BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 7(2017), 379-385
DOI:10.7251/BIMVI1702379A

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ON THE AVERAGE LOWER INDEPENDENCE NUMBER OF SOME GRAPHS

Vecdi Aytaç

Abstract. In a communication network, several vulnerability measures are used to determine the resistance of the network to disruption of operation after the failure of certain stations or communication links. If the communication network is modeled as a simple, undirected, connected and unweighted graph G, then average lower independence number of a graph G can be considered as a measure of graph vulnerability and is defined by

$$i_{av}(G) = \frac{1}{|V(G)|} \sum\nolimits_{v \in V(G)} i_v(G)$$

where $i_v(G)$ is the minimum cardinality of a maximal independent set that contains v. In this paper, I consider the average lower independence number of the thorn graphs of special graphs.

1. Introduction

The stability of a communication network, composed of processing nodes and communication links, is of prime importance to network designers. When network begins losing links or nodes, eventually there is a loss in its effectiveness. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. If the communication network is modeled as a simple, undirected, connected and unweighted graph G, then there exist many graph theoretical parameters which have been used in the past to describe the stability of communication networks. Most notably, the vertex-connectivity and the edge-connectivity [**5**] have been frequently used. Consequently, a number of other parameters have been introduced including toughness [**6**], integrity [**2, 3**], tenacity [**7**], residual closeness

---

2010 *Mathematics Subject Classification.* 05C40, 68M10, 68R10, 05C69, 94C15.
Key words and phrases. vulnerability, connectivity, network design and communication, lower average independence number, thorn graphs.

[**14**], domination and its variations [**10, 12**]. In this paper, I introduce a new graph parameter, *average lower independence number.* The concept of average independence is closely related to the problem of finding large independent sets in graphs. In a graph $G = (V(G), E(G))$, a subset $S \subseteq V$ of vertices is a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex of S. The *dominating number* $\gamma(G)$ is the minimum cardinality of a dominating set. The *independence number* $\beta(G)$ of G is the maximum cardinality of an independent set in G which is a set of vertices of G whose elements are pairwise nonadjacent. The *independent domination number* $i(G)$ of G is the minimum cardinality of a set that is both independent and dominating. It is easy to see that $\gamma(G) \leqslant i(G) \leqslant \beta(G)$ holds for every graph G.

Henning [**11**] introduced the concept of average independence. For a vertex v of a graph G, the *lower independence number*, denoted by $i_v(G)$, is the minimum cardinality of a maximal independent set of G that contains v. The *average lower independence number* of G denoted by $i_{av}(G)$, is the value $\frac{1}{|V(G)|}\sum_{v\in V(G)} i_v(G)$ [**1, 4, 11, 13**]. It is clear that $i(G) = min\{i_v(G) \mid v \in V(G)\}$ and so $i(G) \leqslant i_{av}(G)$.

Throughout this paper, for any graph G, connectivity, covering number and independence number of G are denoted by $\kappa(G)$, $\alpha(G)$, $\beta(G)$, respectively [**9**].

Consider the graphs G_1 and G_2 both having the same edges and vertices. The reliability of these graphs can be measured by average lower independence number. If $i_{av}(G_1) < i_{av}(G_2)$, then we can say that graph G_1 is more reliable than graph G_2. As a result, for measuring the vulnerability of a graph G, average lower independence number is a better parameter. In [**1**], Aytac and Turaci determined the average lower independence number of total graphs.
In this paper, I consider the average lower independence number of the thorn graphs of special graphs.

2. Upper Bounds for $i_{av}(G)$

In this section, I will review some of the known upper bounds on average lower independence number.

Theorem 2.1. [**4, 11**] *For every vertex v in a graph,*

a) $i(G) \leqslant i_v(G) \leqslant \beta(G)$*;*
b) $i(G) \leqslant i_{av}(G) \leqslant \beta(G)$.

Theorem 2.2. [**11**] *For any graph G of order n with independent domination number i and independence number β,*

$$i_{av}(G) \leqslant \beta - \frac{i(\beta - i)}{n}.$$

Theorem 2.3. [**11**] *If T is a tree of order $n \geqslant 2$, then*

$$i_{av}(G) \leqslant n - 2 + \frac{2}{n}.$$

THEOREM 2.4. [**4**] *For any graph G with n vertices, m edges and for any vertex $v \in V(G)$, $\beta_v(G)$ is the maximum cardinality of a matching in the graph,*

$$i_{av}(G) \leqslant n - \frac{2m}{n} - \frac{1}{n} \sum_{v \in V(G)} \beta_v(G).$$

THEOREM 2.5. [**11**] *For every tree T and for any vertex $v \in V(T)$, if $\beta_v(T)$ is the maximum cardinality of a matching in the tree induced by the vertices of $V(T) - N[v]$,then*

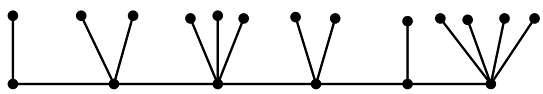
$$i_{av}(G) \leqslant n - 2 + \frac{2}{n} - \frac{1}{n} \sum_{v \in V(T)} \beta_v(T).$$

3. Average Lower Independence Number in Thorn Graphs

In this section, I give some results on the average lower independence number of thorn graphs.

DEFINITION 3.1. [**8**] *Let $p_1, p_2, \ldots, p_n$ be non-negative integers and G be such a graph, $V(G) = n$. The* thorn graph *of the graph G, with parameters $p_1, p_2, \ldots, p_n$ is obtained by attaching p_i new vertices of degree one to the vertex u_i of the graph G, $i = 1, 2, \ldots, n$.*

The thorn graph of the graph G will be denoted by G^*, or if the respective parameters need to be specified, by $G^*(p_1, p_2, \ldots, p_n)$.



$P_6^*(1,2,3,2,1,4)$

FIGURE 1. A thorn graph of P_6

THEOREM 3.1. *Let G be a graph of order n. If G^* is a thorn graph of G with every $p_i = 1$, then the average lower independence number of G^* is $i_{av}(G^*) = n$.*

PROOF. When $p_i = 1$ for all $v_i \in V(G)$, the thorn graph is usually called the *corona*. The number of vertices of graphs G^* is $2n$. It is easy to see that if G^* is the corona of a graph G of order n, then every maximal independent set S^* of G^* consists of an independent set S of G and the thorn of every vertex of $V(G) \backslash S$. Thus every maximal independent set has size $|S| + (n \ |S|) = n$. So, $i_{av}(G^*) = n$. The proof is completed. $\square$

THEOREM 3.2. *Let G be a connected graph C_n (cycle) or K_n (complete) of order n. If G^* is a thorn graph of G with $p_i \geqslant k$ for $k \geqslant 2$, then the average lower independence number of G^* is*

$$i_{av}(G^*) = kn - (k-1)\beta(G).$$

PROOF. $v_i \in V(G)$ for $1 \leqslant i \leqslant n$. Let $p_{i,j}$ be the jth thorn of vertex v_i for $1 \leqslant j \leqslant k$. $S^* \subset V(G^*)$ and S^* ensures the lower independence number of G^*. A graph is vertex-transitive if and only if its graph complement is, since the group actions are identical. Informally, a graph is vertex-transitive if every vertex has the same local environment, so that no vertex can be distinguished from any other based on the vertices and edges surrounding it. Consequently, every vertex-transitive graph is regular. Hence, C_n and K_n are vertex transitive graphs. All vertices of those graphs play the same role, so if $G = C_n$ or $G = K_n$, then the value of $i_v(G^*)$ is the same for every vertex v of G^*, and $i_{av}(G^*) = i_v(G^*)$. Every maximal independent set S^* of G^* consist of an independent set S of G and all the thorns of the vertices of $V(G)\backslash S$. So $i_v = |S^*| = |S| + k(n|S|) = kn - (k-1)|S|$. This value is minimized when S is maximum. We have known that $|S| \leqslant \beta(G)$, so

$$i_{av}(G^*) = i_v(G^*) = kn - (k-1)\beta(G).$$

The proof is completed.

Particularly, if G is a complete graph K_n , then $i_{av}(G^*) = i_v(G^*) = kn - k + 1$ or if G is a cycle graph C_n , then $i_{av}(G^*) = i_v(G^*) = kn - (k-1)\left\lfloor \frac{n}{2} \right\rfloor$. □

THEOREM 3.3. *Let G be a connected graph P_n (path) order of n. If G^* is a thorn graph of G with $p_i \geqslant k$ for $k \geqslant 2$, then the average lower independence number of G^* is*

$$i_{av}(G^*) = \frac{1}{n(k+1)}\left[n^2(k^2+1) - 2\alpha(G)\beta(G)(k-1)^2\right].$$

PROOF. $v_i \in G$ for all integer i $(1 \leqslant i \leqslant n)$. Let $p_{i,j}$ be the thorns of v_i $(1 \leqslant j \leqslant k)$. $S^* \subset V(G^*)$ and S^* ensured the lower independence number of G^* , that is, S^* is an maximal independent set. Let we search the value of i_v for all v_i and its thorns. There are only four types of vertices.

Case 1. If v vertex is any all the thorns of the vertices of $v_i \in \beta(G)$, which the number of this is $k\beta(G)$, then every maximal independent set S^* of G^* consist of all thorns of the vertices in an independent set S of G and the vertices of $V(G)\backslash S$. So $i_v = |S^*| = k|S| + (n - |S|)$. Since the set S is the independent set of G, the value is

$$i_v = |S^*| = k\beta(G) + \alpha(G).$$

Case 2. If v vertex is any all the thorns of the vertices of $v_i \in \alpha(G)$, which the number of this is $k\alpha(G)$, then every maximal independent set S^* of G^* consist of an independent set S of G and all thorns of the vertices of $V(G)\backslash S$. So $i_v = |S^*| = |S| + k(n - |S|)$. Since the set S is the independent set of G, the value is

$$i_v = |S^*| = \beta(G) + k\alpha(G).$$

Case 3. If v vertex is any vertices of $v_i \in \beta(G)$, which the number of this is $\beta(G)$, then every maximal independent set S^* of G^* is similar to *Case 2*. So

$i_v = |S^*| = |S| + k\,(n - |S|)\,.$ Since the set S is the independent set of G, the value is

$$i_v = |S^*| = \beta(G) + k\alpha(G).$$

Case 4. If v vertex is any vertices of $v_i \in \alpha(G)$, which the number of this is $\alpha(G)$, then every maximal independent set S^* of G^* is similar to *Case 1.* So $i_v = |S^*| = k\,|S| + (n - |S|)\,.$ Since the set S is the independent set of G, the value is

$$i_v = |S^*| = k\beta(G) + \alpha(G).$$

Finally, by *Case 1, 2, 3* and *4*, we have,

$$i_{av}(G^*) = \frac{1}{n(k+1)}\left[(\beta(G) + k\alpha(G))^2 + (\alpha(G) + k\beta(G))^2\right].$$

It is well-known that $\alpha(G) = n - \beta(G)$ for all graph G. If we write these values in the definition, we have

$$\begin{aligned} i_{av}(G^*) &= \frac{1}{n(k+1)}\left[(\beta(G) + k(n - \beta(G)))^2 + ((n - \beta(G)) + k\beta(G))^2\right] \\ &= \frac{1}{n(k+1)}\left[n^2(k^2+1) - 2\beta(G)(k-1)^2(n - \beta(G))\right] \\ &= \frac{1}{n(k+1)}\left[n^2(k^2+1) - 2\alpha(G)\beta(G)(k-1)^2\right] \end{aligned}$$

The proof is completed. □

In particular: If G is a path graph P_n with n even, then since $\beta(G) = \alpha(G) = \frac{n}{2}$, this value is

$$i_{av}(G^*) = \frac{(k+1)n}{2}.$$

If G is a path graph P_n with n odd, then since $\beta(G) = \frac{n+1}{2}$ and $\alpha(G) = \frac{n-1}{2}$, this value is

$$i_{av}(G^*) = \frac{n^2(k+1)^2 + (k-1)^2}{2n(k+1)}.$$

THEOREM 3.4. *Let G be a connected graph $S_{1,n-1}$ (star) order of n. If G^* is a thorn graph of G with $p_i \geqslant k$ for $k \geqslant 2$ and $n > 4$, then the average lower independence number of G^* is*

$$i_{av}(G^*) = \frac{1}{n}\left[(\beta(G))^2 + 2k\beta(G)\right] + \frac{1+k^2}{n(k+1)}.$$

PROOF. $v_i \in G$ for all integer i ($1 \leqslant i \leqslant n$). Let $p_{i,j}$ be the thorns of v_i ($1 \leqslant j \leqslant k$). $S^* \subset V(G^*)$ and S^* is an maximal independent set S^* of G^*. Let we search the value of i_v for all v_i and its thorns. There are only four types of vertices like Theorem 3.3.

Case 1. If v vertex is any all the thorns of the vertices of $v_i \in \beta(G)$, which the number of this is $k\beta(G)$, then every maximal independent set S^* of G^* consist of an independent set S of G, except this v vertex, and all thorns of the vertices of $V\,(G)\setminus S$. Moreover, S^* includes the thorns of the vertex

v. So $i_v = |S^*| = (|S|-1) + k(n-|S|) + k$. Since the set S is the independent set of G, the value is

$$i_v = |S^*| = \beta(G) - 1 + k\alpha(G) + k.$$

It is well-known that $\alpha(G) = 1$ and $\beta(G) = n-1$ for all graph G. So, we obtain

$$i_v = |S^*| = \beta(G) - 1 + 2k.$$

Case 2. If v vertex is any all the thorns of the vertices of $v_i \in \alpha(G)$, which the number of this is k because of $\alpha(G) = 1$, then every maximal independent set S^* of G^* consist of an independent set S of G and all thorns of the vertices of $V(G) \setminus S$. So

$$i_v = |S^*| = |S| + k(n\,|S|).$$

Since the set S is the independent set of G, the value is

$$i_v = |S^*| = \beta(G) + k\alpha(G) = \beta(G) + k.$$

Case 3. If v vertex is any vertices of $v_i \in \beta(G)$, which the number of this is $\beta(G)$, then every maximal independent set S^* of G^* is similar to *Case 2*. So $i_v = |S^*| = |S| + k(n-|S|)$. Since the set S is the independent set of G, the value is

$$i_v = |S^*| = \beta(G) + k\alpha(G) = \beta(G) + k.$$

Case 4. If v vertex is any vertices of $v_i \in \alpha(G)$, which the number of this is $\alpha(G) = 1$, then every maximal independent set S^* of G^* consist of all thorns of the vertices in an independent set S of G and the vertices of $V(G) \setminus S$. So $i_v = |S^*| = k|S| + (n-|S|)$. Since the set S is the independent set of G, the value is

$$i_v = |S^*| = k\beta(G) + \alpha(G) = k\beta(G) + 1.$$

Finally, by *Case 1, 2, 3* and *4*, we have,

$$\begin{aligned} i_{av}(G^*) &= \tfrac{1}{n(k+1)}[(\alpha(G) + k\beta(G)) + \beta(G)(\beta(G)+k) + k(k+\beta(G)) \\ &\quad + k\beta(G)((\beta(G)-1)+2k)] \\ &= \frac{1}{n(k+1)}\left[\alpha(G) + k\beta(G)(1+\beta(G)-1+2k) + (\beta(G)+k)^2\right] \\ &= \frac{1}{n(k+1)}\left[(k+1)\left(2k\beta(G)+\beta^2(G)\right) + \left(1+k^2\right)\right] \\ &= \frac{1}{n}\left[\beta^2(G) + 2k\beta(G)\right] + \frac{1+k^2}{n(k+1)} \end{aligned}$$

The proof is completed. □

4. Conclusion

For a graph, its connectivity is the minimum number of vertices whose removal results with a disconnected or trivial graph. Connectivity of a graph is the best known measure of reliability. Since connectivity does not always take into account what happens throughout the graph, its a worst case measure. Recently, there is an interest in vulnerability and reliability of networks. It gives rise to most of other measures. Some of these measures are more global in nature. In this paper, a new measure for reliability of a graph, the average lower independence number is investigated. This graph parameter which is recently introduced by Henning, is closely relative to the problem of finding large independent sets of graphs.
Therefore the vulnerability of the thorn graphs of some special graphs is investigated by using the average lower independence number.

References

[1] A. Aytac and T. Turaci. The average lower independence number of total graphs, *Bull. Int. Math. Virtual Inst.*, **2**(1)(2012), 17-27.

[2] K.S. Bagga and L.W. Beineke. Survey of Integrity, *Discrete Applied Math.*, **37-38** (1992), 13-28.

[3] C.A Barefoot, R. Entringer and H. Swart. Vulnerability in Graphs-A Comparative Survey, *J. Comb. Math. Comb. Comput.*, **1** (1987), 13-22.

[4] M. Blidia, M. Chellali and F. Maffray. On Average Lower Independence and Domination Numbers in Graphs, *Discrete Math.*, **295**(1-3)(2005), 1-11.

[5] J. A. Bondy and U. S. R. Murty. Graph theory with applications, American Elsevier Publishing Co. Inc. New York (1976).

[6] V. Chvtal, Tough graphs and Hamiltonian circuits. *Discrete Math.*, **306**(10-11)(2006), 910917.

[7] Cozzens, M.B., D. Moazzami, S. Stueckle, The tenacity of a Graph, In Y Alavi and A. Schwenk (Eds.): Graph Theory, Combinatorics, and Algorithms (pp. 1111-1122) Wiley, New York, (1995).

[8] I. Gutman. Distance Of Thorny Graphs, *Publ. de L'Institute Math., N.S.*, **63**(77)(1998), 31-36.

[9] F. Harary and F. Buckley. *Distance in Graphs,* Addison-Wesley Publishing Company (1989).

[10] J. Haviland. Independent domination in regular graphs, *Discrete Math.*, **143**(1-3)(1995), 275-280.

[11] M. A. Henning. Trees with equal average domination and independent domination numbers, *Ars Combinatoria*, **71**(2004), 305-318.

[12] L. Sun and J. Wang. An Upper Bound for the Independent Domination Number, *J. of Comb. Theory series B*, **76**(2)(1999), 240-246.

[13] H. Tuncel, T. Turaci and B. Coskun. On the average lower domination number and some results of complementary prisms and graph join, *J. Adv. Res. Appl. Math.*, **7**(1)(2015), 52-61.

[14] T. Turaci and M. Okten. Vulnerability of Mycielski graphs via residual closeness, *Ars Combinatoria*, **118**(2015), 419-427.

Received by editors 27.01.2017; Available online 06.03.2017.

Ege University, Faculty of Engineering, Computer Engineering Department, Bornova IZMIR, 35100, TURKEY
E-mail address: vecdi.aytac@ege.edu.tr