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LOCAL CONNECTIVE CHROMATIC NUMBER OF DIRECT PRODUCT OF PATHS AND CYCLES

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Abstract. Graph coloring is one of the most important concept in graph theory. There are many types of coloring. We study on the local connective chromatic number of a graph G that is defined by us. In this paper, we determine the local connective chromatic number of the direct product of two paths $P_m \times P_n$, two cycles $C_m \times C_n$ and for the direct product of a cycle and a path $C_m \times P_n$, where m and n are the number of vertices.

1. Introduction

Let G be a simple undirected graph, where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. For two vertices $u, v \in V(G)$, u and v are adjacent if they are joined by an edge. Two vertices that are not adjacent in a graph G are said to be *independent*. The *independence number* $\beta(G)$ of a graph G is the maximum cardinality among the independent sets of vertices of G. For the notations and terminology not defined here, we follow [**6**].

The *connectivity* $\kappa = \kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. Two paths are *internally disjoint (vertex disjoint)* if they do not share a common vertex except their end vertices. The *local connectivity* $\kappa_G(u, v) = \kappa(u, v)$ between two distinct vertices u and v of a graph G is defined as the smallest number of vertices whose removal separates u and v. By Menger' s theorem [**14**], $\kappa(u, v)$ equals the maximum number of internally disjoint $u - v$ paths in G and $\kappa(G) = \min\{\kappa(u, v) : u, v \in V(G)\}$. It is straightforward to verify that $\kappa(G) \leqslant \delta(G)$ and $\kappa(u, v) \leqslant \min\{\deg(u), \deg(v)\}$ [**16**].

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The local connective coloring is defined by us by inspiring the notion of packing coloring [**5, 9, 12, 18**].

Routing is the process of delivering messages among vertices and selecting the best paths in a network. Efficiency and reliability of routing can be achieved by using internally disjoint paths because the failure of a path would not affect the performance of other paths. Then the more internally disjoint paths are the better for a network [**13**]. Thus, we use the term internally disjoint path in our coloring and color the vertices depending on the number of internally disjoint paths between two vertices.

A graph G which has a local connective k- coloring can be partitioned into disjoint color classes $X_1, X_2, ..., X_k$ and can be drawn as a k-partite graph. Thereby, the graph is partitioned into the subsets which have disjoint paths. Looking for a secure disjoint path between two vertices u and v in any color class X_i, we make this search with the vertices in the other color classes. This indicate that we look for disjoint paths starting from u and ending to v using the vertices in the other color classes. Thus, this search can be made with $V(G) - (|X_i| - 2)$ vertices. Thereby, NP-complete problem can be solved more easily. Local connective coloring provides to facilitate the routing of non–adjacent vertices to communicate with each other.

The *direct product* $G \times H$ of two graphs G and H is a graph with $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G) \text{ and } v_1v_2 \in E(H)\}$. It is also known as Kronecker product, tensor product, categorical product and graph conjunction. This graph product is commutative and associative [**3**]. The direct product of graphs has been extensively investigated concerning graph recognition and decomposition, graph embeddings, matching theory and stability in graphs [**1, 4**]. More generally, the direct product is a widely used tool in the area of graph colorings [**11**].

LEMMA 1.1. [**17**] *Let G be a connected graph. If G has no odd cycle, then $G \times K_2$ has exactly two connected components isomorphic to G.*

THEOREM 1.1. [**17**] *Let G and H be connected graphs. The graph $G \times H$ is connected if and only if any G or H contains an odd cycle.*

COROLLARY 1.1. [**17**] *If G and H are connected graphs with no odd cycles then $G \times H$ has exactly two connected components.*

THEOREM 1.2. [**15**] *Let $G = (V, E)$ be a connected graph, and $H = (V_1, V_2, E^{'})$ be a bipartite connected graph, then $G \times H$ is a bipartite graph, the partition of the vertex set is $(V \times V_1)$ and $(V \times V_2)$.*

THEOREM 1.3. [**8**] *The direct product of two connected graphs is a non–connected graph if and only if both are bipartite.*

LEMMA 1.2. [**10**] *If $G = (V_0 \cup V_1, E)$ and $H = (W_0 \cup W_1, F)$ are bipartite graphs, then $(V_0 \times W_0) \cup (V_1 \times W_1)$ and $(V_0 \times W_1) \cup (V_1 \times W_0)$ are vertex sets of the two components of $G \times H$.*

LEMMA 1.3. [**10**] *If G is a connected, bipartite graph and $n \geqslant 4$ is an even integer, then the graph $G \times C_n$ consists of two isomorphic connected components.*

THEOREM 1.4. [**2**] *If G and H are regular graphs then $G \times H$ is also a regular graph.*

2. Local Connective Chromatic Number of Direct Product Graphs

DEFINITION 2.1. *A local connective k-coloring of a graph G is a mapping $c : V(G) \longrightarrow \{1, 2, ..., k\}$ such that*

(1) *If $uv \in E(G)$, then $c(u) \neq c(v)$, and*
(2) *If $uv \notin E(G)$ and $\kappa(u, v) \geqslant i$, then $c(u) = c(v) = i$, where $\kappa(u, v)$ is the maximum number of internally disjoint paths between u and v.*

The smallest integer k for which there exists a local connective k- coloring of G is called the local connective chromatic number of G, and it is denoted by $\chi_{lc}(G)$.

The first condition characterizes proper coloring. Thus, every local connective coloring is a proper coloring.

The vertices of G are partitioned into disjoint color classes $X_1, X_2, ..., X_k$, where each color class X_i consists of distinct vertices $u, v \in X_i$ such that $\kappa(u, v) \geqslant i$ and $\bigcup_{i=1}^{n} X_i = V(G)$. The maximum cardinality of X_i in G is denoted by k_i.

In this section, we give local connective chromatic number of direct product of paths and cycles. Let G and H be any two graphs with vertex sets $V(G) = \{u_1, u_2, ..., u_m\}$, $V(H) = \{v_1, v_2, ..., v_n\}$, respectively. A vertex (u_i, v_j) is abbreviated as w_{ij}, where $w_{ij} \in V(G \times H)$, $i \in \{1, 2, ..., m\}$, $j \in \{1, 2, ..., n\}$.

THEOREM 2.1. *Let P_m and P_n be two paths of order m and n, respectively. Then, $4 \leqslant \chi_{lc}(P_m \times P_n) \leqslant \max\{2n + 5, m + n + 7\}$ for $m \leqslant n$.*

PROOF. It is known that paths are bipartite graphs. By Theorem 1.3, the graph $P_m \times P_n$ is non–connected for m and n being odd or even. Further by Corollary 1.1, $P_m \times P_n$ has two connected components as G_1 and G_2. Since $\delta(G_1) = \delta(G_2) = 2$ and $\Delta(G_1) = \Delta(G_2) = 4$, we have $2 \leqslant \kappa(w_{ij}, w_{kl}) \leqslant 4$, where $w_{ij}, w_{kl} \in V(G_1)$(or $V(G_2)$) , $i, k \in \{1, 2, ..., m\}$, $j, l \in \{1, 2, ..., n\}$. Thus, $k_i \leqslant 1$ for $i \geqslant 5$. That is, the pair of vertices can be colored with the same color at most color 4, and the remaining uncolored vertices receive different colors.

We prove this theorem in four cases for m and n being odd or even.

Case 1. Let m and n be odd.

Since $|V(P_m \times P_n)| = mn$, we have $|V(G_1)| = \lceil \frac{mn}{2} \rceil$, $|V(G_2)| = \lfloor \frac{mn}{2} \rfloor$. The graph $P_m \times P_n$ has four vertices of each of degree one in the only one component. Assume that these vertices be in the component G_1. Since there is one internally disjoint path between these vertices, they can be colored with color 1. The vertex w_{ij} can be colored with color 1, where $i \in \{1, 3, ..., m\}$, $j \in \{1, 3, ..., n\}$. Thus, $\beta(G_1) = \lceil \frac{n}{2} \rceil \lceil \frac{m}{2} \rceil$ vertices in G_1 are colored with color 1, and $\lfloor \frac{n}{2} \rfloor \lfloor \frac{m}{2} \rfloor$ vertices in G_1 remain uncolored. $\lfloor \frac{n}{2} \rfloor \lfloor \frac{m}{2} \rfloor < \lfloor \frac{mn}{2} \rfloor$ and thus color the graph G_2 with color 2. When we start coloring from the vertex w_{12} and color all vertices which are not adjacent with each other in G_2, then maximum $\lfloor \frac{n}{2} \rfloor \lceil \frac{m}{2} \rceil$ vertices in G_2 are colored

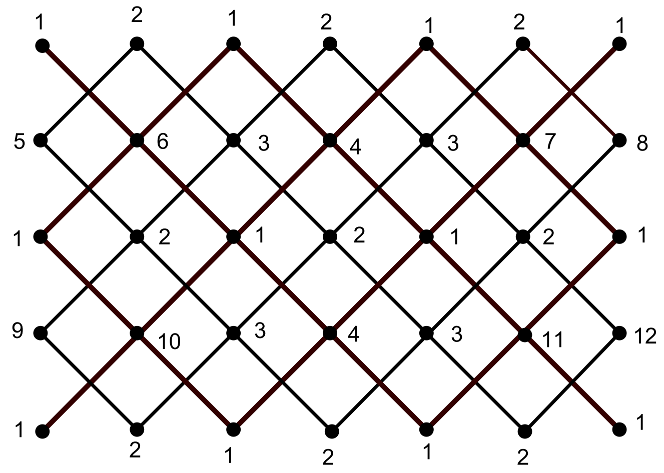

FIGURE 1. Local connective coloring of $P_5 \times P_7$

with color 2. Thus, there are $\lceil \frac{n}{2} \rceil \lfloor \frac{m}{2} \rfloor$ and $n\lfloor \frac{m}{2} \rfloor$ uncolored vertices in G_2 and G, respectively.

Case 1.1. Let $m = 3$ and $n \geqslant 3$.

For all vertices w_{ij}, w_{kl} in G_1 or G_2, we have $\kappa(w_{ij}, w_{kl}) \leqslant 2$, where $i, k \in \{1, 2, 3\}$, $j, l \in \{1, 2, ..., n\}$. Thus, there is not any vertex colored with color 3 and color 4. Hence, the remaining $n\lfloor \frac{m}{2} \rfloor$ vertices are colored with different colors. Then we have $\chi_{lc}(P_3 \times P_n) = 2 + n\lfloor \frac{m}{2} \rfloor = n + 2$.

Case 1.2. Let $m = 5$ and $n \geqslant 5$.

Since $\kappa(G_1) = 1$, $\kappa(G_2) = 2$, we have $\kappa(w_{ij}, w_{kl}) \geqslant 2$, where $w_{ij}, w_{kl} \in V(G_2)$. Four vertices in $P_5 \times P_n$ are colored with color 3. Color the vertex w_{ij} in G_2 with color 3 for the minimum local connective coloring number, where $i \in \{2, 4\}$, $j \in \{3, 5\}$.

Case 1.2.1. If $m = n = 5$, no vertices that remain uncolored in G_1 or G_2 are colored with color 4. Thus, the remaining vertices receive different colors and we have $\chi_{lc}(P_5 \times P_5) = 3 + n\lfloor \frac{m}{2} \rfloor - 4 = 9$.

Case 1.2.2. If $m = 5$ and $n \geqslant 7$, the vertex w_{i4} for $i \in \{2, 4\}$ in G_1 is colored with color 4. The remaining $n\lfloor \frac{m}{2} \rfloor - 6 = 2n - 6$ vertices receive different colors. Thus, we have $\chi_{lc}(P_5 \times P_n) = 2n - 2$.

Case 1.3. Let $m \geqslant 7$ and $n \geqslant 7$.

Take the vertex w_{ij} in G_1, where $i \in \{2, 4, ..., m-1\}, j \in \{2, 4, ..., n-1\}$. The number of internally disjoint paths between these vertices is at most 4. Thus, maximum $(\frac{m-1}{2})(\frac{n-1}{2})$ vertices are colored with color 3, and there is no vertex in G_1 remains uncolored.

Case 1.3.1. If $m = n = 7$, the number of internally disjoint paths between only two vertices in G_2 is 4. Hence, these vertices receive color 4, and we have $\chi_{lc}(P_7 \times P_7) = 4 + \lfloor \frac{m}{2} \rfloor \lceil \frac{n}{2} \rceil - 2 = 14$.

Case 1.3.2. Let $m = 7$ and $n \geqslant 9$. Since $m = 7$, there are four vertices in G_2 that the number of internally disjoint paths between them is 4. Thus, these four vertices receive color 4, and we have $\chi_{lc}(P_7 \times P_n) = 4 + \lfloor \frac{m}{2} \rfloor \lceil \frac{n}{2} \rceil - 4 = \frac{3}{2}(n+1)$.

Case 1.3.3. Let $m \geqslant 9$ and $n \geqslant 9$.

Take the vertices w_{ij}, where $i \in \{4, 6, 8, ..., m-3\}, j \in \{3, 5, 7, ..., n-2\}$ and w_{kl}, where $k \in \{2, m-1\}, l \in \{5, 7, 9, ..., n-4\}$ in G_2. Since the number of internally disjoint paths between them is 4, $\lfloor \frac{m-4}{2} \rfloor (\frac{n-3}{2}) + 2\lfloor \frac{n-6}{2} \rfloor$ vertices are colored with color 4. Hence, the remaining $m+3$ vertices receive $m+3$ different colors, and we have $\chi_{lc}(P_m \times P_n) = 4 + m + 3 = m + 7$.

Consequently, if m and n are odd, then we have

$$\chi_{lc}(P_m \times P_n) = \begin{cases} n+2, & m = 3, n \geqslant 3 \\ 9, & m = n = 5 \\ 2n-2, & m = 5, n \geqslant 7 \\ 14, & m = n = 7 \\ \frac{3}{2}(n+1), & m = 7, n \geqslant 9 \\ m+7, & m \geqslant 9, n \geqslant 9. \end{cases}$$

Case 2. Let m be even and n be odd.

$|V(P_m \times P_n)| = mn$ and $|V(G_1)| = |V(G_2)| = \frac{mn}{2}$. The graph $P_m \times P_n$ has four vertices of each of degree one. Two of them are in G_1 and the other two vertices are in G_2. Assume that we start coloring from the vertex w_{11} in G_1. Assign color 1 to every vertex when i and j are odd. Thus, $\beta(G_1) = \frac{m}{2}\lceil \frac{n}{2} \rceil$ vertices are colored with color 1. The graph G_1 has $\lfloor \frac{n}{2} \rfloor$ uncolored vertices of degree 2 and $\lfloor \frac{n}{2} \rfloor (\frac{m}{2} - 1)$ uncolored vertices each of degree 4. Then the total number of the remaining uncolored vertices in G_1 is $\lfloor \frac{n}{2} \rfloor \frac{m}{2}$.

Start coloring the graph G_2 with color 2 for the minimum number of local connective coloring. Assume that we start coloring from the vertex w_{21}. The graph G_2 has total $\frac{mn}{2} - 2$ vertices of degree two and four, and since the number of internally disjoint paths between these vertices is at least 2, each vertex which is not adjacent with each other is colored with color 2. Thus, the vertices w_{ij}, where $i \in \{2, 4, 6, ..., m-2\}$, $j \in \{1, 3, 5, ..., n\}$ and w_{nl}, where $l \in \{3, 5, 7, ..., n-2\}$ are colored with color 2. Then maximum $\lceil \frac{n}{2} \rceil (\frac{m-2}{2}) + \frac{n-3}{2} = \frac{m}{2}\lceil \frac{n}{2} \rceil - 2$ vertices in G_2 receive color 2. Among the remaining $2 + \lfloor \frac{n}{2} \rfloor \frac{m}{2}$ vertices in G_2, two of them have degree one, $\lfloor \frac{n}{2} \rfloor$ of them are each of degree 2 and $\lfloor \frac{n}{2} \rfloor (\frac{m}{2} - 1)$ of them are each of degree 4. Thus, there are $2 + \lfloor \frac{n}{2} \rfloor m$ vertices in the graph $P_m \times P_n$ remain uncolored.

Case 2.1. Let $m = 2$ and $n \geqslant 3$.

In this case, $P_n \cong G_1$, $P_n \cong G_2$ and $P_2 \times P_n \cong 2P_n$. Since $\chi_{lc}(P_n) = 1 + \lfloor \frac{n}{2} \rfloor$

by [**7**], we get

$$\chi_{lc}(P_2 \times P_n) = n + \chi_{lc}(P_n) = n + 1 + \lfloor \frac{n}{2} \rfloor = \frac{3n+1}{2}.$$

Case 2.2. Let $m = 4$ and $n \geqslant 5$.

Since $\kappa(w_{ij}, w_{kl}) \leqslant 2$, where $w_{ij}, w_{kl} \in V(G_1)$(or $V(G_2)$) there is not any vertex colored with color 3 and 4. Hence, the remaining $m\lfloor \frac{n}{2} \rfloor + 2 = 2n$ vertices receive different colors. Then we have$\chi_{lc}(P_4 \times P_n) = 2n + 2$.

Case 2.3. Let $m \geqslant 6$ and $n \geqslant 7$.

Every pair of vertices in G_1 each of degree 4 satisfy the condition $\kappa(w_{ij}, w_{kl}) \geqslant 3$, where $i, k \in \{2, 4, ..., m-2\}$, $j, l \in \{2, 4, ..., n-1\}$. Thus, $\lfloor \frac{n}{2} \rfloor(\frac{m}{2} - 1)$ vertices are colored with color 3. The number of the remaining vertices in G_1 is $\lfloor \frac{n}{2} \rfloor$ and each of them has degree two. Further, these vertices satisfy the condition $\kappa(w_{mj}, w_{ml}) \leqslant 2$, where $j, l \in \{2, 4, ..., n-1\}$,$j \neq l$. Then they receive different colors. Hence, we have $\chi_{lc}(G_1) = 2 + \lfloor \frac{n}{2} \rfloor$.

Consider coloring the graph G_2. For the vertices w_{ij}, where $i \in \{5, 7, 9, ..., m-3\}$, $j \in \{2, 4, 6, ..., n-1\}$ and w_{kl}, where $k \in \{3, m-1\}$, $l \in \{4, 6, 8, ..., n-3\}$, the number of internally disjoint paths between them are 4 and so $\lfloor \frac{m-5}{2} \rfloor(\frac{n-1}{2}) + 2\lfloor \frac{n-4}{2} \rfloor$ vertices are colored with color 4. The remaining $\frac{n+11}{2}$ vertices in G_2 receive different colors. Thus, $\chi_{lc}(G_2) = 2 + \frac{n+11}{2}$, and we have

$$\chi_{lc}(P_m \times P_n) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = n + 9.$$

Consequently, if m is even and n is odd, then we have

$$\chi_{lc}(P_m \times P_n) = \begin{cases} \frac{3n+1}{2}, & m = 2, n \geqslant 3 \\ 2n + 2, & m = 4, n \geqslant 5 \\ n + 9, & m \geqslant 6, n \geqslant 7. \end{cases}$$

Case 3. Let m be odd and n be even.

The graph $P_m \times P_n$ has four vertices of each of degree one. Two of them are in G_1 and the other two vertices are in G_2. If we color the vertices of G_1 starting from the vertex w_{11} as Case 2, $\frac{n}{2}\lceil \frac{m}{2} \rceil$ vertices receive color 1. For color 2, assume that we start coloring from the vertex w_{12} in G_2. The vertices w_{ij}, where $i \in \{3, 5, 7, ..., m-2\}$, $j \in \{2, 4, 6, ..., n\}$ and w_{kl}, where $k \in \{1, m\}$, $l \in \{2, 4, 6, ..., n-2\}$ are colored with color 2. There are at most $\frac{(m-3)n}{4} + \frac{2(n-2)}{2} = \lceil \frac{m}{2} \rceil \frac{n}{2} - 2$ vertices which receive color 2. Thus, $\lfloor \frac{m}{2} \rfloor n + 2$ vertices remain uncolored in $P_m \times P_n$.

Case 3.1. Let $m = 3$ and $n \geqslant 4$.

In this case, $\kappa(w_{ij}, w_{kl}) \leqslant 2$, where $w_{ij}, w_{kl} \in V(G_1)$(or $V(G_2)$), $i, k \in \{1, 2, 3\}$, $j, l \in \{1, 2, ..., n\}$. Thus, there is not any pair of vertices that is colored with color 3 or color 4. Then we have

$$\chi_{lc}(P_3 \times P_n) = 2 + \lfloor \frac{m}{2} \rfloor n + 2 = n + 4.$$

Case 3.2. Let $m = 5$ and $n \geqslant 6$.

If $\kappa(w_{ij}, w_{kl}) \geqslant 3$ or $\deg(w_{ij}) \geqslant 3$, where $i, k \in \{1, 2, ..., m\}$, $j, l \in \{1, 2, ..., n\}$, any two vertices w_{ij} and w_{kl} can be colored with color 3. Thus, there are $\lfloor \frac{m}{2} \rfloor(\frac{n}{2} - 1)$

vertices in G_1 each of degree 4 that can be colored with color 3. Since $m = 5$, only four vertices of them are colored with color 3.

Case 3.2.1. If $m = 5$ and $n = 6$, there is not any pair of vertices which receives color 4. Hence, the remaining vertices are colored with different colors, and so we have $\chi_{lc}(P_5 \times P_6) = 3 + \lfloor \frac{m}{2} \rfloor n + 2 - 4 = 13$.

Case 3.2.2. Let $m = 5$ and $n \geqslant 8$. The number of internally disjoint paths between only two vertices in G_2 is 4. Thus, these two vertices are colored with color 4. Since the remaining $2n-4$ vertices receive different colors, we have $\chi_{lc}(P_5 \times P_n) = 2n$.

Case 3.3. Let $m \geqslant 7$ and $n \geqslant 8$.

Since $m \geqslant 7$, in G_1 there are $\lfloor \frac{m}{2} \rfloor (\frac{n}{2} - 1)$ vertices each of degree 4 which are colored with color 3. Thus, we have

$$\chi_{lc}(G_1) = 2 + \lfloor \frac{m}{2} \rfloor \frac{n}{2} - \lfloor \frac{m}{2} \rfloor (\frac{n}{2} - 1) = 2 + \lfloor \frac{m}{2} \rfloor.$$

Case 3.3.1. If $m = 7$ and $n \geqslant 8$, only four vertices in G_2 can be colored with color 4. Since the remaining $\frac{n}{2} \lfloor \frac{m}{2} \rfloor - 2$ vertices in G_2 receive different colors, $\chi_{lc}(G_2) = 2 + \frac{n}{2} \lfloor \frac{m}{2} \rfloor - 2 = \frac{3n}{2}$, and we have $\chi_{lc}(P_7 \times P_n) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = \frac{3n}{2} + 5$.

Case 3.3.2. Let $m \geqslant 9$ and $n \geqslant 10$.

For the minimum number of local connective coloring consider the vertices w_{ij}, where $i \in \{4, 6, ..., m-3\}$, $j \in \{3, 5, ..., n-1\}$ and w_{kl}, where $k \in \{2, m-1\}$, $l \in \{5, 7, ..., n-3\}$ in G_2. Since the number of internally disjoint paths between these $\lfloor \frac{m-4}{2} \rfloor (\frac{n-2}{2}) + 2 \lfloor \frac{n-5}{2} \rfloor$ vertices is 4, they are colored with color 4. The remaining $\frac{m+11}{2}$ vertices in G_2 receive different colors. Hence, $\chi_{lc}(G_2) = 2 + \frac{m+11}{2}$ and we have $\chi_{lc}(P_m \times P_n) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = m + 9$.

Consequently, if m is odd and n is even, then we get

$$\chi_{lc}(P_m \times P_n) = \begin{cases} n + 4, & m = 3, n \geqslant 4 \\ 13, & m = 5, n = 6 \\ 2n, & m = 5, n \geqslant 8 \\ \frac{3n}{2} + 5, & m = 7, n \geqslant 8 \\ m + 9, & m \geqslant 9, n \geqslant 10. \end{cases}$$

Case 4. Let m and n be even.

In this case, $|V(G_1)| = |V(G_2)| = \frac{mn}{2}$. The graph $P_m \times P_n$ has four vertices of each of degree one. Two of them are in G_1 and the other two vertices are in G_2. Assume that we start coloring the graph from the vertex w_{11} in G_1, and assign color 1 to every vertex w_{ij} when i and j are not both even. Thus, $\beta(G_1) = \frac{mn}{4}$ vertices are colored with color 1 and $\frac{mn}{4}$ vertices in G_1 remain uncolored.

For color 2, assume that we start coloring the graph G_2 from the vertex w_{21}, and consider the vertices w_{ij}, where $i \in \{2, 4, ..., m-2\}$, $j \in \{1, 3, ..., n-1\}$ and w_{nl}, where $l \in \{3, 5, ..., n-1\}$. Thus, $(\frac{m-2}{2}) \lceil \frac{n-1}{2} \rceil + \frac{n-2}{2} = \frac{mn}{4} - 1$ vertices in G_2

are colored with color 2, and the number of the remaining uncolored vertices in G_2 is $\frac{mn}{4}+1$.

Case 4.1. Let $m=2$ and $n \geqslant 2$.

Since $P_2 \times P_n \cong 2P_n$ and $\chi_{lc}(P_n) = 1 + \lfloor \frac{n}{2} \rfloor$ by [**7**], we have

$$\chi_{lc}(P_2 \times P_n) = n + \chi_{lc}(P_n) = n + 1 + \lfloor \frac{n}{2} \rfloor = \lceil \frac{3n+1}{2} \rceil.$$

Case 4.2. Let $m=4$ and $n \geqslant 4$.

Since $m=4$, there is not any pair of vertices which is colored with color 3 and color 4. Hence, we have

$$\chi_{lc}(P_4 \times P_n) = 2 + \frac{mn}{2} + 1 = 2n + 3.$$

Case 4.3. Let $m \geqslant 6$ and $n \geqslant 6$.

For every vertex of degree 4 in G_1 which is not adjacent with each other $\kappa(w_{ij}, w_{kl}) \geqslant 3$ is satisfied, where $i,k \in \{2,4,...,m-2\}$, $j,l \in \{2,4,...,n-2\}$. Thus, $(\frac{m-2}{2})(\frac{n-2}{2})$ vertices are colored with color 3, and $\frac{m+n}{2}-1$ vertices in G_1 remain uncolored. Since the number of internally disjoint paths between these remaining vertices is at most 2, all of them receive different colors. Hence, we have $\chi_{lc}(G_1) = 1 + \frac{m+n}{2}$.

Case 4.3.1. Let $m=n=6$. Since there is not any pair of vertices in G_2 which is colored with color 4, we have $\chi_{lc}(P_6 \times P_6) = \chi_{lc}(G_1) + 1 + \frac{mn}{4} + 1 = 18$.

Case 4.3.2. Let $m=6$, $n \geqslant 8$. The number of internally disjoint paths between only two vertices in G_2 is 4. Thus, the remaining $\frac{mn}{4}-1$ vertices receive different colors, and we have $\chi_{lc}(P_6 \times P_n) = \chi_{lc}(G_1) + 2 + \frac{mn}{4} - 1 = 2n+5$.

Case 4.3.3. Let $m \geqslant 8$ and $n \geqslant 8$.

Consider the vertices w_{ij}, where $i \in \{5,7,...,m-3\}, j \in \{2,4,...,n-2\}$ and w_{kl}, where $k \in \{3, m-1\}, l \in \{4,6,...,n-4\}$ in G_2. The number of internally disjoint paths between these vertices is 4. Thus, $\lfloor \frac{m-5}{2} \rfloor (\frac{n-2}{2}) + 2\lfloor \frac{n-5}{2} \rfloor = \frac{mn}{4} - \frac{m+n}{2} - 3$ vertices receive color 4. The remaining $\frac{m+n}{2}+4$ vertices receive different colors. Then $\chi_{lc}(G_2) = 6 + \frac{m+n}{2}$ and we have $\chi_{lc}(P_m \times P_n) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = m+n+7$.

As a result, if m and n is even, we have

$$\chi_{lc}(P_m \times P_n) = \begin{cases} \lceil \frac{3n+1}{2} \rceil, & m=2, n \geqslant 2 \\ 2n+3, & m=5, n \geqslant 4 \\ 18, & m=n=6 \\ 2n+5, & m=6, n \geqslant 6 \\ m+n+7, & m \geqslant 8, n \geqslant 8. \end{cases}$$

By summing up four cases we have the statement of Theorem. □

THEOREM 2.2. *Let C_m and P_n be cycle and path of order m and n, respectively. Then,*

$$\chi_{lc}(C_m \times P_n) = \begin{cases} 2, & \textit{if } m \textit{ is odd} \\ m+2, & \textit{if } m \textit{ is even, } n=2 \\ m+3, & \textit{if } m \textit{ is even, } n=3 \\ m+4, & \textit{if } m \textit{ is even, } n \geqslant 4 \textit{ even} \\ & \textit{or } m \textit{ is even, } n \geqslant 5 \textit{ odd} \end{cases}$$

PROOF. Since $\deg(w_{ij}) = \deg(u_i) \cdot \deg(v_j)$, we have

$$\kappa(w_{ij}, w_{kl}) \leqslant min\{w_{ij}, w_{kl}\} \leqslant 4,$$

where $w_{ij}, w_{kl} \in V(C_m \times P_n)$, $i, k \in \{1, 2, ..., m\}$, $j, l \in \{1, 2, ..., n\}$. Thus, $k_i \leqslant 1$ for $i \geqslant 5$, and the pair of vertices can be colored with the same color at most color 4. We have following two cases for coloring the graph $C_m \times P_n$.

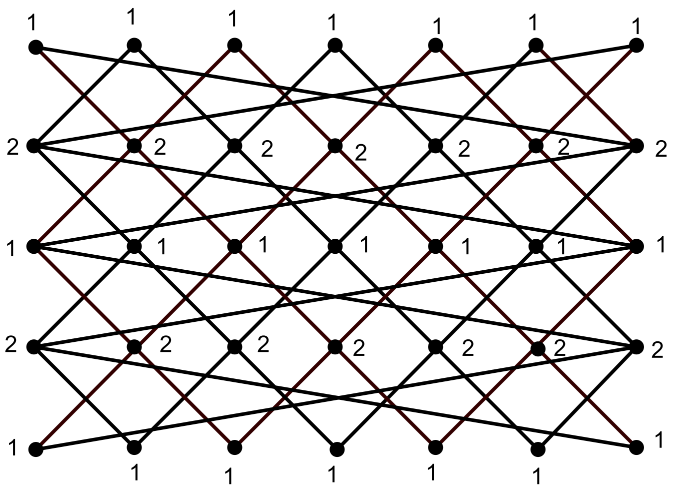

FIGURE 2. Local connective coloring of $C_5 \times P_7$

Case 1. Let m and n be odd or m be odd and n be even.

By Theorem 1.1, since m is odd, the graph $C_m \times P_n$ is connected, and by Theorem 1.2, $C_m \times P_n$ is bipartite graph. Let $C_m \times P_n = (V_1 \cup V_2, E)$. Since $\kappa(w_{ij}, w_{kl}) \leqslant 4$, where $w_{ij}, w_{kl} \in V(C_m \times P_n)$, $i, k \in \{1, 2, ..., m\}$, $j, l \in \{1, 2, ..., n\}$, the vertices of V_1 and V_2 can be colored with color 1 and color 2, respectively. Then we have $\chi_{lc}(C_m \times P_n) = 2$.

Case 2. Let m and n be even or m be even and n be odd.

In this case, since cycles and paths are bipartite graphs, let $C_m = (V_0 \cup V_1, E), P_n = (W_0 \cup W_1, F)$. By Theorem 1.3 and Theorem 1.2, the graph $C_m \times P_n$ is non-connected bipartite graph. Further, by Lemma 1.2 and 1.3, $G_1 = ((V_0 \times W_0) \cup (V_1 \times W_1), \frac{E \cdot F}{2})$ and $G_2 = ((V_0 \times W_1) \cup (V_1 \times W_0), \frac{E \cdot F}{2})$ are two bipartite components of $C_m \times P_n$. Since G_1 is bipartite graph, $\chi_{lc}(G_1) = 2$ by Case 1. Let's start coloring the graph G_2 with color $\chi_{lc}(G_1) + 1 = 3$.

Since $|V(C_m \times P_n)| = |V(C_m)||V(P_n)|$, it is obvious that $|V(C_n)| = |V(P_m)| = \frac{mn}{2}$.

Case 2.1. Let $n = 2$ and m be even.

Since $P_2 = K_2$, by Lemma 1.1 the graph $C_m \times K_2$ has exactly two connected components G_1 and G_2 isomorphic to C_m.

Since G_2 is 2–regular graph, the number of internally disjoint paths between all two vertices in G_2 is at most 2. Thus, all vertices of G_2 receive different colors. Then, $\chi_{lc}(C_m \times P_2) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = m + 2$.

Case 2.2. Let $n = 3$ and m be even.

In this case, the graph G_2 has $\frac{m(n-2)}{2} = \frac{m}{2}$ vertices of degree 4, and these vertices are either in the vertex set $V_0 \times W_1$ or in $V_1 \times W_0$. That is, these vertices are not adjacent. Thus, $\frac{m}{2}$ vertices are colored with color 3. The number of the remaining uncolored vertices is m. Since the degree of these remaining vertices is 2, these m vertices receive different colors. Thus, we have

$$\chi_{lc}(C_m \times P_3) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = 2 + 1 + m = m + 3.$$

Case 2.3. Let m be even and $n \geqslant 4$ even or m be even and $n \geqslant 5$ odd.

The graph $C_m \times P_n$ has $m(n-2)$ vertices each of degree 4, and G_2 has half of these vertices. The vertex sets W_0 and W_1 have $\lfloor \frac{n-2}{2} \rfloor$ and $\lceil \frac{n-2}{2} \rceil$ internal vertices of P_n, respectively.

Since $\kappa(w_{ij}, w_{kl}) \leqslant 4$, where $w_{ij}, w_{kl} \in V(G_2)$, we color $\frac{m}{2}\lfloor \frac{n-2}{2} \rfloor$ vertices of G_2 with color 3 and $\frac{m}{2}\lceil \frac{n-2}{2} \rceil$ vertices of G_2 with color 4. The remaining m vertices receive different colors. Thus, we have

$$\chi_{lc}(C_m \times P_n) = \chi_{lc}(G_1) + \chi_{lc}(G_2) = 2 + 2 + m = m + 4$$

. □

THEOREM 2.3. *Let C_m and C_n be two cycles of order m and n, respectively. Then,*

$$\chi_{lc}(C_m \times C_n) = \begin{cases} 2, & \text{if } m \text{ is odd, } n \text{ is even or } m \text{ is even, } n \text{ is odd} \\ 3, & \text{if } m \text{ and } n \text{ are odd} \\ 4, & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

PROOF. By proof of Theorem 1.4, the graph $C_m \times C_n$ is 4–regular graph. Then the number of internally disjoint paths between any two vertices in $C_m \times C_n$ is at most 4. Hence, $k_i \leqslant 1$ for $i \geqslant 5$. We have following three cases for coloring the graph $C_m \times C_n$.

Case 1. Let m be odd and n be even or m be even and n be odd.

It is known that if the order of a cycle is even, it is bipartite graph. Thus, by Theorem 1.1 and Theorem 1.2, the graph $C_m \times C_n$ is bipartite connected graph. Since the number of internally disjoint paths between any two vertices in $C_m \times C_n$ is at most 4, we have $\chi_{lc}(C_m \times C_n) = 2$.

Case 2. Let m and n be even.

Since m and n are even, C_m and C_n are bipartite graphs. By Theorem 1.2, Theorem 1.3 and Lemma 1.3, the graph $C_m \times C_n$ is bipartite non-connected graph

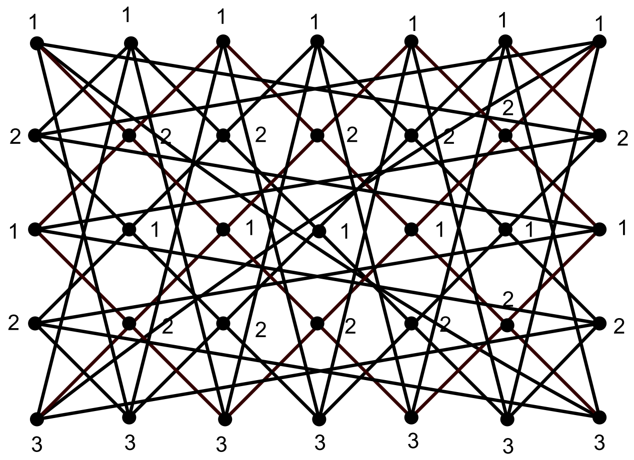

FIGURE 3. Local connective coloring of $C_5 \times C_7$

and has exactly two isomorpic connected components G_1 and G_2. Thus, these components are also bipartite graphs. Since G_1 is bipartite graph and the number of internally disjoint paths between any two vertices of G_1 is at most 4, we have $\chi_{lc}(G_1) = 2$. Let's start coloring the graph G_2 with color $\chi_{lc}(G_1)+1 = 3$. Since G_2 is also bipartite graph and the number of internally disjoint paths between any two vertices of G_2 is at most 4, the vertices of G_2 receive color 3 and color 4. Hence, $\chi_{lc}(C_m \times C_n) = 4$.

Case 3. Let m and n be odd.

Since m and n are odd, the graph $C_m \times C_n$ is connected by Theorem 1.1. Assume that we start coloring the graph from the vertex w_{11}. By definition of direct product, the vertex w_{1j} can be colored with color 1 for $j \in \{1, 2, ..., n\}$. Thus we color all non–adjacent vertices w_{1j}, w_{3j}, w_{5j},...,$w_{(m-2)j}$ with color 1. Since the vertices w_{2j}, w_{4j}, w_{6j},...,$w_{(m-1)j}$ are not adjacent with each other, and the number of internally disjoint paths between them is at most 4, these vertices receive color 2. Then total $n(m-1)$ vertices in $C_m \times C_n$ are colored with color 1 and color 2. Hence, the vertex w_{mj} remains uncolored and the number of its vertices is $mn - n(m-1) = n$. Since the vertex w_{mj} is adjacent to the vertices w_{1j} and $w_{(m-1)j}$, the vertex w_{mj} is colored with different color other than color 1 and color 2. Thus, all vertices in the graph $C_m \times C_n$ are colored with three local connective colors. □

3. Conclusion

In this paper, we define a new type of graph coloring called local connective coloring. It is known that a communication network is fault-tolerant if it has alternative paths (internally disjoint paths) between vertices and the internally disjoint paths are used to transmit messages among vertices. Thus, we use the term internally disjoint path in our coloring and color the vertices depending on the number of internally disjoint paths between two vertices. In our work, we study on the local connective chromatic number of direct product of some cycles and paths. We can consider the local connective chromatic number of Cartesian product of graphs in further study.

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