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Effect of Permeabilities on the Translational Motion of a Spherical Particle with Porous Core in a Concentric Spherical Cavity

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ABSTRACT

An analytical investigation for the creeping motion of a spherically symmetric fluid-permeable composite sphere composed by a uniform porous core and a uniformly surrounded porous shell located at the center of a spherical cavity filled with an incompressible Newtonian fluid is presented here. In the limit of small Reynolds number, the Stokes and Brinkman equations are solved for the flow field of the system. The hydrodynamic drag force exerted by the fluid on the composite sphere and wall correction factors are also obtained here. For a given geometry and permeability ratio, the variations of the wall correction factor are discussed. However, Keh and Chou [3] studied translation and rotation of a spherical particle composed by solid core and a surrounding porous shell located at the center of a spherical cavity filled with a fluid but this paper is different from that paper as core is taken porous here in place of solid core taken by above author. Another interesting thing is that permeability of core and surface layer on the core are taken unequal here. For the purpose of verification of results, in particular for the limiting cases, the analytical solutions describing the drag force on a composite sphere in the spherical cavity are reduced here for a simple solid sphere and simple porous sphere and obtained the results similar to Keh and Chou [3].

Key words: Composite sphere, porous core, spherical cavity, permeability ratio, drag, wall correction factor, analytical solution

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INTRODUCTION

The problem addressed in this paper is to obtain the wall effects on the creeping motion of an arbitrary composite sphere in concentric spherical cavity. The flow inside cavity wall and outside composite sphere is governed by the Stokes' equation. The flow within the porous layer and porous core (with different permeabilities k_2 and k_1 respectively) are governed by Brinkman equations. Boundary conditions e.g. no slip and matching conditions are employed on flow governing equations to obtain solution of the problem. Our objective here is to determine the hydrodynamic drag force exerted on the composite sphere (porous core). The wall correction factor is evaluated and its variation is studied numerically.

The problem has many applications in nature e.g. transport phenomena in environment, flotation, sedimentation, electrophoresis, spray drying, agglomeration and motion of blood cells in an artery or vein., transport of radio-nuclide from deposits of nuclear waste materials and other forced and convective flow associated with the fundamental geometries of internal (cavities, annulus, etc.) and external (over surfaces) flows.

MATHEMATICAL FORMULATION

Referring to Fig. 1, consider the creeping motion of a non-deformable composite sphere of radius b, consisting of a homogeneous porous core of radius a and permeability k_1 covered by a homogeneous porous shell of thickness b - a with permeability k_2 in a concentric spherical cavity of radius c filled with an incompressible Newtonian fluid of viscosity μ. We shall suppose that the composite sphere to be non-deformable and its centre translate with constant velocity U in the positive z direction. Apart from a constant velocity U, the problem is same to that of a spherical cavity moving in the negative z direction with uniform velocity U. Let us introduce a spherical co-ordinate system (r, θ, ϕ) with the origin located at the cavity centre and the line θ=0 as the axis of symmetry, in the direction of the sphere velocity U approaching the system. The Reynolds number is assumed to be sufficiently small so that the inertial terms in the fluid momentum equation can be neglected, in comparison with the viscous terms. The porous core region ($r \le a$), the porous surface layer region ($a \le r \le b$), and the region outside composite sphere and inside spherical cavity ($b \le r \le c$), are denoted as regions *I*, *II* and *III* respectively. Then, the fluid flow in regions *I* and *II* is governed by Brinkmen equation and the equation of continuity:

$$\mu\nabla^2 \mathrm{v}_i - (\mu / k_i)\mathrm{v}_i = \nabla p_i \tag{1}$$

$$\nabla.\mathrm{v}_i = 0 \tag{2}$$

where $i = 1, 2$.

For the fluid flow in region *III* is governed by Stokes equation and the equation of continuity:

$$\mu\nabla^2 \mathrm{v}_3 = \nabla p_3 \tag{3}$$

$$\nabla.\mathrm{v}_3 = 0 \tag{4}$$

The subscripts 1, 2 and3 refers to the physical quantities in regions *I*, *II* and *III* respectively.

Here, we have assumed that the fluid has the same viscosity inside and outside the composite sphere [5].

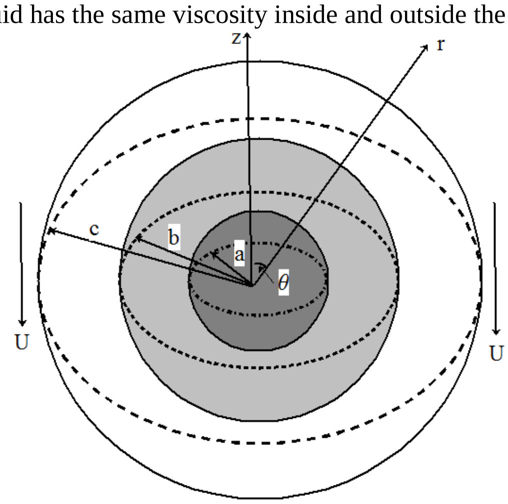

Fig.1 Composite sphere of radius *b* with a porous core of radius *a* in a concentric spherical cavity of radius *c*

BOUNDARY CONDITIONS

The following boundary conditions are used to analyze the flow in the three regions. The four matching conditions are imposed on the surface of porous core $(r = a)$ [2 and 7].

$$v_{r1} = v_{r2} \tag{5}$$

$$v_{\theta 1} = v_{\theta 2} \tag{6}$$

$$\tau_{rr(1)} = \tau_{rr(2)} \tag{7}$$

$$\tau_{r\theta(1)} = \tau_{r\theta(2)} \tag{8}$$

The boundary conditions at the outer surface of the porous surface layer $(r = b)$ due to the continuity of velocity and stress components, which is physically realistic and mathematically consistent for the present problem [1, 5-6, 8].

$$v_{r2} = v_{r3} \tag{9}$$

$$v_{\theta 2} = v_{\theta 3} \tag{10}$$

$$\tau_{rr(2)} = \tau_{rr(3)} \tag{11}$$

$$\tau_{r\theta(2)} = \tau_{r\theta(3)} \tag{12}$$

The no-slip boundary condition at the spherical cavity surface $(r = c)$ is

$$v_{r3} = -U\cos\theta \tag{13}$$

$$v_{\theta 3} = U\sin\theta \tag{14}$$

Here, τ_{rr} and $\tau_{r\theta}$ are the normal and shear stresses for the fluid flow relevant to the particle surfaces. These conditions take a reference frame that the composite sphere is at rest and velocity of the fluid at cavity wall is the particle velocity in the opposite direction. Since we take the same fluid viscosity inside and outside the composite sphere, use the fluid velocity continuity, and neglect the possible osmotic effect in composite sphere, normal component of stress is equivalent to the continuity of pressure.

SOLUTION OF THE PROBLEM AND DETERMINATION OF ARBITRARY CONSTANTS

As the flow is axially symmetric, we introduce the Stokes stream function $\psi_i(r,\theta)$ satisfying the equation of continuity on taking

$$v_{ri} = -\frac{1}{r^2 \sin\theta}\frac{\partial \psi_i}{\partial \theta} \tag{15}$$

$$v_{\theta i} = \frac{1}{r\sin\theta}\frac{\partial \psi_i}{\partial r} \tag{16}$$

where $\psi_1(r,\theta)$, $\psi_2(r,\theta)$ and $\psi_3(r,\theta)$ correspond respectively to regions *I*, *II* and *III*. Eliminating pressure p_3 from equation (3) by taking the curl and making use of equation (4), we get

$$E^4\psi_3 = 0\,, \quad (b \le r \le c), \tag{17}$$

Where E^2 denotes the Stokes stream function operator given by

$$E^2 = \frac{\partial}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) \tag{18}$$

Accordingly, Eq. (1) and (2) can be expressed for *i*=1 and 2 in terms of the stream functions, as

$$E^4\psi_1 - (1/k_1)E^2\psi_1 = 0\,, \quad \left(r \le a\right) \tag{19}$$

$$E^4\psi_2 - (1/k_2)E^2\psi_2 = 0, \quad \left(a \le r \le b\right) \tag{20}$$

A solution to Eq. (17), (19) and (20) suitable for satisfying boundary conditions on the spherical surfaces is [3-4, 6]

$$\psi_1 = \varepsilon(A_1\lambda^2 + B_1(\lambda^{-1}\kappa\sinh(\kappa^{-1}\lambda) - \cosh(\kappa^{-1}\lambda)))\sin^2\theta\,, \left(\lambda \le \alpha\right) \tag{21}$$

$$\psi_2 = \varepsilon(C_2\lambda^{-1} + A_2\lambda^2 + G(\kappa^{-1}\lambda^{-1}\cosh(\kappa\lambda) - \sinh(\kappa\lambda)) + B_2(\kappa^{-1}\lambda^{-1}\sinh(\kappa\lambda) - \cosh(\kappa\lambda)))\sin^2\theta, \left(\alpha \le \lambda \le \beta\right) \tag{22}$$

$$\psi_3 = \varepsilon\left(C_3\lambda^{-1} + E\lambda + A_3\lambda^2 + F\lambda^4\right)\sin^2\theta\,, \quad \left(\beta \le \lambda \le \gamma\right) \tag{23}$$

where the dimensionless variables and constants $\lambda = r\left(k_1k_2\right)^{-1/4}$, $\alpha = a\left(k_1k_2\right)^{-1/4}$, $\beta = b\left(k_1k_2\right)^{-1/4}$, $\gamma = c\left(k_1k_2\right)^{-1/4}$, $2\varepsilon = U\left(k_1k_2\right)^{1/2}$ and $\kappa = \left(k_1/k_2\right)^{1/4}$. We denote the ratio of permeability of porous core to porous shell of composite sphere by κ^4. The dimensionless constants A_1, A_2, A_3, B_1, B_2, C_2, C_3, *E*, *F* and *G* are found from Eq. (5) to (14). The procedure is straightforward but tedious, and the expressions for these constants are lengthy, we do not present them here except *E* which is required for the drag to the composite sphere by fluid external to composite sphere given by

$$\begin{aligned}E = {} & (6\gamma\kappa^2(s_2((\kappa^3\beta\gamma^5 - \kappa^3\beta^6 - 45\kappa\beta^4)s_9 + (\kappa^2\gamma^5 - 6\kappa^2\beta^5 - 45\beta^3)s_{10}) + 30\beta^6(s_1s_7(\kappa\beta s_9 + s_{10}) + \kappa\alpha s_7s_{11}s_{13} + (s_{13} \\ & + \alpha^2\kappa^3s_0)s_7s_{12})))/(3s_1s_3s_6s_7 + \kappa(\kappa^2(180\beta^3(\beta-\gamma) + (\beta-\gamma)^4(4\beta^2 + 7\beta\gamma + 4\gamma^2)\kappa^2)s_2 + 6\beta(-45\beta^3\gamma + (-20\beta^6 + 9\beta^5\gamma \\ & + 10\beta^3\gamma^3 + \gamma^6)\kappa^2)s_1s_7)s_9 + (3(60\kappa^2\beta^3 + (\beta-\gamma)^3(8\beta^2 + 9\beta\gamma + 3\gamma^2)\kappa^4)s_2 - 6(45\beta^3\gamma + (20\beta^6 + 36\beta^5\gamma - 10\beta^3\gamma^3 \\ & - \gamma^6)\kappa^2)s_1s_7)s_{10} - 6s_7(-90\beta^3\gamma + (20\beta^6 - 27\beta^5\gamma + 5\beta^3\gamma^3 + 2\gamma^6)\kappa^2)(\kappa\alpha s_{11}s_{13} + (s_{13} + \alpha^2\kappa^3s_0)s_{12}))\end{aligned}$$

where the dimensionless parameters s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14} and s_{15} are given in appendix.

EVALUATION OF DRAG ON COMPOSITE SPHERE

Evaluation of drag force is important in the applications of the flow problem we are investigating. Drag on the sphere is the force exerted on it by the moving fluid. The drag force (in the *z* direction) exerted by the external fluid on the composite sphere (porous core) with the spherical boundary $r = b$ can be evaluated as:

$$D = \pi\mu\int_0^{\pi} r^3\sin^3\theta\frac{\partial}{\partial r}\left(\frac{\Im^2\psi_3}{r^2\sin^2\theta}\right)rd\theta \tag{24}$$

Substitution of Eq. (23) into the above integral results in the simple relation

$$D = 4\pi\mu U\left(k_1k_2\right)^{1/4}E \tag{25}$$

where *E* is same as for equation (23).

RESULTS AND DISCUSSION

Some Cases and Known Results:

(A) DRAG

- When $\gamma \to \infty$, the expression for the drag force *D* is the reduced result for the translation of an isolated composite sphere in an unbounded fluid is given by $$D_\infty = 4\pi\mu U\left(k_1k_2\right)^{1/4}E_\infty \tag{26}$$

Where

$$\begin{aligned}E_\infty = {} & (3s_2\kappa^2(\beta\kappa s_9 + s_{10}))/(3\beta^3\kappa^4s_1^2s_3s_7^2 + 2\kappa^3s_2s_9 + 3\alpha^2\kappa^3s_0s_7(3\kappa s_1s_3s_7(\beta\kappa\cosh\left(\kappa(\alpha-\beta)\right) + \alpha\kappa s_8 \\ & + \sinh(\kappa(\alpha-\beta))) - 2s_{12}) - 6s_7(\alpha\kappa s_{11} + s_{12})s_{13} + 3s_1s_7(\beta\kappa s_9 + s_{10} - \kappa s_3s_7(3(\alpha-\beta)\kappa\cosh\left(\kappa(\alpha-\beta)\right) \\ & + 3(\alpha\beta\kappa^2 - 1)\sinh(\kappa(\alpha-\beta)) + \alpha^3\kappa^3s_8)s_{13}))\end{aligned}$$

- When permeability of core and surface layer of composite spherical particle are equal i.e. $k_1 = k_2 = k$ (say), (equivalently the radius of porous core is equal to radius of outer surface of composite spherical particle i.e. a=b) the expression for the drag force *D* reduces as

$$D^{\kappa_1} = 4\pi\mu U k^{1/2} E^{\kappa_1} \tag{27}$$

for the translation of an isolated porous sphere of radius b in a spherical cavity, here

$$E^{\kappa_1} = -(6\beta^3\gamma(\beta(15\beta^3+\beta^5-\gamma^5)\cosh(\beta)-(15\beta^3+6\beta^5-\gamma^5)\sinh(\beta)))/(\beta(60\beta^6+4\beta^8-9\beta^7\gamma+6\gamma^6+4\beta^2\gamma^6$$
$$+2\beta^5\gamma(5\gamma^2-63)-3\beta^3\gamma(90-20\gamma^2+3\gamma^4))\cosh(\beta)-3(20\beta^6+8\beta^8-15\beta^7\gamma+2\gamma^6+2\beta^5\gamma(5\gamma^2-36)$$
$$-\beta^3\gamma(90-20\gamma^2+3\gamma^4))\sinh(\beta))$$

Moreover, when c is very large, we have $\gamma \to \infty$, the expression for the drag force *D* reduces as

$$D_\infty^{\kappa_1} = 12\pi\mu U\left(\frac{\sqrt{k}\beta^3\left(\beta\cosh(\beta)-\sinh(\beta)\right)}{\beta\left(3+2\beta^2\right)\cosh(\beta)-3\sinh(\beta)}\right) \tag{28}$$

for the translation of an isolated porous sphere in an unbounded fluid. The expression conform with the physics of flow as it shows that the magnitude of drag force on the particle decreases on increasing radius of the outer cavity sphere and is least for unbounded medium. In addition, if radius of particle is small so that its forth order can be neglected in the expression for the magnitude of drag force D^{κ_1} is approximated as

$$D_\infty^{\kappa_1} = 4\pi\mu U\sqrt{k}\beta^3 \tag{29}$$

This shows that under above limitations, the drag increases cubically on increasing radius of the inner particle.

- If we have impermeability condition i.e. $k \to 0$, in the expression (27)for the drag force, the results conform with the physics of flow due solid particle. Remember that β is function of permeability k. Now, the drag force $D_\infty^{k_0}$ becomes

$$D_\infty^{k_0} = 6\pi\mu U b \tag{30}$$

which is same as classical result for the translation of an isolated solid sphere of radius b in an unbounded fluid. Moreover, when $k \to \infty$, the expression (28) for the drag force reduces as

$$D_\infty^{k_\infty} = 0 \tag{31}$$

(B) WALL EFFECTS

- The wall correction factor K is ratio of the actual drag D experienced by the porous particle in the concentric spherical cavity and the drag D_∞ on the porous particle in an infinite expanse of fluid. Observe that $K = 1$ as $\beta/\gamma = 0$ and $1 \le K$ as $0 < \beta/\gamma \le 1$. The presence of the cavity wall always enhances the hydrodynamic drag on the composite sphere since the fluid flow vanishes at the wall as required by no slip boundary conditions appeared in eq.(14).
- When $\kappa = 1$ $(k_1 = k_2 = k)$ (permeability of core and surface layer of composite sphere are equal), the expression for the wall correction factor of a composite sphere (porous core) K reduces for the wall correction factor K^{κ_1} of an isolated porous sphere in a spherical cavity

$$K^{\kappa_1} = D^{\kappa_1} / D_\infty^{\kappa_1} \tag{32}$$

where D^{κ_1} and $D_\infty^{\kappa_1}$ are given in expressions (27) and (28).

- When $k = 0$ in above case we get the translation of a solid sphere in a spherical cavity.

GENERAL CASES OF THE WALL CORRECTION FACTOR

We now examine the some general cases of the wall correction factor K. This depends upon β/γ also. The ratio β/γ ranges from 0 (when radius of outer cavity sphere tends to infinity i.e. $\gamma \to \infty$) to 1 (when no cavity). So this ratio reflects the extent of closeness between the particle and cavity wall. Figs (2-8) depict the motion of translating composite sphere (porous core) in a concentric spherical cavity. These Figs are drawn for describing the relationship between the wall correction factor K and radii of spheres for various values of permeabilities.

In Figs (2-7), is plotted for different cases for β and α/β, as a function of β/γ (on horizontal axis) over the entire ranges of the separation and some values between 0 to 1of the parameter α/β. Fig. (2) describes the relationship between the wall correction factor K (on vertical axis) and ratio β/γ (on horizontal axis) keeping $\beta = 1$ fix $\alpha/\beta = 0.2$ Observe that six curves in this fig are characterized for six values of κ. In the Fig., it is evident that value of K increases on increasing β/γ. Further, it may be interesting to observe that when $k_1 > k_2$ i.e. when $\kappa > 1$, the curve for greater vales of κ are above the curve for lesser values (shows that K increases with the increasing values of κ when $\kappa > 1$.), however, when $k_1 < k_2$ i.e. when $\kappa < 1$ the curve for greater vales of κ are below the curve for lesser values. (shows that *K* decreases with the increasing values of κ when $\kappa<1$).Observe that cases for $\kappa=$ 0.9 and 0.1, the curves are much closed, so that approximately coincident. Fig. (3) is similar as Fig. (2) except the value of β which is now 4 (instead of 1 as in fig. 2) Further, it may be interesting to observe that the slope of curve in fig 3 are greater than the slope of curves in fig 2. So increment in β also increases the value of *K*. Now the curve for $\kappa=$ 0.05 and 0.1, are very much closed. The fig. 4-8 are succession of the above but the pattern have some peculiarities, e.g. in fig. 4 it may be interesting to observe that the curve for $\kappa=2$ lies between curves for $\kappa=0.9$ and $\kappa=0.1$. In fig. 5 the curve for $\kappa=0.9$ intersect to the curves $\kappa=0.05$ and $\kappa=0.1$ at α/β = 0.8475 and 0.8805 respectively. In fig. 6 the curve for $\kappa=10$ is intersecting to the curves $\kappa=$ 0.1 at $\alpha/\beta=0.456$. In fig. 7 curve for $\kappa=5$ is intersecting to the curves $\kappa=0.05$ and 0.1 at $\alpha/\beta=0.6004$ and 0.6344and curve for $\kappa=0.9$ is intersecting to the curves $\kappa=2$ $\alpha/\beta=0.7646$ and in this Fig. curve for $\kappa=$ 10 is not analysed here as *K* is very large in amplitude showing fluctuation about initial line *K*.

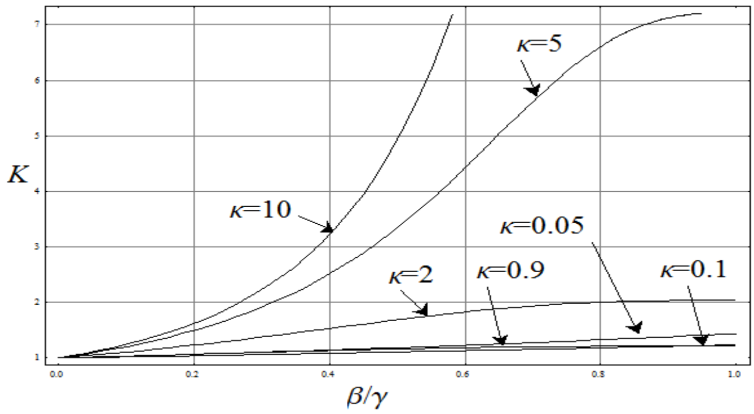

Fig.2 *K* versus *β/γ* for *α/β*=0.2 and *β* =1 and six values of κ

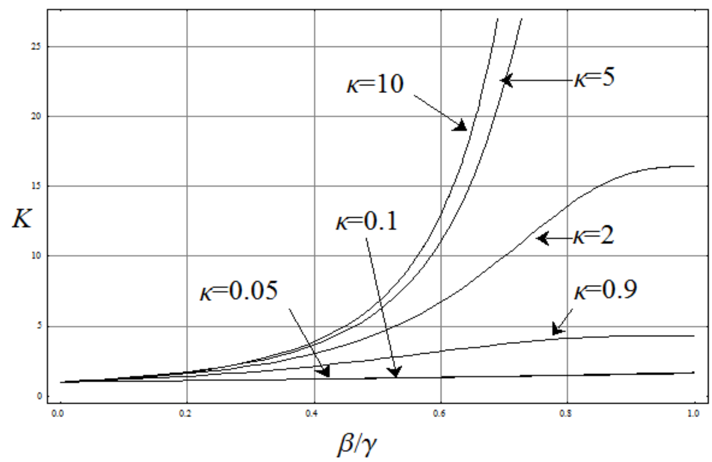

Fig. 3 *K* versus *β/γ* for *α/β*=0.2 and *β* =4 and six values of κ

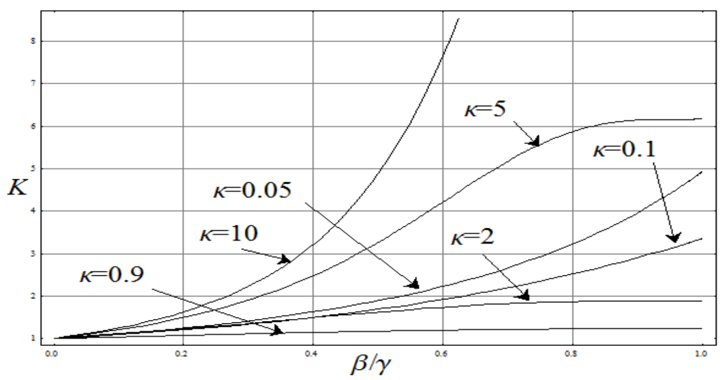

Fig.4. *K* versus *β/γ* for *α/β*=0.5 and *β* =1 and six values of κ

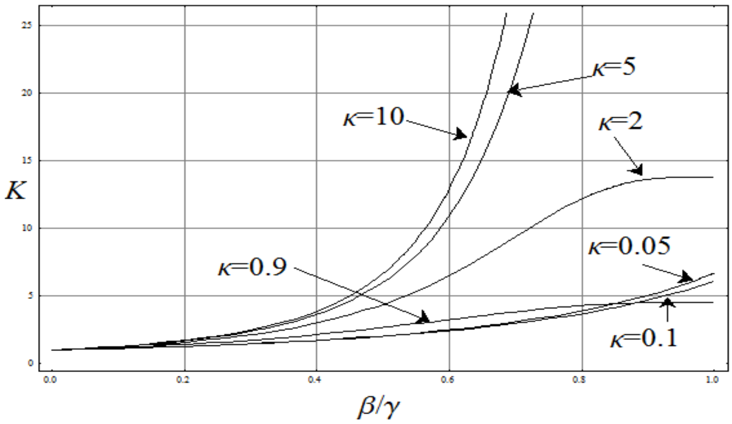

Fig.5. ***K*** **versus** ***β/γ*** **for** ***α/β*****=0.5 and** ***β*** **=4 and six values of κ**

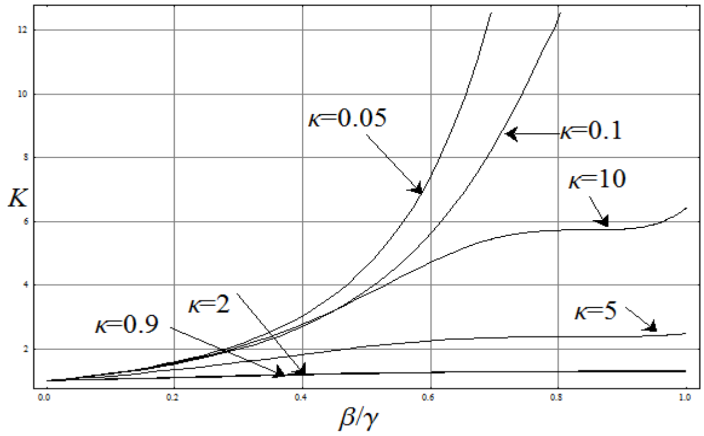

Fig.6. ***K*** **versus** ***β/γ*** **for** ***α/β*****=0.9 and** ***β*** **=1 and six values of κ**

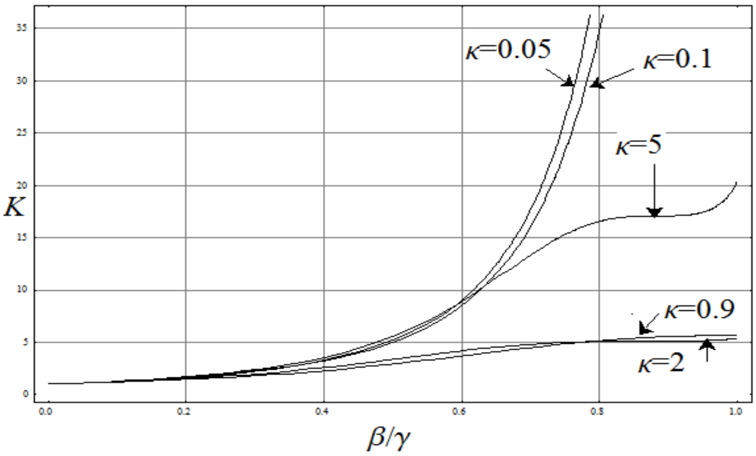

Fig.7 ***K*** **versus** ***β/γ*** **for** ***α/β*****=0.9 and** ***β*** **=4 and five values of κ**

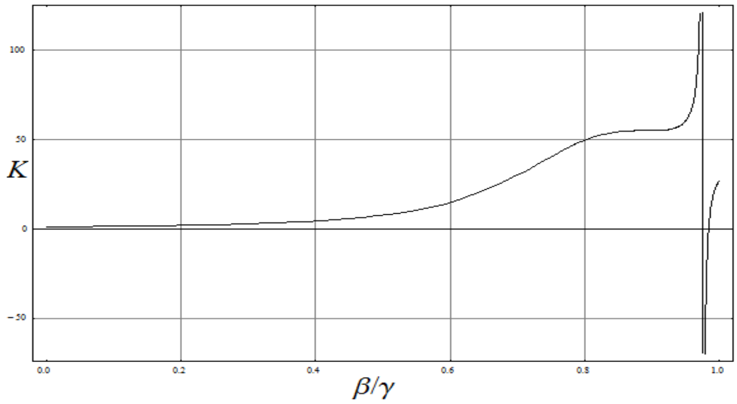

Fig.8 ***K*** **versus** ***β/γ*** **for** ***α/β*****=0.9 and** ***β*** **=4 and κ=9.8938688**

In fig. 7 various curves show that when permeability of composite sphere core is less than permeability of shell of composite sphere (κ <1) , the wall correction factor increases on decreasing κ. However when permeability of composite sphere core is greater than permeability of shell of composite sphere (1<κ <9.8938688 approx.), the wall correction factor increases with increasing κ. In Fig. 8 , it may be observe that the wall correction factor K for κ =9.89387 initially increases with maxima at the point β/γ=0.972 (approx). For β/γ >0.972. The wall correction factor decreases and had been negative value for 0.9748< β/γ <0.9778(approx). The negative K is caused by the high permeability produced by porous core of the composite sphere with respect to of porous shell of the composite sphere. Case $\alpha/\beta = 0$ (similar as case κ =1 or $\alpha/\beta = 1$) provides the results for simple porous particle that is discussed by Keh and Chou [3] and Khe and Lu [4], so we do not discuss here.

CONCLUSION

An analytic solution of the governing equations for the problem of the motion of a composite sphere in a spherical cavity filled with an incompressible Newtonian fluid has been obtained. Brinkman's model is used in porous region and Stokes' equations in the liquid region to analyze the problem. An expression for the hydrodynamic drag on the composite sphere in a spherical cavity is obtained. The wall effect is computed and presented the whole range of influences of the considered porous parameter from the limiting case of nearly porous sphere to solid sphere by Fig.s. It has been found that, the wall correction factor of the composite sphere is increasing function of separation parameter (ratio of radius of composite sphere to spherical cavity). The analysis assumes that composite sphere and its core are non deformable. We believe that our results provide useful insights into the actual phenomena of the motion of a composite sphere in a spherical cavity/container, also these results are more realistic to pore geometries for the spherical cavity and wall effects of the cavity wall on this motion can be significant in appropriate situations.

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Appendix

Entities used above are as:

$$s_0 = \alpha\cosh(\alpha/\kappa) - \kappa\sinh(\alpha/\kappa),\ s_1 = s_7 s_{13} - 3\kappa^5 s_0,\ s_2 = \alpha^3 s_7 s_8 (3\kappa s_0 - s_{13}) + \beta^3 s_1 s_7,\ s_3 = 6\alpha^3 s_0 s_8,$$

$$s_4 = -\kappa s_1 s_7 (s_0(\kappa^2\beta^3(3\kappa^2 s_8 s_{14} + \alpha^2 s_7\sinh(\kappa\alpha)) - \alpha^3(3s_8 s_{14} - 6\kappa\beta\cosh(\kappa\beta) + \kappa^2\alpha^2\sinh(\kappa\alpha) + 6\sinh(\beta\kappa))s_8) - \kappa^3(\beta^3 s_7 - \alpha^3 s_8)s_{13}s_{14}),$$

$$s_5 = -\kappa s_1 s_7 (s_0(\kappa^2\beta^3(3\kappa^2 s_8 s_{15} + \alpha^2 s_7\cosh(\kappa\alpha)) - \alpha^3(3s_8 s_{15} - 6\kappa\beta\sinh(\kappa\beta) + \kappa^2\alpha^2\cosh(\kappa\alpha) + 6\cosh(\beta\kappa))s_8) - \kappa^3(\beta^3 s_7 - \alpha^3 s_8)s_{13}s_{15}),$$

$$s_6 = \gamma\kappa s_7(s_1\kappa^5\beta^3(3\beta^5 - 5\beta^3\gamma^2 + 2\gamma^5) + (3s_0\kappa - s_{13})\alpha^3\kappa^3(3\beta^5\kappa^2 + 2\gamma^5\kappa^2 - 5\beta^3(18+\gamma^2\kappa^2))s_8 + 6((\alpha^2\beta\kappa^4(-6\beta^5\kappa^2 + \gamma^5\kappa^2$$
$$+5\beta^3(-9+\gamma^2\kappa^2))s_0 + \kappa(45(\alpha-\beta)\beta^3 + (3(7\alpha - 2\beta)\beta^5 + (\beta-\alpha)\gamma^2(5\beta^3+\gamma^3))\kappa^2)s_{13})\cosh(\kappa(\alpha-\beta)) + (\alpha^2\kappa^3(-21\beta^5\kappa^2$$
$$+\gamma^5\kappa^2 + 5\beta^3(-9+\gamma^2\kappa^2))s_0 + (-45\beta^3 + (45\alpha\beta^4 - 21\beta^5 + 5\beta^3\gamma^2 + \gamma^5)\kappa^2 + \alpha\beta(6\beta^5 - 5\beta^3\gamma^2 - \gamma^5)\kappa^4)s_{13})\sinh(\kappa(\alpha-\beta))))$$

$$s_7 = 2 + \kappa^4,\ s_8 = -1 + \kappa^4,\ s_9 = s_4\sinh(\beta\kappa) - s_5\cosh(\beta\kappa),\ s_{10} = s_5\sinh(\beta\kappa) - s_4\cosh(\beta\kappa),\ s_{11} = s_4\sinh(\alpha\kappa) - s_5\cosh(\alpha\kappa),$$

$$s_{12} = s_5\sinh(\alpha\kappa) - s_4\cosh(\alpha\kappa),\ s_{13} = \alpha^2\sinh(\alpha/\kappa),\ s_{14} = \alpha\kappa\cosh(\alpha\kappa) - \sinh(\alpha\kappa),\ s_{15} = -\cosh(\alpha\kappa) + \alpha\kappa\sinh(\alpha\kappa)$$