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NONSPLIT DOMINATION EDGE CRITICAL GRAPHS

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Abstract. A set of vertices S is said to *dominate* the graph G if for each $v \notin S$, there is a vertex $u \in S$ with u adjacent to v. The minimum cardinality of any *dominating set* is called the *domination number* of the graph G and is denoted by $\gamma(G)$. A *dominating set* D of a graph $G = (V, E)$ is a *nonsplit dominating set* if the induced graph $\langle V - D \rangle$ is connected. The *nonsplit domination number* $\gamma_{ns}(G)$ of the graph G is the minimum cardinality of a *nonsplit domination set*. The aim of this paper is to investigate of those graphs which are critical in the sense that: A graph G is called *edge domination critical* if $\gamma(G + e) < \gamma(G)$ for every edge e in $\overline{G}$. A graph G is called *edge nonsplit domination critical* if $\gamma_{ns}(G + e) < \gamma_{ns}(G)$ for every edge e in $\overline{G}$. Initially we verify whether some particular classes of graphs are γ_{ns} critical or not. Later 2-γ_{ns}-critical and 3-γ_{ns}-critical graphs are characterized.

1. Introduction

In this paper all our graphs will be finite, connected, undirected and without loops or multiple edges. Terminology not defined here will conform to that in [**3**]. Let $P_n, C_n, K_{1,n}, K_n, K_{m,n}$ denote the *path, cycle, star, complete and bipartite graph.*

An *end vertex* of a graph G is a vertex of *degree one* and an *support vertex* of a graph G is a vertex adjacent to end vertex. The eccentricity of the vertex v is the maximum distance from v to any vertex of G. That is

$$e(v) = max\{d(v, w); w \in V(G)\}.$$

The diameter of G is the maximum eccentricity among the vertices of G. Thus

$$diameter(G) = max\{e(v); v \in V(G)\}.$$

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A vertex $v \in V(G)$ is called a cut-vertex of a graph G, if $G - v$ is the disconnected graph. The neighborhood of a vertex in the graph G is the set of vertices adjacent to v. The neighborhood is denoted by $N(v)$ and $\kappa(G)$ is the vertex connectivity of the graph G.

A set of vertices S is said to *dominate* the graph G if for each $v \notin S$, there is a vertex $u \in S$ with u adajcent to v. The minimum cardinality of any dominating set is called the *domination number* of G and is denoted by $\gamma(G)$. The concept of *nonsplit domination* has introduced by Kulli V.R. and B. Janakiram [**5**]. A *dominating set* D of a graph $G = (V, E)$ is a *nonsplit dominating set* if the induced graph $\langle V - D\rangle$ is connected. *The nonsplit domination number* $\gamma_s(G)$ of the graph G is the minimum cardinality of a *nonsplit domination set.* The concept of *domination* has been studied by T. W. Haynes [**4**] and *domination critical graphs* has been studied by Sumner and Blitch [**7**] and Sumner [**8**] and also refer [**6, 1, 2**].

In this paper, we study the *nonsplit domination edge critical graph.* A graph G is called *edge nonsplit domination critical* if $\gamma_{ns}(G + e) < \gamma_{ns}(G)$ for every edge e in $\overline{G}$. Thus, G is k-γ_{ns} critical if $\gamma_{ns}(G) = k$ for each edge $e \in \overline{G}$, $\gamma_{ns}(G + e) < k$.

First we discuss whether some particular classes of graphs are γ_{ns}-critical or not and then 2-γ_{ns}-critical and 3-γ_{ns}-critical are characterized with respect to *diameter* of the graph G.

2. We Require the Following Theorems to Prove the Later Results

In [**5**] the following theorems has been proved.

THEOREM 2.1. *For any cycle C_n, $\gamma_{ns}(C_n) = n - 2$.*

THEOREM 2.2. *For any path P_n, $\gamma_{ns}(P_n) = n - 2, n > 3$, otherwise $\gamma_{ns}(P_n) = n - 1, n \leqslant 3$.*

THEOREM 2.3. *For any complete graph K_n, $\gamma_{ns}(K_n) = 1, n > 1$.*

3. The Main Results

THEOREM 3.1. *Let G be a connected graph. Then for any edge $e \in E(\overline{G})$*

$$\gamma_{ns}(G) - \lfloor \frac{n}{2} \rfloor + 1 \leqslant \gamma_{ns}(G + e) \leqslant \gamma_{ns}(G)$$

.

PROOF. Let D be the minimum non-split dominating set of graph G. Clearly $\gamma_{ns}(G + e) \leqslant \gamma_{ns}(G)$. For $e = v_1v_2, v_1 \in D$ and $v_2 \notin D$.

Case 1: Suppose if $d(v_2) = 2$ and if $< G - v_2 >$ is disconnected into two components G_1 and G_2 such that $n_1 + n_2 + 1 = n$. If $n_1 = n_2$ and if the graph G_1 and G_2 are complete graphs or G_1 and G_2 have atleast one vertex say $v_3 \notin N(v_2), d(v_3) = n_1$ or G_1 is complete graph and G_2 has a at least one vertex say $v_3 \notin N(v_2), d(v_3) = n_1$, then $\gamma_{ns}(G + e) = \gamma_{ns}(G) - n_1 + 1 = \gamma_{ns}(G) - \lfloor \frac{n}{2} \rfloor + 1$. Otherwise $\gamma_{ns}(G + e) > \gamma_{ns}(G) - \lfloor \frac{n}{2} \rfloor + 1$.

Case 2: Suppose $d(v_2) = 2$ and $< G - v_2 >$ is connected or $d(v_2) \geqslant 2$. If $V(G) - (D \cup N(D - v_4)) \neq \phi, v_4 \in N(v_2) \cap D$ or v_4 is end vertex, then $\gamma_{ns}(G+e) = \gamma_{ns}(G)$. Otherwise $\gamma_{ns}(G+e) < \gamma_{ns}(G)$.

Therefore from Case 1 and Case 2 , we have

$$\gamma_{ns}(G) - \lfloor \frac{n}{2} \rfloor + 1 \leqslant \gamma_{ns}(G+e) \leqslant \gamma_{ns}(G).$$

□

THEOREM 3.2. *If T is not a star, then T is not γ_{ns}-edge critical.*

PROOF. Assume that the tree $T \neq K_{1,n}$ is γ_{ns}-edge critical. Then $\gamma_{ns}(T+e) < \gamma_{ns}(T)$ for every edge $e \in E(\overline{G})$. Let $S = N \cup B \cup R$ is a vertex set of a tree T, where $N = \{v_i, v_i$ is an end vertex of a tree $T\}$, $B = \{v_j, v_j$ is an support vertex of a tree T and

$R = \{v_k, v_k$ is an neither a support vertex nor a end vertex of a tree $T\}$.

Let D be the γ_{ns} set of a tree T. we consider the following cases:

Case 1: If every vertex of a tree T is adjacent to an end vertex. Then $\gamma_{ns}(G) = N$. Now consider the graph $G + e, e = v_1v_2, v_1 \in N$ and $v_2 \in B$. Then v_2 dominates $N(v_2)$. Let $A = \{D - N(v_2)\} \cup v_2$. Then $< A >$ is disconnected. Therefore $\gamma_{ns}(G+e) = |D| = \gamma_{ns}(G)$, which is a contradiction.

Case 2: If atleast one vertex of a tree T is not adjacent to an end vertex say v_1. Now consider the graph $G + e, e = v_1v_2, v_2 \in B$ and $v_1 \in N$. Then either we can remove v_1 or v_2 if $v_2 \in D$ or remove $N(v_2)$ if $v_2 \notin D$ from D. Removal of v_1 from D, then there exists atleast one vertex say v_k which is not covered by any of the vertex of $(D - v_1)$ or the graph $G + e$ is disconnected, otherwise removal of v_2 from D makes the graph $G + e$ disconnected or otherwise removal of $N(v_2)$, Since $N(v_2)$ is a support vertex, $N(v_2) \in D$. Therefore $\gamma_{ns}(G+e) = |D| = \gamma_{ns}(G)$, which is a contradiction.

From the above cases, we can say that the tree T is not γ_{ns}-edge critical, if T is not a star. □

THEOREM 3.3. *The graph $G = C_n$, $n \geqslant 4$ is γ_{ns}-edge critical for nonsplit domination.*

PROOF. Let us consider the graph $G = C_n$ and $G + e$ where $e \in \overline{G}$ is a graph consists of two cycles $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $n_1 + n_2 - 2 = n$ such that $|V_1| \leqslant |V_2|$. Let $A = V(G_1) \cap V(G_2) = \{v_i, v_j\}$. We consider the following cases:

Case 1: If G_1 and G_2 are the cycles of length 3, then $G = C_4$ and $\gamma_{ns}(G) = 2$. Then $\gamma_{ns}(G+e) = |v_i| = 1$, where $v_i \in A$. Therefore $\gamma_{ns}(G+e) < \gamma_{ns}(G)$.

Case 2: If G_1 and G_2 are the cycles of length 4, then $G = C_6$ and $\gamma_{ns}(G) = 4$. Let D_2 be the nonsplit dominating set of the graph $G + e, e \in \overline{G}$. Then $D_1 = \{v_r, v_s\}$ where $v_r \in N(v_i) \cap V(G_1), v_s \in N(v_j) \cap V(G_2)$. So that $\gamma_{ns}(G + e) = |D_1| = 2$. Therefore $\gamma_{ns}(G + e) < \gamma_{ns}(G)$.

Case 3: If G_1 and G_2 are the cycles of length 3 and length 4, then the graph $G = C_n$ will be C_5 and $\gamma_{ns}(G) = 3$. Let D_2 be the nonsplit dominating set of the graph $G+e, e \in \overline{G}$. Then $D_2 = \{v_r, v_s\}$, where $v_r \in V(G_1)-A, v_s \in V(G_2) - A$. Then $\gamma_{ns}(G+e) = |D_2| = 2$. Therefore $\gamma_{ns}(G+e) < \gamma_{ns}(G)$.

Case 4: If G_1 and G_2 are the cycles of length $\geqslant 3$ and length > 4, then $\gamma_{ns}(G) = n - 2$. Let D_3 be the nonsplit dominating set of the graph $G + e, e \in \overline{G}$. Then $D_3 = B \cup C\{v_r, v_l\}$, where $\{(v_r, v_l)\} \in N(A) \cap V(G_2)$, $B = \{v_s/v_s \in V(G_1) - A\}, B = \{v_m/v_m \in V(G_2) - A\}$. Then
$\gamma_{ns}(G + e) = (n_1 - 2) + (n_2 - 2) - 2$
$=n_1 + n_2 - 2 - 4 = n - 4$
since $n - 4 < n - 2$, therefore $\gamma_{ns}(G + e) < \gamma_{ns}(G)$.

The result follows from the above cases. □

Theorem 3.4. *The graph $G = P_n$,$n > 3$ is not γ_{ns}-edge critical for nonsplit domination.*

Proof. Let D be the γ_{ns} set of the graph G and let $G + e$ be the graph where $e \in \overline{G}$. we consider the following cases.

case 1: If $e \in \overline{G}$ joins $\{v_1, v_2\} \in D$ and $v_2 \neq N(v_1)$,then either we can remove v_1 or v_2 from D, then either there exists atleast one vertex say v_k which is not covered by any of the vertex of $(D - (v_1 \text{ or } v_2))$ or the graph G is disconnected. Therefore $\gamma_{ns}(G + e) = |D| = \gamma_{ns}(G)$.

Case 2: If $e \in \overline{G}$ joins $v_1 \in D, v_2 \notin D, v_2$ is a not support vertex, then we can remove $v_r, v_r \in N(V(T) - D), v_r \in D, v_r$ covers v_2. Then $\gamma_{ns}(G + e) = |D - 1| < \gamma_{ns}(G)$. Otherwise if v_2 is a support vertex, then removal of $v_r, v_r \in N(v_2) \cap D$, then v_r is not dominated by any of the vertex of $D - v_r$. Therefore $\gamma_{ns}(G + e) = |D|$. Hence $\gamma_{ns}(G + e) = \gamma_{ns}(G)$.

The result follows from the above cases. □

Lemma 3.1. *K_n is not γ_{ns}-edge critical for $n \geqslant 2$.*

Lemma 3.2. *$K_{m,n}$ is not γ_{ns}-edge critical for $m, n \geqslant 2, m, n \neq 2$ and γ_{ns}-critical for $m, n = 2$.*

Lemma 3.3. *$K_{1,n}$ is γ_{ns}-edge critical for $n \geqslant 3$.*

Theorem 3.5. *A connected graph G is $2 - \gamma_{ns}$-edge critical if and only if $\overline{G} = \cup_{i=1}^{i=m} K_{1,m_i}$ for $m_i \geqslant 1$ and $m \geqslant 2$.*

Proof. If G is a connected $2-\gamma_{ns}$-critical graph, then for any edge $e \in E(\overline{G})$, say $e = ab$, we have $\gamma_{ns}(G + e) = 1$. Thus, it follows that $\{a\}$ dominates $G + e$ and so a is an isolated vertex of $\overline{G} - e$. Hence, we have shown that every edge of

G is incident with an end vertex of $\overline{G}$. Since G is a connected graph, it follows that $\overline{G} = \cup_{i=1}^{1=m}$ for $m_i \geqslant 1$ and $m \geqslant 2$. Now, we prove the sufficiency condition, if $\overline{G} = \cup_{i=1}^{i=m} K_{1,m_i}$ for $m_i \geqslant 1$ and $m \geqslant 2$ then it is obvious that no vertex can dominate G. Hence, $\gamma_{ns}(G) > 1$. Let b be an end vertex of $\overline{G}$ and a be a center vertex of $\overline{G}$ and $ab \notin E(\overline{G})$. Then $\{a, b\}$ is a nonsplit dominating set of G. Hence, $\gamma_{ns}(G) \leqslant 2$, that is, $\gamma_{ns}(G) = 2$. For arbitrary $e = ab \in E(\overline{G})$, assume that b is an end vertex and a is a center. It is clear that $d(G + e)(b) = 1$ and $\gamma_{ns}(G + e) = 1$. So, G is a connected $2 - \gamma_{ns}$-edge critical graph. □

THEOREM 3.6. *If G is γ_{ns}-edge critical with n vertices, then there is no support vertex of degree at most $n - 2$ in G.*

PROOF. Assume that the the graph G is γ_{ns}-edge critical in which the degree of the support vertex say v is at most $n - 2$ which is adjacent to an end-vertex say x of a graph G. Since the degree of v is atmost $n - 2$ there exists atleast one vertex say $v_1, v_1 \notin N(v)$. Let D and D_1 be the minimum non-split dominating set of the graph G and $G_1 = G + e, e \in E(\overline{G})$. Since v is support vertex then either $v \in D$ or $v \notin D$ and x is a support vertex $x \in D_1$. we consider the following cases:

Case 1: If $v \notin D$, then consider the graph $G_1 = G + e, e = vv_1$. If $v_1 \notin D$, then clearly $\gamma_{ns}(G_1) = \gamma_{ns}(G)$. Otherwise if $v_1 \in D$, then we can remove $N(v) \in D$, then there exists atleast one vertex say v_k which is not dominated by any vertex of $[D - N(v)]$. Therefore $\gamma_{ns}(G_1) = \gamma_{ns}(G)$ which is a contradiction.

Case 2: If $v \in D$, then there exists atleast one vertex say $v_3 \notin D$. Then consider the graph $G_1 = G + e, e = xv_3$, $v_3 \in N(V(G) - D)$, then we can remove $N(v_3)$ or x from D, then there exists atleast one vertex say v_k which is not dominated by any vertex of $(D - v_3)$ or the graph G_1 disconnected. Therefore $\gamma_{ns}(G_1) = \gamma_{ns}(G)$, which is a contradiction.

Hence the proof. □

THEOREM 3.7. *For a graph $G \neq K_{1,n}$ with n vertices, if:*

(i) $\kappa(G) = 1$.

(ii) *$G - v$ has exactly two components.*

(iii) $d(v) = n - 1$.

Then, G is γ_{ns}-edge critical.

PROOF. Let us consider the graph G with $\kappa(G) = 1$ and let D be the γ_{ns} set of the graph G. Let v be the cut-vertex of the graph G. If $G - v$ has two components G_1 and G_2 and $d(v) = n - 1$, then $\gamma_{ns}(G) \geqslant 2$. Now consider the graph $G_1 = G + e, e = v_1v_2, v_1 \in V(G_1), v_2 \in V(G_2)$, then $\gamma_{ns}(G_1) = |v| = 1$. Therefore $\gamma_{ns}(G_1) < \gamma_{ns}(G)$. Hence it is γ_{ns}-edge critical. □

THEOREM 3.8. *Let G be a connected $2 - \gamma_{ns}$ and $3 - \gamma_{ns}$-edge critical graph, then $dia(G) = 2$.*

PROOF. we consider the following cases:

Case 1: Let G be connected $2-\gamma_{ns}$-edge critical graph and suppose G has a diameter atleast 3. Assume that $p = v_1, v_2, ..., v_d$ is a longest path with the diameter equal to the diameter of the graph G. Let D be the γ_{ns} set of $G+v_1v_d$. Since G is a connected $2-\gamma_{ns}$ critical graph, $\gamma_{ns}(G+v_1v_d) = 1$. If suppose $v_1 \in D$ then the vertex v_{d-1} cannot be dominated by v_1 because $v_{d-1}v_1$ is at a distance of 2 which is a contradiction. Otherwise if $v_d \in D$, the vertex v_2 cannot be dominated by v_d because v_dv_2 is at a distance of 2 which is a contradiction. Therefore for a $2-\gamma_{ns}$-edge critical graph the $dia(G) \leqslant 2$. If $dia(G) = 1$, then G is not γ_{ns}-edge critical. Therefore $dia(G) = 2$.

Case 2: Let G be a connected $3-\gamma_{ns}$-edge critical graph and suppose G has a diameter atleast 3. Assume that $p = v_1, v_2, ..., v_d$ is a longest path with the diameter equal to the diameter of the graph G. Since G is a connected $3-\gamma_{ns}$critical graph, $\gamma_{ns}(G+v_1v_d) \leqslant 2$. Let D be the γ_{ns} set of the $G_1 = (G+v_1v_d)$. The set D has to contain two vertices say $v_i, v_j \in p$. Since G is a connected $3-\gamma_{ns}$critical graph, $\gamma_{ns}(G+v_1v_d) \leqslant 2$.If $(v_i, v_j) \in D$ then there exists atleast one vertex say v_k cannot be dominated by any of the vertex of D, since v_k is at a distance of 2 from (v_i, v_j) or $G_1 - (v_i, v_j)$ results a disconnected graph, which is a contradiction. Therefore for a $3-\gamma_{ns}$critical graph the $dia(G) \leqslant 2$. If $dia(G) = 1$, then G is not γ_{ns}-edge critical. Therefore $dia(G) = 2$.

□

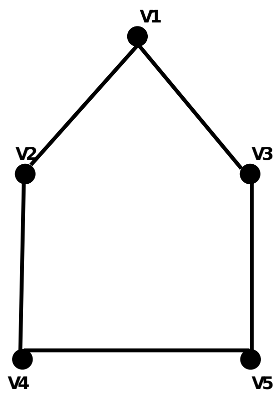

FIGURE 1. A $3-\gamma_{ns}$-edge critical graph with diameter =2

4. Construction of $2-\gamma_{ns}$-critical graph and $3-\gamma_{ns}$-critical graph

1. Construction of $2-\gamma_{ns}$-critical graph

(i) A graph in which $n-1$ vertices of degree $n-2$ and n^{th} vertex is of degree 2 is always a critical graph.

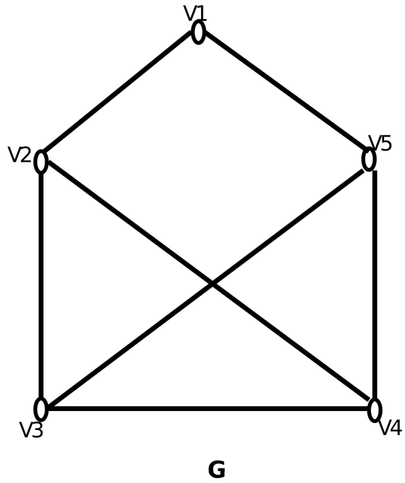

FIGURE 2. $\gamma_{ns}(G) = \{v_3, v_2\} = 2, \gamma_{ns}(G + v_3v_1) = \{v_3\} = 1$

(ii) Let us consider the graph $G_1 = K_n$ and $G_2 = K_2 = u_1u_2$, then we can the construct the critical graph G with,
   (a) the vertex set $V(G) = V(G_1) \cup V(G_2), v_1 = u_1, v_1$ is any vertex G_1.
   (b) the edge set $E(G) = E(G_1) \cup E(G_2)$.

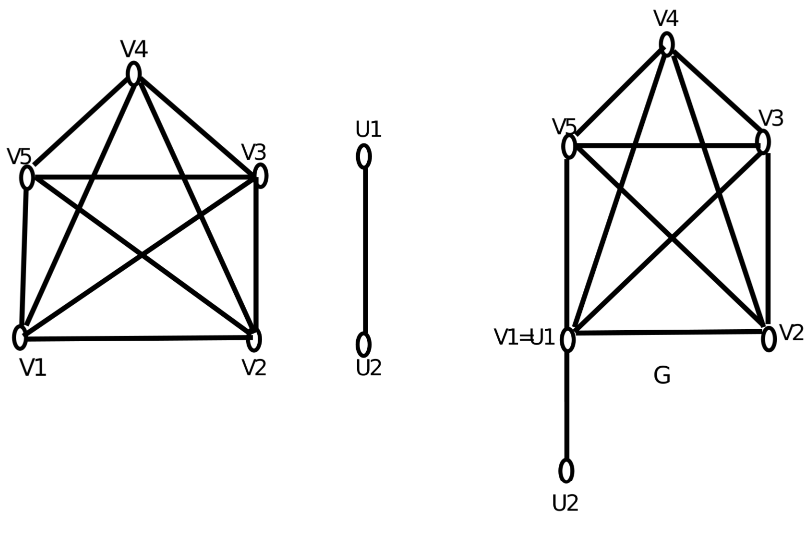

FIGURE 3. $\gamma_{ns}(G) = \{v_1, u_2\} = 2, \gamma_{ns}(G + u_2v_2) = \{v_1\} = 1$

2. Construction of $3-\gamma_{ns}$-critical graph.
Let the consider graph G_1 in which $n-1$ vertices of degree $n-2$ and n^{th} vertex is of degree 2 and $G_2 = K_2 = u_1u_2$. Then we can construct the critical graph G with

(1) the vertex set $V(G) = V(G_1) \cup V(G_2), v_1 = u_1, v_1 \in G_1, d(v_1) = n-2$.
(2) $E(G) = E(G_1) \cup E(G_2)$.

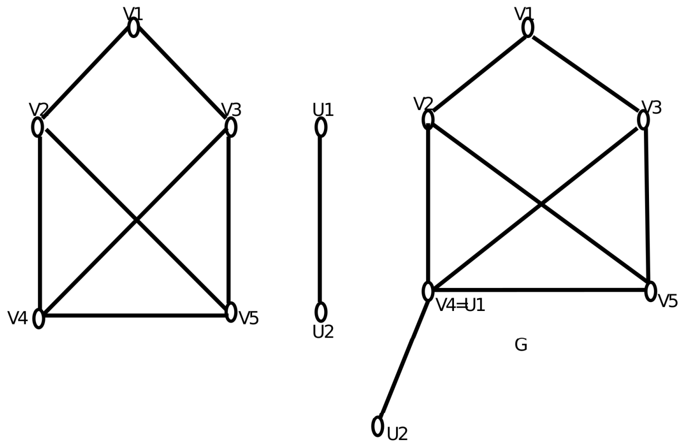

FIGURE 4. $\gamma_{ns}(G) = \{v_4, v_2, u_2\} = 3, \gamma_{ns}(G + v_4v_1) = \{v_4, u_2\} = 2$

References

[1] R. C. Brigham, P. Z. Chinn and R. D. Dutton. Vertex domination critical graphs. *Networks*, **18**(3)(1988), 173-179.
[2] X-G. Chen, L. Sun and De-X. Ma. Connected domination critical graphs. *Appl. Math. Letters*, **17**(5)(2004), 503-507.
[3] F. Harary. *Graph theory.* Addison-Wesley Publishing Co. Inc., Reading, Mass., 1969.
[4] T. W. Haynes, S. T. Hedetniemi and P. J. Slater. *Fundamental of domination of graphs.* Marcel Dekker Inc., New York, 1998.
[5] V. R. Kulli and B.Janikiram. Nonsplit domination number of a graph. *Indian J. Pure Appl. Math.*, **31**(4)(2000), 441-447.
[6] M. Lemanska and A. Patyk. Weakly connected domination critical graphs. *Opuscula Mathematica*, **28**(3)(2008), 325-330.
[7] D.P.Summer and P.Blitch. Domination critical graphs. *Journal of combinatorial theory series B*, **34**(1)(1983), 65-76.
[8] D.P.Sumner. Critical concepts in domination. *Discrete Math.*, **86**(1-3)(1990), 33-46.

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