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POLYGON DISSECTIONS COMPLEXES ARE SHELLABLE

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Abstract. All dissections of a convex $(mn+2)$-gons into $(m+2)$-gons are facets of a simplicial complex. This complex is introduced by S. Fomin and A.V. Zelevinsky in [**7**]. We reprove the result of E. Tzanaki about shellability of such complex by finding a concrete shelling order. Also, we use this shelling order to find a combinatorial interpretation of h-vector and to describe the generating facets of these complexes.

1. Introduction

The problem of enumeration of triangulations of a convex polygon by noncrossing diagonals goes back to Leonhard Euler, who first found a closed formula for what we now call the Catalan numbers, see Appendix B in [**14**]. The enumeration of certain special types of dissections of a polygon by its diagonals are interesting combinatorial problems. The paper by Przytycki and Sikora [**11**] offers a nice review of the problems of this type.

In our paper we will consider all dissections of a convex $(mn+2)$-gon P_{mn+2} by non-crossing diagonals into $(m+2)$-gons. A set of non-crossing diagonals of P_{mn+2} is *m-divisible* if the partial dissection of P_{mn+2} defined by these diagonals can be completed to a dissection of P_{mn+2} into $(m+2)$-gons.

The number of m-divisible sets with exactly i-diagonals in P_{mn+2} is (see Corollary 2 in [**11**]) given by

$$\frac{1}{i+1}\binom{mn+i+1}{i}\binom{n-1}{i}. \tag{1.1}$$

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The above numbers are known as Fuss-Kirkman numbers.

The investigation of geometrical and topological questions related to the triangulations of a convex polygon started by Tamari, Milnor, Stasheff and others in the mid-twentieth century. The associahedron K_{n+2} is a well-known n-dimensional convex polytope whose vertices correspond to the triangulations of a convex $(n+3)$-gon. The facets of K_{n+2} correspond to the diagonals of this $(n+3)$-gon. For the history of the construction and some generalization of associahedron see chapter 9 in [**16**] and [**12**].

An *abstract simplicial complex* is a collection Δ of finite nonempty subsets such that $A \subseteq B \in \Delta \Rightarrow A \in \Delta$. The element F of Δ is called a *face* (or *simplex*) of Δ and its dimension is $|F| - 1$. The *dimension* of the complex Δ is defined as the largest dimension of any of its faces. For a d-dimensional simplicial complex Δ we denote the number of i-dimensional faces of Δ by f_i, and call $f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_d)$ the *f-vector*. A new invariant, the *h-vector* of d-dimensional complex Δ is $h(\Delta) = (h_0, h_1, \ldots, h_d, h_{d+1})$ defined by the formula

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}.$$

A simplicial complex is *pure* if all its maximal sets (*facets*) have the same cardinality. The interested reader can find more about simplicial complexes and other topological concepts used in this paper in [**3**] and [**10**].

Very often we define a simplicial complex in a natural way from a combinatorial or a geometrical object: graph, poset, polytope, matroid, etc. Finding relations between some properties of a combinatorial object X and the topology of the corresponding simplicial complex $\Delta(X)$ is a great source of the problems in combinatorial topology, see in [**8**].

An easy combinatorial way to obtain lots of information about algebraic, combinatorial and topological properties of a simplicial complex is to establishing the shellability of this complex, see [**4**] or [**5**]. A simplicial complex is *shellable* if it is pure and its facets can be ordered so that each one (other than the first) intersects the union of its predecessors in a nonempty union of maximal proper faces. Formally, this can be described by the following definition.

DEFINITION 1.1. *A simplicial complex Δ is shellable if Δ is pure and there exists a linear ordering (shelling order) of its facets $F_1, F_2, \ldots, F_k$ such that for all $i < j \leqslant k$, there exists some $l < j$ and a vertex v of F_j, such that*

$$F_i \cap F_j \subseteq F_l \cap F_j = F_j \setminus \{v\}.$$

For a fixed shelling order $F_1, F_2, \ldots, F_k$ of Δ, the *restriction* $\mathcal{R}(F_j)$ of the facet F_j is defined by:

$$\mathcal{R}(F_j) = \{v \text{ is a vertex of } F_j : F_j \setminus \{v\} \subset F_i \text{ for some } 1 \leqslant i < j\}.$$

Geometrically, if we build up Δ from its facets according to the shelling order, then $\mathcal{R}(F_j)$ is the unique minimal new face added at the j-th step.

The *type* of the facet F_j in the given shelling order is $type(F_j) = |\mathcal{R}(F_j)|$. If a simplicial complex Δ is shellable, there is a nice combinatorial interpretation of its h-vector:

$$h_k(\Delta) = |\{F \text{ is a facet of } \Delta : type(F) = k\}|.$$

This interpretation of the h-vector was the key argument in the proof of the upper-bound theorem and in the characterization of f-vectors of simplicial polytopes (see chapter 8 in [**16**]). Further, a shellable d-dimensional simplicial complex is homotopy equivalent to a wedge of h_{d+1} spheres of dimension d. For a given shelling order of a complex Δ we can describe a set of *generating simplices* of Δ (a set of facets of Δ such that the removal of their interiors makes Δ contractible). A facet F is a generating facet if and only if $\mathcal{R}(F) = F$, or equivalently:

(1.2) $\quad \forall v \in F$ there exists a facet F' before F such that $F \cap F' = F \setminus \{v\}$.

2. Shelling of dissection complexes

Definition 2.1. *For a convex polygon P_{mn+2} with $mn + 2$ vertices let Δ_n^m denote the abstract simplicial complex whose vertices are the diagonals that divide P_{mn+2} into an $(sm + 2)$-gon and an $((n - s)m + 2)$-gon, for some $1 \leqslant s \leqslant n - 1$. The facets of Δ_n^m are the sets of non-crossing diagonals that dissect P_{mn+2} into $(m + 2)$-gons.*

Note that each dissection of P_{mn+2} into $(m + 2)$-gons contains exactly $n - 1$ appropriate diagonals, and therefore Δ_n^m is a pure $(n - 2)$-dimensional simplicial complex. Further, there is an obvious correspondence between the set of $(i - 1)$-dimensional faces of Δ_n^m and all m-divisible sets with exactly i non-crossing diagonals of P_{mn+2}. In other words, the faces of Δ_n^m are all partial dissection that can be completed to a dissection of P_{mn+2} into $(m + 2)$-gons.

So, we can recognize that the entries of f-vector of Δ_n^m are Fuss-Kirkman numbers given by (1.1).

It is well-known that for $m = 1$ the complex Δ_n^1 is the boundary of the dual of associahedron, i.e., $\Delta_n^1 \cong \partial K_{n+1}^*$. The complex Δ_n^m also appears in [**7**] as the generalized cluster complex (a simplicial complex associated to a crystallographic root system). E. Tzanaki proved that Δ_n^m is vertex-decomposable (Proposition 4.1 in [**15**]) and therefore shellable. Athanasiadis and Tzanaki later proved in [**2**] that generalized cluster complexes are shellable for all finite root systems.

Theorem 2.1 (Tzanaki, [**15**]). *The complex Δ_n^m is shellable.*

We reprove this result by finding a concrete shelling order for Δ_n^m.

Proof. Assume that the vertices of P_{mn+2} are labelled by $v_1, v_2, \ldots, v_{mn+2}$ in the clockwise direction, and fix the linear order

$$v_1 < v_2 < \ldots < v_{mn+1} < v_{mn+2}$$

on the vertices of P_{mn+2}. Recall that the vertices of Δ_n^m are appropriate diagonals xy, where x and y are vertices of P_{mn+2}. In this notation we always assume that $x < y$, and we say that the diagonal xy starts from x.

Now,we use the above defined linear order $<$ on the set of vertices of P_{mn+2} to define the lexicographical order $<_L$ on the set of vertices of Δ_n^m:

$$v_a v_b <_L v_c v_d \iff v_a < v_c \text{ or } v_a = v_c, v_b < v_d.$$

Finally, we order the set of facets of Δ_n^m (these facets are $(n-1)$-element subsets of appropriate non-crossing diagonals of P_{mn+2}) anti-lexicographically:

For two facets F and F' we let

$$F <_{AL} F' \Leftrightarrow max_{<_L}(F \triangle F') \in F'. \tag{2.1}$$

Now, we prove that the linear order defined in (2.1) satisfies the conditions described in Definition 1.1, i.e., $<_{AL}$ is a shelling order for Δ_n^m.

REMARK 2.1. *Assume that $v_p v_q = max_{<_L}(F \triangle F') \in F'$. In the partition of P_{mn+2} into $(m+2)$-gons defined by $(n-1)$ diagonals from F', the diagonal $v_p v_q$ lies at the boundary of exactly two convex $(m+2)$-gons. We assume that these $(m+2)$-gons are labeled by $x_1 x_2 \dots x_{m+2}$ and $y_1 y_2 \dots y_{m+2}$, where $x_1 = v_p = y_{m+2}$ and $x_{m+2} = v_q = y_1$, see Figure 1.*

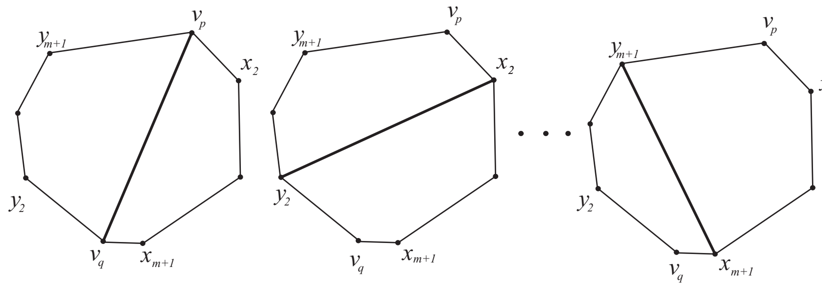

FIGURE 1. Polygons that contain $v_p v_q$ and all facets containing $F' \setminus \{v_p v_q\}$

Note that $F' \setminus \{v_p v_q\}$ is an $(n-2)$-dimensional face of Γ_n^m contained in exactly $(m+1)$ facets

$$F', F' \setminus \{v_p v_q\} \cup \{x_2 y_2\}, \dots, F' \setminus \{v_p v_q\} \cup \{x_{m+1} y_{m+1}\}.$$

All of these facets of Δ_n^m are obtained by "rotation" of the diagonal $v_p v_q$ inside $(2m+2)$-gon $v_p x_2 \dots x_{m+1} v_q y_2 \dots y_{m+1}$, see Figure 1.

If there exists $x_j y_j <_L v_p v_q$ for some j (in that case we have $x_j = v_r$ for some $v_r < v_p$), then we let $F'' = F' \setminus \{v_p v_q\} \cup \{x_j y_j\}$. Obviously, we have that $F'' <_{AL} F'$ and

$$F \cap F' \subseteq F' \cap F'' = F' \setminus \{v_p v_q\}.$$

We obtain the same if there is a diagonal $y_j x_j <_L v_p v_q$.

If $v_p < x_i$ (for all $i > 2$) and $v_p < y_j$ (for all $j < m+2$), then (because we know that $v_p v_q = max_{<_L}(F \triangle F'))$ all of segments

$$x_2x_3, x_3x_4, \ldots, x_{m+1}v_q, v_qy_2, \ldots, y_{m+1}v_r$$

are edges of P_{mn+2}, or appropriate diagonals contained in both F and F'.

As we know that $v_pv_q \notin F$, and the diagonals from F dissect P_{mn+2} into $(m+2)$-gons, then there exist a vertex x_i such that $v_p < x_i$ and the diagonal x_iy_i is contained in F. But x_iy_i is not contained in F' (otherwise it would cross v_pv_q), and this is a contradiction with assumption that $F <_{AL} F'$. □

The shelling order defined in (2.1) enables us to determine the restriction for each facets of Δ_n^m in this order. If $F = \{x_1y_1, x_2y_2, \ldots, x_{n-1}y_{n-1}\}$ is a facet of Δ_n^m recall that $x_iy_i \in \mathcal{R}(F)$ if and only if we can exchange x_iy_i with a diagonal xy such that $F' = F \setminus \{x_iy_i\} \cup \{xy\} <_{AL} F$.

In other words, $x_iy_i \in \mathcal{R}(F)$ if and only if we can rotate x_iy_i inside a $(2m+2)$-gon defined by partition of P_{mn+2} by F (see Remark 2.1 and Figure 1) in order to obtain an antilexicographically smaller facet. Therefore, we conclude that $x_iy_i \in \mathcal{R}(F)$ if and only if:

$$x_i \neq v_1, \text{ and there is no } x_iy \in F \text{ for some } y > y_i.$$

Now, we obtain the following combinatorial interpretation for the h-vector of Δ_n^m: $h_i(\Delta_n^m)$ counts the number of dissections F of P_{mn+2} into $(m+2)$-gons by $n-1$ diagonals so that there are exactly i of diagonals $xy \in F$ starting with $x > v_1$ for which does not exist the diagonal $xz \in F$ such that $x < y < z$.

The h-vector of Δ_n^m is obtained in [**15**] by elementary calculation, and its entries are

$$h_i = \frac{1}{i+1}\binom{mn}{i}\binom{n-1}{i}. \tag{2.2}$$

In [**15**] (see also in [**1**] and [**13**]) these numbers are called generalized Narayana numbers. However, these numbers appeared in many situations. We point out some of their combinatorial interpretation mentioned in [**6**].

(a) An r-ary tree is a finite set of nodes consists of a root s that is connected with exactly r disjoint r-ary trees $R_1, R_2, \ldots, R_r$ (some of them may be empty) in this order. The edge connecting an inner node with R_j is labelled by j. We say that this edge is strong if $R_j \neq \emptyset$. An r-ary tree with n inner nodes has $n(r-1)+1$ leaves and $n-1$ strong edges. The formula (2.2) counts the number of $(m+1)$-ary trees with n inner nodes and exactly $n-i-1$ strong edges labelled by a fixed label, see Theorem 1 and Theorem 2 in [**6**].

(b) The $(m+1)$-Catalan path is a nonnegative path in the integer lattice from $(0,0)$ to $(0,(m+1)n)$ with steps $(1,1)$ and $(1,1-m)$. The number of $(m+1)$-Catalan paths with exactly $i+1$ peaks is given by (2.2), see Theorem 2 in [**6**].

(c) The number of noncrossing partitions of $[n]$ into i blocks $B_1, B_2, \dots, B_i$ (there do not exist $a < b < c < d$ such that $a, c \in B_i$ and $b, d \in B_j$ for $i \neq j$) in which the cardinality of each block is divisible by m. These partitions correspond with Catalan permutations (in the sense of D. Knuth, see [**9**]).

The bijections between all of the above defined sets are non-trivial and can be found in [**6**]. Our consideration of h-vector of Δ_n^m offers another combinatorial interpretation for the numbers given by (2.2).

At the end of this section we describe a bijection between the facets of Δ_n^m and the set of $(m+1)$-ary trees with n-nodes. For a given dissection $F \in \Delta_n^m$ we associate the $(m+1)$-ary tree $T(F)$ in the following way:

- Let the edge v_1v_2 be the root of $T(F)$.
- Choose the unique $(m+2)$-gon $M = v_1v_2x_1 \dots x_m$ that contains v_1v_2.
- Connect v_1v_2 with segments $v_1x_m, x_{m-1}x_m, \dots, v_2x_3$ in this order (the edge that connect v_1v_2 and v_1x_m is the most left edge in $T(F)$, labeled by 1).
- For each diagonal $xy \in M$, we continue to build $T(F)$ in the same way taking into account the orientation of diagonals.

Assume that the diagonal xy is a new inner node of $T(F)$. If its neighbors are segments in the $(m+2)$-gon $xyz_1z_2, \dots, z_m$, the leftmost edge from the node xy (this edge of $T(F)$ is labeled by 1) is xz_m and the rightmost edge from xy is yz_1 (recall that $x < y$). It is not too complicated to define the inverse map for $F \mapsto T(F)$.

The above bijection is graded in the following sense: A dissection of P_{mn+2} defined by $F \in \Delta_n^m$ of the type i maps to the $(m+1)$-ary tree $T(F)$ with exactly $n - i - 1$ strong edges labelled by 1.

3. Generating facets of dissection complexes

In this section we describe the set of generating facets of Δ_n^m in our shelling order.

Remark 3.1. *From (1.2) and Remark 2.1 we have the following characterization:*
A facet $F \in \Delta_n^m$ is a generating facet if and only if

(a) *there is no diagonal in F starting from v_1; and*
(b) *for any vertex $v_i > v_1$ there is at most one diagonal in F starting from v_i, i.e., there is at most one $v_iv_j \in F$ (recall that here $v_i < v_j$).*

Further, from the formulas (1.1) and (2.2) we easily obtain that $h_{n-1}(\Delta_n^{m+1}) = f_{n-2}(\Delta_n^m)$, i.e., the number of generating facets of Δ_n^{m+1} is equal to the number of all facets of Δ_n^m. In particular, we conclude that the Betti number of Δ_n^2 is the Catalan number C_n.

Theorem 3.1. *There is a bijection between all facets of Δ_n^m and generating facets of Δ_n^{m+1}.*

We will prove the above theorem by using our combinatorial interpretation of generating facets of Δ_n^{m+1}.

Proof. A generating facet F of Δ_n^{m+1} defines the dissection of $P_{(m+1)n+2}$ into $(m+3)$-gons. Each of these $(m+3)$-gons (there are exactly n of them) contains at least one edge of $P_{(m+1)n+2}$.

Indeed, assume that in the dissection defined by generating facet F there is an $(m+3)$-gon $v_{i_1}v_{i_2}\dots v_{i_{m+3}}$ such that all of its edges are diagonal of $P_{(m+1)n+2}$.

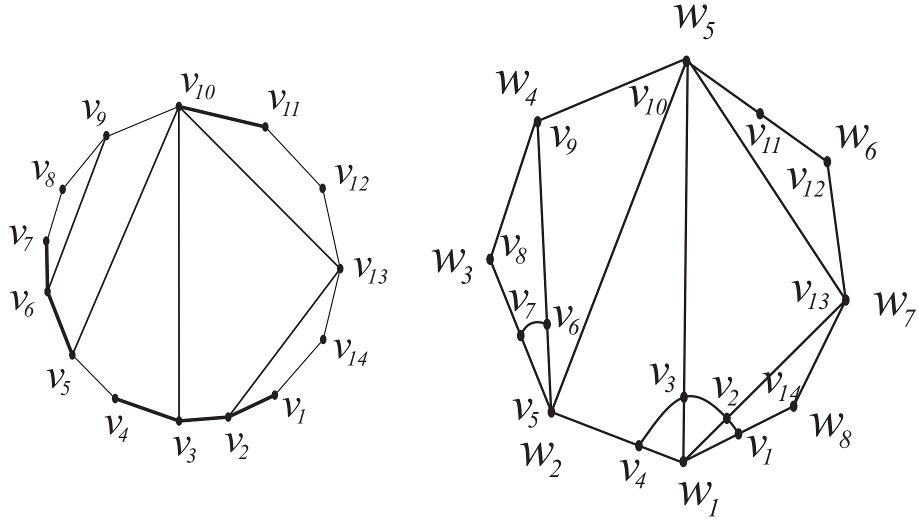

Figure 2. A generating facet of Δ_6^2 and corresponding facet of Δ_6^1

In that case we have that two diagonals in F started from the smallest vertex in $v_{i_1}v_{i_2}\dots v_{i_{m+3}}$, and this is a contradiction with the assumption that F is a generating facet, see (b) in Remark 3.1.

Now, for each of the $(m+3)$-gon in the considered dissection, we choose the minimal vertex v_i such that the edge v_iv_{i+1} is in this $(m+3)$-gon, and contract this edge. In the example on Figure 2 all of the thick edges of 14-gon will be contracted. After this operation, we obtain a dissection of an convex $(mn+2)$-gon into $(m+2)$-gons by $n-1$ diagonals.

We label the vertices of this new polygon with $w_1, w_2, \dots, w_{mn+2}$, starting with $v_1 = v_2 = w_1$. Note that the edge v_1v_2 is contained in an $(m+3)$-gon in the dissection of $P_{(m+1)n+2}$ defined by a generating facet F, see (a) in Remark 3.1. Also, we take into account that all of identified vertices share the same new label, see right part of Figure 2.

It is easy to describe the inverse of the above defined map. Consider a dissection of P_{mn+2} defined by a facet $F \in \Delta_n^m$. We describe a geometric way to construct corresponding generating facet of Δ_n^{m+1}. If F contains k diagonals from v_1, first we draw a small arc around v_1. Now, we take the end points of this arc and its intersections with diagonals from v_1 to define $k+2$ new vertices of $P_{(m+1)n+2}$, see right part of Figure 2.

We continue in a similar manner. For each vertex $v_i, i > 1$ that is the beginning of s diagonals, we draw a small arc around v_i that intersects all diagonals from v_i except the diagonal with the largest end vertex. There are s intersection points (we count the end point of this arc on the edge of P_{mn+2}), and these s points are new vertices of $P_{(m+1)n+2}$. Note that each $(m+2)$-gon of dissection of P_{mn+2} defined by F now is transformed into an generating facet of Δ_n^{m+1}. □

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