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INTEGRAL SOLUTIONS OF NON-HOMOGENEOUS QUINTIC EQUATION WITH FIVE UNKNOWNS $3\left(x^4 - y^4\right) = 26(z^2 - w^2)p^3$

M.A.Gopalan [1], V.Krithika [2], Dr.A.VijayaSankar [3]
[1] Professor, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu, India
[2] Research Scholar, Department of Mathematics, National College, Trichy-620001, Tamilnadu, India
[3] Assistant Professor, National College, Trichy-620001, India

Abstract

The non-homogeneous quantic equation with five unknowns represented by $3\left(x^4 - y^4\right) = 26(z^2 - w^2)p^3$ is analyzed for its non-zero distinct integral solutions. A few interesting relations among the solutions are exhibited.

Keywords: Quintic Equation with Five Unknowns; Integral Solutions.

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1. Introduction

Diophantine equations, homogeneous and non-homogeneous, have aroused the interest of numerous mathematicians since antiquity as can be seen from [1-4]. The problem of finding all integer solutions of a Diophantine equation with three or more variables and degree atleast three, in general, presents a good deal of difficulties. Cubic equations in two variables fall into the theory of elliptic curves which is a very developed theory but still an important topic of current research. A lot is known about equations in two variables in higher degrees. For equations with more than three variables and degree atleast three, very little is known. It is worth to note that undesirability appears in equations, even perhaps at degree four with fairly small coefficients. In [5-7], ternary quintic diophantine equations are studied and in [8-9], quintic diophantine equations are analyzed. In [10-11], a few quintic Diophantine equation with five unknowns are observed.

In this communication, a typical fifth degree Diophantine equation $3\left(x^4 - y^4\right) = 26(z^2 - w^2)p^3$ is considered and different Choices of integral solutions are obtained. A few interesting relations among the solutions are presented.

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2. Method of Analysis

The Diophantine equation to be solved for its non-zero distinct integral solutions is given by

$$3(x^4 - y^4) = 26(z^2 - w^2)p^3 \tag{1}$$

Introducing the transformations

$$x = u + v, y = u - v, z = 6u + v, w = 6u - v \;,\quad u \neq v \neq 0 \tag{2}$$

in (1), can be written as

$$u^2 + v^2 = 26p^3 \tag{3}$$

(3) can be solved through different methods and we obtain different choices of integer solutions to (1).

Choice 1:

Assume $p = a^2 + b^2$ (4)

where a and b are non-zero distinct integers.

Write 26 as $26 = (5+i)(5-i)$ (5)

Substituting (4) & (5) in (3) and applying the method of factorization , define

$$u + iv = (5+i)(a+ib)^3$$

Equating the real and imaginary parts, we have

$$u = 5a^3 + b^3 - 3a^2b - 15ab^2$$

$$v = a^3 - 5b^3 + 15a^2b - 3ab^2$$

From (2) , the integer solutions of (1) are

$$x(a,b) = 6a^3 - 4b^3 + 12a^2b - 18ab^2$$

$$y(a,b) = 4a^3 + 6b^3 - 18a^2b - 12ab^2$$

$$z(a,b) = 31a^3 + b^3 - 3a^2b - 93ab^2$$

$$w(a,b) = 29a^3 + 11b^3 - 33a^2b - 87ab^2$$

$$p(a,b) = a^2 + b^2$$

Note 1:

Instead of (5), 26 can also be written as

$$26 = (1+5i)(1-5i) \tag{6}$$

By using (4) & (6) in (3), and applying the same procedure as mentioned above we get the corresponding integer solutions to (1) are given below

$$x(a,b) = 6a^3 + 4b^3 - 12a^2b - 18ab^2$$

$$y(a,b) = 4a^3 + 6b^3 - 18a^2b + 12ab^2$$

$$z(a,b) = 11a^3 + 29b^3 - 87a^2b - 33ab^2$$

$$w(a,b) = a^3 + 31b^3 - 93a^2b - 3ab^2$$

$$p(a,b) = a^2 + b^2$$

Choice 2:

Rewrite 26 as $26 = \dfrac{(23+11i)(23-11i)}{25}$ (7)

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Substituting (4) & (7) in (3) and applying the method of factorization , define

$$u+iv=\left[\frac{(23+11i)}{5}\right](a+ib)^3$$

Equating the real and imaginary parts, we have

$$u=\frac{1}{5}\left[23a^3+11b^3-33a^2b-69ab^2\right]$$
$$v=\frac{1}{5}\left[11a^3-23b^3+69a^2b-33ab^2\right] \qquad (8)$$

As our aim is to find integer solutions, choose $a=5A,\ b=5B$ in (8), we obtain

$$u=575A^3+275B^3-825A^2B-1725AB^2$$
$$v=275A^3-575B^3+1725A^2B-825AB^2$$

From (2) , the integer solutions of (1) are

$$x(A,B)=850A^3-300B^3+900A^2B-2550AB^2$$
$$y(A,B)=300A^3+850B^3-2550A^2B-900AB^2$$
$$z(A,B)=3725A^3+1075B^3-3225A^2B-11175AB^2$$
$$w(A,B)=3175A^3+2225B^3-6675A^2B-9525AB^2$$
$$p(A,B)=25(A^2+B^2)$$

Note 2:

Instead of (7), 26 can also be written as

$$26=\frac{(11+23i)(11-23i)}{25} \qquad (9)$$

By using (4) & (9) in (3), and proceeding as above we get the corresponding integer solutions to (1) are given below

$$x(A,B)=850A^3+300B^3-900A^2B-2550AB^2$$
$$y(A,B)=-300A^3+850B^3-2550A^2B+900AB^2$$
$$z(A,B)=2225A^3+3175B^3-9525A^2B-6675AB^2$$
$$w(A,B)=1075A^3+3725B^3-11175A^2B-3225AB^2$$
$$p(A,B)=25(A^2+B^2)$$

Choice 3:

One may write (3) as

$$u^2+v^2=26p^3*1 \qquad (10)$$

Write 1 as

$$1=\frac{(4+3i)(4-3i)}{25} \qquad (11)$$

Using (4), (5) & (11) in (10) and applying the method of factorization , define

$$u+iv=\left[\frac{(4+3i)}{5}\right](5+i)(a+ib)^3$$

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Equating the real and imaginary parts, we have

$$u=\frac{1}{5}\left[17a^3+19b^3-57a^2b-51ab^2\right]$$
$$v=\frac{1}{5}\left[19a^3-17b^3+51a^2b-57ab^2\right] \tag{12}$$

As our interest is on finding integer solutions, choose $a=5A,\ b=5B$ in (12), we have

$$u=425A^3+475B^3-1425A^2B-1275AB^2$$
$$v=475A^3-425B^3+1275A^2B-1425AB^2$$

From (2) , the integer solutions of (1) are

$$x(A,B)=900A^3+50B^3-150A^2B-2700AB^2$$
$$y(A,B)=-50A^3+900B^3-2700A^2B+150AB^2$$
$$z(A,B)=3025A^3+2425B^3-7275A^2B-9075AB^2$$
$$w(A,B)=2075A^3+3275B^3-9825A^2B-6225AB^2$$
$$p(A,B)=25(A^2+B^2)$$

The above solution patterns satisfy the following relations respectively:

$$\left[z(y+6w)+wy+6x^2\right]\equiv 0\,(\text{mod } 39)$$
$$(zw-xy)\equiv 0\,(\text{mod } 35)$$
$$6(x+y)=z+w$$
$$\left(zw+xy+x^2+y^2\right)\equiv 0\,(\text{mod } 39)$$
$$(x+w)=z+y$$
$$\left[6\left(y^2+zw\right)+x(z+w)\right]\equiv 0\,(\text{mod } 39)$$

Remark 1:

Instead of (2), we can take another set of transformations as

$$x=u+v,\ y=u-v,\ z=6uv+1,\ w=6uv-1,\quad u\neq v\neq 0 \tag{13}$$

in (1), we get

$$u^2+v^2=26p^3 \tag{14}$$

By applying the same process as mentioned in the above choices, we obtain different choices of integral solutions to (1) using (13).

Choice 4:

$$x(a,b)=6a^3-4b^3+12a^2b-18ab^2$$
$$y(a,b)=4a^3+6b^3-18a^2b-12ab^2$$
$$z(a,b)=30\left(a^6-b^6-48a^3b^3\right)+432\left(a^5b+ab^5\right)+450\left(a^2b^4-a^4b^2\right)+1$$
$$w(a,b)=30\left(a^6-b^6-48a^3b^3\right)+432\left(a^5b+ab^5\right)+450\left(a^2b^4-a^4b^2\right)-1$$
$$p(a,b)=a^2+b^2$$

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Choice 5:

$$x(A,B) = 850A^3 - 300B^3 + 900A^2B - 2550AB^2$$
$$y(A,B) = 300A^3 + 850B^3 - 2550A^2B - 900AB^2$$
$$z(A,B) = 3750(253A^6 - 253B^6 - 4080A^3B^3) + 4590000(A^5B + AB^5) + 14231250(A^2B^4 - A^4B^2) + 1$$
$$w(A,B) = 3750(253A^6 - 253B^6 - 4080A^3B^3) + 4590000(A^5B + AB^5) + 14231250(A^2B^4 - A^4B^2) - 1$$
$$p(A,B) = 25(A^2 + B^2)$$

Choice 6:

$$x(A,B) = 900A^3 + 50B^3 - 150A^2B - 2700AB^2$$
$$y(A,B) = -50A^3 + 900B^3 - 2700A^2B + 150AB^2$$
$$z(A,B) = 3750(323A^6 - 323B^6 + 720A^3B^3) - 810000(A^5B + AB^5) + 18168750(A^2B^4 - A^4B^2) + 1$$
$$w(A,B) = 3750(323A^6 - 323B^6 + 720A^3B^3) - 810000(A^5B + AB^5) + 18168750(A^2B^4 - A^4B^2) - 1$$
$$p(A,B) = 25(A^2 + B^2)$$

The above solution patterns satisfy the following relations respectively:

$[6x(x-y)-(z+w)] \equiv 0 \pmod{12}$

$3(x^2 - y^2) = z + w$

$z^2 - 3(x^2 - y^2) - 1$ is a perfect square.

$6(zw+1)$ is a nasty number.

$6(x^2 - y^2) = z^2 - w^2$

$z^2 - 2z - zw = 0.$

Remark 2:

By using the different set of transformations as

$$x = u + v,\ y = u - v,\ z = 4u + 3v,\ w = 4u - 3v,\quad u \neq v \neq 0 \qquad (15)$$

in (1), it leads to

$$u^2 + v^2 = 52p^3 \qquad (16)$$

By applying the same process as mentioned in the above choices, we obtain different choices of integral solutions to (1) using (15).

Choice 7:

Write 52 as $52 = (6+4i)(6-4i)$ (17)

Substituting (4) & (17) in (16) and applying the method of factorization , define

$$u + iv = (6+4i)(a+ib)^3$$

Equating the real and imaginary parts, we have

$$u = 6a^3 + 4b^3 - 12a^2b - 18ab^2$$
$$v = 4a^3 - 6b^3 + 18a^2b - 12ab^2$$

From (2) , the integer solutions of (1) are

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$$x(a,b) = 10a^3 - 2b^3 + 6a^2b - 30ab^2$$
$$y(a,b) = 2a^3 + 10b^3 - 30a^2b - 6ab^2$$
$$z(a,b) = 36a^3 - 2b^3 + 6a^2b - 108ab^2$$
$$w(a,b) = 12a^3 + 34b^3 - 102a^2b - 36ab^2$$
$$p(a,b) = a^2 + b^2$$

Note 3:
In the place of (17), 52 can also be written as

$$52 = (4+6i)(4-6i) \quad (18)$$

By using (4) & (18) in (16), and applying the same procedure as mentioned above we get the corresponding integer solutions to (1) are given below

$$x(a,b) = 10a^3 + 2b^3 - 6a^2b - 30ab^2$$
$$y(a,b) = -2a^3 + 10b^3 - 30a^2b + 6ab^2$$
$$z(a,b) = 34a^3 + 38b^3 - 114a^2b - 102ab^2$$
$$w(a,b) = 46a^3 - 22b^3 + 66a^2b - 138ab^2$$
$$p(a,b) = a^2 + b^2$$

Choice 8:
Rewrite 52 as $52 = \frac{(12+34i)(12-34i)}{25}$ (19)

Substituting (4) & (19) in (16) and applying the method of factorization , define

$$u + iv = \left[\frac{(12+34i)}{5}\right](a+ib)^3$$

Equating the real and imaginary parts, we have

$$u = \frac{1}{5}\left[12a^3 + 34b^3 - 102a^2b - 36ab^2\right]$$
$$v = \frac{1}{5}\left[34a^3 - 12b^3 + 36a^2b - 102ab^2\right] \quad (20)$$

As our aim is to find integer solutions, choose $a = 5A,\ b = 5B$ in (20), we obtain

$$u = 300A^3 + 850B^3 - 2550A^2B - 900AB^2$$
$$v = 850A^3 - 300B^3 + 900A^2B - 2550AB^2$$

From (15) , the integer solutions of (1) are

$$x(A,B) = 1150A^3 + 550B^3 - 1650A^2B - 3450AB^2$$
$$y(A,B) = -550A^3 + 1150B^3 - 3450A^2B + 1650AB^2$$
$$z(A,B) = 3750A^3 + 2500B^3 - 7500A^2B - 11250AB^2$$
$$w(A,B) = -1350A^3 + 4300B^3 - 12900A^2B + 4050AB^2$$
$$p(A,B) = 25(A^2 + B^2)$$

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Note 4:
Instead of (19), 52 can also be written as

$$52 = \frac{(34+12i)(34-12i)}{25} \tag{21}$$

By using (4) & (21) in (16), and proceeding as above we get the corresponding integer solutions to (1) are given below

$$x(A,B) = 1150A^3 - 550B^3 + 1650A^2B - 3450AB^2$$
$$y(A,B) = 550A^3 + 1150B^3 - 3450A^2B - 1650AB^2$$
$$z(A,B) = 4300A^3 - 1350B^3 + 4050A^2B - 12900AB^2$$
$$w(A,B) = 2500A^3 + 3750B^3 - 11250A^2B - 7500AB^2$$
$$p(A,B) = 25(A^2 + B^2)$$

Choice 9:
One may write (16) as

$$u^2 + v^2 = 52p^3 * 1 \tag{22}$$

Write 1 as

$$1 = \frac{(12+5i)(12-5i)}{169} \tag{23}$$

Using (4), (17) & (23) in (22) and applying the method of factorization , define

$$u + iv = \left[\frac{(12+5i)}{13}\right](6+4i)(a+ib)^3$$

Equating the real and imaginary parts, we have

$$\begin{aligned} u &= \frac{1}{13}\left[52a^3 + 78b^3 - 234a^2b - 156ab^2\right] \\ v &= \frac{1}{13}\left[78a^3 - 52b^3 + 156a^2b - 234ab^2\right] \end{aligned} \tag{24}$$

As our interest is on finding integer solutions, choose $a = 13A,\ b = 13B$ in (24), we have

$$u = 8788A^3 + 13182B^3 - 39546A^2B - 26364AB^2$$
$$v = 13182A^3 - 8788B^3 + 26364A^2B - 39546AB^2$$

From (2) , the integer solutions of (1) are

$$x(A,B) = 21970A^3 + 4394B^3 - 13182A^2B + 65910AB^2$$
$$y(A,B) = -4394A^3 + 21970B^3 - 65910A^2B + 13182AB^2$$
$$z(A,B) = 74698A^3 + 26364B^3 - 79092A^2B - 224094AB^2$$
$$w(A,B) = -4394A^3 + 79092B^3 - 237276A^2B + 13182AB^2$$
$$p(A,B) = 169(A^2 + B^2)$$

The above solution patterns satisfy the following relations respectively:

$$4(x+y) = z + w$$
$$w^2 - 4(x+y)^2 + 2(x+y)(z-w) \equiv 0 \pmod 9$$

$4(x+y)^2 - 2(x-y)^2 - zw$ is a perfect square.

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$6\{4(x+y)^2 - zw\}$ is a nasty number.

$12(x^2 - y^2) = z^2 - w^2$

Remark 3:

If we take different set of transformations in the place of (2), as

$$x = u + v,\ y = u - v,\ z = 12u + v,\ w = 12u - v,\ \ u \neq v \neq 0 \tag{25}$$

in (1), we get

$$u^2 + v^2 = 52p^3 \tag{26}$$

By applying the same process as mentioned in the above choices, we obtain different choices of integral solutions to (1) using (25).

Choice 10:

$x(a,b) = 10a^3 - 2b^3 + 6a^2b - 30ab^2$

$y(a,b) = 2a^3 + 10b^3 - 30a^2b - 6ab^2$

$z(a,b) = 444a^3 + 10b^3 - 30a^2b - 133ab^2$

$w(a,b) = 420a^3 - 58b^3 + 174a^2b - 1260ab^2$

$p(a,b) = a^2 + b^2$

Choice 11:

$x(A,B) = 1150A^3 + 550B^3 - 1650A^2B - 3450AB^2$

$y(A,B) = -550A^3 + 1150B^3 - 3450A^2B + 1650AB^2$

$z(A,B) = 4450A^3 + 9900B^3 - 29700A^2B - 13350AB^2$

$w(A,B) = 2750A^3 + 10500B^3 - 31500A^2B - 8250AB^2$

$p(A,B) = 25(A^2 + B^2)$

Choice 12:

$x(A,B) = 21970A^3 + 4394B^3 - 13182A^2B + 65910AB^2$

$y(A,B) = -4394A^3 + 21970B^3 - 65910A^2B + 13182AB^2$

$z(A,B) = 118638A^3 + 149396B^3 - 448188A^2B - 355914AB^2$

$w(A,B) = 92274A^3 + 166972B^3 - 500916A^2B - 276822AB^2$

$p(A,B) = 169(A^2 + B^2)$

The above solution patterns satisfy the following relations respectively:

$2(zw - z^2) + (z - w)\{x - y + z + w\} = 0.$

$(z + w) - 12(x + y) = 0.$

$xy - x^2 + x(z - w) = 0.$

$2x - 2y - z + w = 0.$

$36(x+y)^2 - zw$ is a perfect square.

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3. Conclusion

In this paper, we have presented sets of infinitely many non-zero distinct integer solutions to the quintic equation with five unknowns given by $3\left(x^4-y^4\right)=26(z^2-w^2)p^3$. As Diophantine equations are rich in variety due to their definition. One may attempt to find integer solutions to higher degree Diophantine equation with multiple variables.

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