **22**(1)(2008), 145-160.

[16] D. A. Romano, On Quasi-Antiorder Relation on Semigroups; *Mat. vesnik*, **64**(3)(2012), 190-199.

[17] D. A. Romano. On Some Mappings between Co-quasiordered Relational Systems. *Bull. Inter. Math. Virtual Inst.*, **7**(2)(2017), 279-291.

[18] D. A. Romano. Weakly Isotone and Strongly Reverse Isotone Mappings of Relational Systems, *Bull. Inter. Math. Virtual Inst.*, **7**(2)(2017), 293-303.

[19] A. Troelstra and D. van Dalen, *Constructivism in Mathematics, An Introduction, Volume II*; North-Holland, Amsterdam 1988.

[20] J. von Plato, Positive lattices, in: P.Schuster, U. Berger, and H. Osswald (eds.), *Reuniting the Antipodes - Constructive and Nonstandard Views of the Continuum*, (pp. 185-197); Kluwer, Dordrecht, The Netherlands, 2001.

6, KORDUNAŠKA STREET, 78000 BANJA LUKA, BOSNIA AND HERZEGOVINA
E-mail address: bato49@hotmail.com