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Existence results for a coupled system of fractional differential equations with multi-point boundary value problems

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ABSTRACT

In this paper, we study existence and uniqueness of solutions for a coupled system of multi-point boundary value problems for fractional differential equations. Applying principle contraction and Shaefer's fixed point theorem new existence results are obtained.

1. Introduction

Differential equations of fractional order have been of great interest for the last three decades. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in the modelling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, and so forth. Therefore, the theory of fractional differential equations has been developed very quickly. Many qualitative theories of fractional differential equations have been obtained. Many important results can be found in [4, 5, 7, 9, 16, 17, 20, 21, 25] and the references cited therein. Multi-point nonlinear boundary value problems, which refer to a different family of boundary conditions in the study of disconjugacy theory [3] and take into account the boundary data at intermediate points of the interval under consideration, have been addressed by many authors, for example, see [1, 6, 8, 10, 11, 24, 28, 29] and the references therein. Multi-point boundary conditions are important in various physical problems of applied science when the controllers at the end points of the interval dissipate or add energy according to the censors located at intermediate points. Moreover, the study of coupled

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systems of fractional order is also important in various problems of applied nature [2, 12, 13, 14, 18, 25, 26, 30, 31]. Recently, many people have established the existence and uniqueness of solutions for the multipoint boundary value problems of some fractional systems, see [19, 27] and the reference therein. This paper deals with the existence and uniqueness of solutions to the following coupled system of fractional differential equations :

$$\begin{cases} D^{\alpha}x(t) = \psi_1(t) f_1(t, y(t), (\phi_1 y)(t), (\phi_2 y)(t)), t \in J, \\ D^{\beta}y(t) = \psi_2(t) f_2(t, x(t), (\varphi_1 x)(t), (\varphi_2 x)(t)), t \in J, \\ x(0) = x_0^*, \left|x'(0)\right| + \left|x''(0)\right| + ... + \left|x^{(n-2)}(0)\right| = 0, x(T) = \sum\limits_{i=1}^{m} \lambda_i x(\eta_i), \\ y(0) = y_0^*, \left|y'(0)\right| + \left|y''(0)\right| + ... + \left|y^{(n-2)}(0)\right| = 0, y(T) = \sum\limits_{j=1}^{m} \mu_j y(\xi_j), \end{cases} \tag{1}$$

where D^{α} and D^{β}, denote the Caputo fractional derivatives, with $n-1 < \alpha < n$ and $n-1 < \beta < n, n \in N^*$, $n \neq 1, J = [0,1], \lambda_i, \mu_j \in \mathbb{R}^+, i,j = 1,...,m$, ψ_1, ψ_2 are two continuous functions, $\phi_h x, \varphi_h y, h = 1,2$ are integral operators, $x_0^*, y_0^* \in \mathbb{R}, 0 < \eta_i, \xi_j < 1, i,j = 1,...,m$, f_1, f_2, are two functions which will be specified later, and for $\gamma_h, \delta_h, h = 1,2 : [0,1] \times [0,1] \to [0,\infty)$,

$$(\varphi_h x)(t) = \int_0^t \gamma_h(t,s) x(s) ds, (\phi_h y)(t) = \int_0^t \delta_h(t,s) y(s) ds, \ h = 1,2.$$

The rest of this paper is organized as follows. In section 2, we present some preliminaries and lemmas. Section 3 is devoted to existence of solution of problem (1). In section 4 an example is treated illustrating our results.

2. Preliminaries

The following notations, definitions, and preliminary facts will be used throughout this paper.

Definition 1 *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[0,\infty[$ is defined as :*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0, \tag{2}$$

$$J^0 f(t) = f(t),$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

Definition 2 *The fractional derivative of $f \in C^n([0,\infty[)$ in the Caputo's sense is defined as :*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, n-1 < \alpha < n, n \in N^*. \tag{3}$$

For more details about fractional calculus, we refer the reader to [22, 23].

Let us now introduce the spaces $X = \{x : x \in C([0,T])\}$ and $Y = \{y : y \in C([0,T])\}$ endowed with the norm $\| x \| = \sup\limits_{t \in J} |x(t)|$ and $\| y \| = \sup\limits_{t \in J} |y(t)|$.

Obviously, $(X, \| \, . \, \|)$ and $(Y, \| \, . \, \|)$, are Banach spases. The product space $(X \times Y, \|(x,y)\|)$ is also Banach space with norm $\|(x,y)\| = \|x\| + \|y\|$.

We give the following lemmas [15] :

Lemma 3 *For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha} x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{4}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1, n = [\alpha] + 1$.

Lemma 4 *Let $\alpha > 0$. Then*

$$J^{\alpha} D^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{5}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, n = [\alpha] + 1$.

We need also the following auxiliary result :

Lemma 5 *Let $\sum_{i=1}^{m} \lambda_i \eta_i^{n-1} \neq T^{n-1}$. Then for a given $g \in C([0,T])$, the solution of the equation*

$$D^{\alpha} x(t) = (\psi_1 g)(t), t \in J, n-1 < \alpha \leq n, \tag{6}$$

subject to the boundary condition

$$x(0) = x_0^*, \left| x^{'}(0) \right| + \left| x^{''}(0) \right| + ... + \left| x^{(n-2)}(0) \right| = 0, x(T) = \sum_{i=1}^{m} \lambda_i x(\eta_i), \tag{7}$$

is given by

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\psi_1 g)(s) \, ds + x_0^* \\ &+ \frac{t^{n-1}}{\Gamma(\alpha) \left(\sum_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1} \right)} \int_0^T (T-s)^{\alpha-1} (\psi_1 g)(s) \, ds \\ &- \frac{t^{n-1}}{\Gamma(\alpha) \left(\sum_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1} \right)} \sum_{i=1}^{m} \lambda_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} (\psi_1 g)(s) \, ds. \end{aligned} \tag{8}$$

Proof. By lemmas 3 and Lemma 4, the general solution of (6 is written as

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\psi_1 g)(s) \, ds - c_0 - c_1 t - c_2 t^2 - ... - c_{n-1} t^{n-1}. \tag{9}$$

Using the boundary condition (7), we have $c_1 = ... = c_{n-2} = 0$ and $c_0 = -x_0^*$.

For c_{n-1}, we have

$$c_{n-1} = -\frac{1}{\Gamma(\alpha)\left(\sum\limits_{i=1}^{m}\lambda_i\eta_i^{n-1}-T^{n-1}\right)}\int_0^T (T-s)^{\alpha-1}(\psi_1 g)(s)\,ds \qquad (10)$$
$$+\frac{1}{\Gamma(\alpha)\left(\sum\limits_{i=1}^{m}\lambda_i\eta_i^{n-1}-T^{n-1}\right)}\sum_{i=1}^{m}\lambda_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}(\psi_1 g)(s)\,ds.$$

Substituting the value of $c_0, c_1, ..., c_{n-2}$ and c_{n-1} in (10), we get (9).

3. Main Results

Let us introduce the quantities :

$$\omega_h = \sup_{t\in[0,T]}\left|\int_0^t \delta_h(t,s)\,ds\right|, \varpi_h = \sup_{t\in[0,T]}\left|\int_0^t \gamma_h(t,s)\,ds\right|, h=1,2,$$
$$\theta_1 = \frac{\|\psi_1\|_\infty}{\Gamma(\alpha+1)}\left(1+\frac{T^\alpha+\sum\limits_{i=1}^{m}|\lambda_i|\,\eta_i^\alpha}{\left|\sum\limits_{i=1}^{m}\lambda_i\eta_i^{n-1}-T^{n-1}\right|}\right), \qquad (11)$$
$$\theta_2 = \frac{\|\psi_2\|_\infty}{\Gamma(\beta+1)}\left(1+\frac{T^\beta+\sum\limits_{j=1}^{m}|\mu_j|\,\xi_j^\beta}{\left|\sum\limits_{j=1}^{m}\mu_j\xi_j^{n-1}-T^{n-1}\right|}\right).$$

We list also the following hypotheses :

(H_1) : The functions $f_1, f_2\ : [0,T]\times\mathbb{R}^3\to\mathbb{R}$ are continuous.

(H_2) : There exist non negative continuous functions $a_i, b_i \in C([0,T]), i = 1,3$ such that for all $t\in[0,T]$ and $(x,y,z), (x_1,y_1,z_1)\in\mathbb{R}^3$, we have

$$|f_1(t,x,y,z)-f_1(t,x_1,y_1,z_1)| \leq a_1(t)|x-x_1|+a_2(t)|y-y_1|+a_3(t)|z-z_1|,$$
$$|f_2(t,x,y,z)-f_1(t,x_1,y_1,z_1)| \leq b_1(t)|x-x_1|+b_2(t)|y-y_1|+b_3(t)|z-z_1|,$$

with

$$A_1 = \sup_{t\in J}a_1(t), A_2=\sup_{t\in J}a_2(t), A_3=\sup_{t\in J}a_3(t)$$
$$B_1 = \sup_{t\in J}b_1(t), B_2=\sup_{t\in J}b_2(t), B_3=\sup_{t\in J}b_3(t).$$

(H_3) : There exists non negative continuous functions l_1 and l_2 such that $|f_1(t,x,y,z)|\leq l_1(t), |f_2(t,x,y,z)|\leq l_2(t)$ for each $t\in J$ and all, $(x,y,z)\in\mathbb{R}^3$ with $L_1=\sup\limits_{t\in J}l_1(t), L_2=\sup\limits_{t\in J}l_2(t)$.

Our first result is based on Banach contraction principle :

Theorem 6 *Suppose that* $\sum\limits_{i=1}^{m}\lambda_i\eta_i^{n-1} \neq T^{n-1}, \sum\limits_{j=1}^{m}\mu_j\xi_j^{n-1} \neq T^{n-1}$ *and assume that the hypothesis* (H_1) *and* (H_2) *hold.*

If

$$\theta_1\left(A_1+A_2\delta_1+A_3\delta_2\right)+\theta_2\left(B_1+B_2\gamma_1+B_3\gamma_2\right)<1, \tag{12}$$

then the boundary value problem (1) has a unique solution on J.

Proof. Consider the operator $R : X\times Y \to X\times Y$ defined by :

$$R\left(x,y\right)\left(t\right):=\left(R_1y\left(t\right),R_2x\left(t\right)\right), \tag{13}$$

where

$$\begin{aligned}
&R_1y\left(t\right)=\\
&\tfrac{1}{\Gamma(\alpha)}\int_0^t\left(t-s\right)^{\alpha-1}\psi_1\left(s\right)f_1\left(s,y\left(s\right),\left(\phi_1y\right)\left(s\right),\left(\phi_2y\right)\left(s\right)\right)ds+x_0^*\\
&+\frac{t^{n-1}}{\Gamma(\alpha)\left(\sum\limits_{i=1}^{m}\lambda_i\eta_i^{n-1}-T^{n-1}\right)}\int_0^T\left(T-s\right)^{\alpha-1}\psi_1\left(s\right)f_1\left(s,y\left(s\right),\left(\phi_1y\right)\left(s\right),\left(\phi_2y\right)\left(s\right)\right)ds\\
&-\frac{t^{n-1}}{\Gamma(\beta)\left(\sum\limits_{j=1}^{m}\mu_j\xi_j^{n-1}-T^{n-1}\right)}\sum_{j=1}^{m}\mu_j\int_0^{\xi_j}\left(\mu_j-s\right)^{\beta-1}\psi_1\left(s\right)f_2\left(s,x\left(s\right),\left(\phi_1x\right)\left(s\right),\left(\phi_2x\right)\left(s\right)\right)ds
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&R_2x\left(t\right)=\\
&\tfrac{1}{\Gamma(\beta)}\int_0^t\left(t-s\right)^{\beta-1}\psi_2\left(s\right)f_2\left(s,x\left(s\right),\left(\varphi_1x\right)\left(s\right),\left(\varphi_2x\right)\left(s\right)\right)ds+y_0^*\\
&+\frac{t^{n-1}}{\Gamma(\beta)\left(\sum\limits_{j=1}^{m}\mu_j\xi_j^{n-1}-T^{n-1}\right)}\int_0^T\left(T-s\right)^{\beta-1}\psi_2\left(s\right)f_2\left(s,x\left(s\right),\left(\varphi_1x\right)\left(s\right),\left(\varphi_2x\right)\left(s\right)\right)ds\\
&-\frac{t^{n-1}}{\Gamma(\beta)\left(\sum\limits_{j=1}^{m}\mu_j\xi_j^{n-1}-T^{n-1}\right)}\sum_{j=1}^{m}\mu_j\int_0^{\xi_j}\left(\mu_j-s\right)^{\beta-1}\psi_1\left(s\right)f_2\left(s,x\left(s\right),\left(\phi_1x\right)\left(s\right),\left(\phi_2x\right)\left(s\right)\right)ds
\end{aligned} \tag{15}$$

We shall prove that R is contraction mapping :

Let us set $\sup_{t\in[0,T]}f_1\left(t,0,0,0\right)=M_1$ and $\sup_{t\in[0,T]}f_2\left(t,0,0,0\right)=M_2$ such that

$$r\geq\frac{\theta_1M_1+\theta_2M_2+\left|x_0^*\right|+\left|y_0^*\right|}{1-\theta_1\left(A_1+A_2\delta_1+A_3\delta_2\right)-\theta_2\left(B_1+B_2\gamma_1+B_3\gamma_2\right)} \tag{16}$$

We show that $RB_r\subset B_r$, where $B_r=\left\{(x,y)\in X\times Y,\left\|(x,y)\right\|\leq r\right\}.$

For $(x,y)\in B_r$, we have :

$$\begin{aligned}
&|R_1 y(t)| \leq \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{0\leq s\leq T} |\psi_1(s)| \, |f_1(s, y(s), (\phi_1 y)(s), (\phi_2 y)(s))| \, ds \\
&+ |x_0^*| + \frac{t^{n-1}}{\Gamma(\alpha)\left|\sum\limits_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1}\right|} \int_0^T (T-s)^{\alpha-1} \\
&\times \sup_{0\leq s\leq T} |\psi_1(s)| \, |f_1(s, y(s), (\phi_1 y)(s), (\phi_2 y)(s))| \, ds \\
&+ \frac{t^{n-1}}{\Gamma(\alpha)\left|\sum\limits_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1}\right|} \sum_{i=1}^{m} |\lambda_i| \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \\
&\times \sup_{0\leq s\leq T} |\psi_1(s)| \, |f_1(s, y(s), (\phi_1 y)(s), (\phi_2 y)(s))| \, ds
\end{aligned} \tag{17}$$

Hence

$$\begin{aligned}
&|R_1 y(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{0\leq s\leq T} |\psi_1(s)| \\
&\times \left(|f_1(s, y(s), (\phi_1 y)(s), (\phi_2 y)(s)) - f_1(s,0,0,0)| + |f_1(s,0,0,0)| \right) ds \\
&+ |x_0^*| + \frac{t^{n-1}}{\Gamma(\alpha)\left|\sum\limits_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1}\right|} \int_0^T (T-s)^{\alpha-1} \left| \sup_{0\leq s\leq T} \psi_1(s) \right| \\
&\times \left(|f_1(s, y(s), (\phi_1 y)(s), (\phi_2 y)(s)) - f_1(s,0,0,0)| + |f_1(s,0,0,0)| \right) ds \\
&+ \frac{t^{n-1}}{\Gamma(\alpha)\left|\sum\limits_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1}\right|} \sum_{i=1}^{m} |\lambda_i| \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \left| \sup_{0\leq s\leq T} \psi_1(s) \right| \\
&\times \left(|f_1(s, y(s), (\phi_1 y)(s), (\phi_2 y)(s)) - f_1(s,0,0,0)| + |f_1(s,0,0,0)| \right) ds
\end{aligned} \tag{18}$$

Thanks to (H_2) ,we obtain

$$\begin{aligned}
&|R_1 y(t)| \leq \\
&\frac{\|\psi_1\|_\infty}{\Gamma(\alpha+1)} \left(\sup_{t\in[0,T]} a_1(t) \|y(s)\| + \sup_{t\in[0,T]} a_2(t) \|\phi_1 y(s)\| + \sup_{t\in[0,T]} a_3(t) \|\phi_3 y(s)\| + M_1 \right) \\
&+ |x_0^*| + \frac{\|\psi_1\|_\infty T^\alpha}{\Gamma(\alpha+1)\left|\sum\limits_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1}\right|} \Big(\sup_{t\in[0,T]} a_1(t) \|y(s)\| + \sup_{t\in[0,T]} a_2(t) \|\phi_1 y(s)\| \\
&+ \sup_{t\in[0,T]} a_3(t) \|\phi_3 y(s)\| + M_1 \Big) + \frac{\|\psi_1\|_\infty \sum\limits_{i=1}^{m} |\lambda_i| \eta_i^\alpha}{\Gamma(\alpha+1)\left|\sum\limits_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1}\right|} \Big(\sup_{t\in[0,T]} a_1(t) \|y(s)\| \\
&+ \sup_{t\in[0,T]} a_2(t) \|\phi_1 y(s)\| + \sup_{t\in[0,T]} a_3(t) \|\phi_3 y(s)\| + M_1 \Big)
\end{aligned} \tag{19}$$

Consequently,

$$\begin{aligned}
&|R_1 y(t)| \leq \frac{\|\psi_1\|_\infty}{\Gamma(\alpha+1)} \left[(A_1 + A_2\delta_1 + A_3\delta_2) \|y\| + M_1 \right] + |x_0^*| \\
&+ \frac{\|\psi_1\|_\infty T^\alpha}{\Gamma(\alpha+1)\left|\sum\limits_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1}\right|} \left[(A_1 + A_2\delta_1 + A_3\delta_2) \|y\| + M_1 \right] \\
&+ \frac{\|\psi_1\|_\infty \sum\limits_{i=1}^{m} |\lambda_i| \eta_i^\alpha}{\Gamma(\alpha+1)\left|\sum\limits_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1}\right|} \left[(A_1 + A_2\delta_1 + A_3\delta_2) \|y\| + M_1 \right] \\
&\leq \frac{\|\psi_1\|_\infty}{\Gamma(\alpha+1)} \left(1 + \frac{T^\alpha + \sum\limits_{i=1}^{m} |\lambda_i| \eta_i^\alpha}{\left|\sum\limits_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1}\right|} \right) \left[(A_1 + A_2\delta_1 + A_3\delta_2) \|y\| + M_1 \right] + |x_0^*|
\end{aligned} \tag{20}$$

which implies that

$$|R_1 y(t)| \leq \theta_1 \left[(A_1 + A_2\delta_1 + A_3\delta_2) r + M_1 \right] + |x_0^*|, t\,[0,T], \tag{21}$$

Hence,

$$\|R_1(y)\| \leq \theta_1 \left[(A_1 + A_2\delta_1 + A_3\delta_2) r + M_1 \right] + |x_0^*|. \tag{22}$$

With the same arguments as before, we have

$$\|R_2(x)\| \leq \theta_2 \left[(B_1 + B_2\gamma_1 + B_3\gamma_2) r + M_2 \right] + |y_0^*|. \tag{23}$$

And by (22) and (23), we obtain

$$\begin{array}{l} \|R(x,y)\| \leq \\ \theta_1 \left[(A_1 + A_2\delta_1 + A_3\delta_2) r + M_1 \right] + \theta_2 \left[(B_1 + B_2\gamma_1 + B_3\gamma_2) r + M_2 \right] + |x_0^*| + |y_0^*| \end{array} \tag{24}$$

Consequently,

$$\|R(x,y)\| \leq r. \tag{25}$$

Now for $(x_1, y_1), (x_2, y_2) \in X \times Y$, and for any $t \in [0,T]$, we get

$$\begin{array}{l} |R_1 y_1(t) - R_1 y_2(t)| \leq \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{0\leq s\leq T} |\psi_1(s)| \\ \times |f_1(s, y_1(s), (\phi_1 y_1)(s), (\phi_2 y_1)(s)) - f_1(s, y_2(s), (\phi_1 y_2)(s), (\phi_2 y_2)(s))| \, ds \\ + \frac{t^{n-1}}{\Gamma(\alpha)\left|\sum\limits_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1}\right|} \int_0^T (T-s)^{\alpha-1} \sup_{0\leq s\leq T} |\psi_1(s)| \\ \times |f_1(s, y_1(s), (\phi_1 y_1)(s), (\phi_2 y_1)(s)) - f_1(s, y_2(s), (\phi_1 y_2)(s), (\phi_2 y_2)(s))| \, ds \\ \frac{t^{n-1}}{\Gamma(\alpha)\left|\sum\limits_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1}\right|} \sum\limits_{i=1}^{m} |\lambda_i| \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \sup_{0\leq s\leq T} |\psi_1(s)| \\ \times |f_1(s, y_1(s), (\phi_1 y_1)(s), (\phi_2 y_1)(s)) - f_1(s, y_2(s), (\phi_1 y_2)(s), (\phi_2 y_2)(s))| \, ds \end{array} \tag{26}$$

Thanks to (H_2), we can write

$$\begin{array}{l} |R_1 y_1(t) - R_1 y_2(t)| \leq \\ \frac{\|\psi_1\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ \times \left(\sup_{t\in[0,T]} a_1(t) \|y_1(s) - y_2(s)\| + \sup_{t\in[0,T]} a_2(t) \|\phi_1 y_1(s) - \phi_1 y_2(s)\| \right. \\ \left. + \sup_{t\in[0,T]} a_3(t) \|\phi_3 y_1(s) - \phi_3 y_2(s)\| \right) ds + \frac{\|\psi_1\|_\infty}{\Gamma(\alpha)\left|\sum\limits_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1}\right|} \int_0^T (T-s)^{\alpha-1} \\ \times \left(\sup_{t\in[0,T]} a_1(t) \|y_1(s) - y_2(s)\| + \sup_{t\in[0,T]} a_2(t) \|\phi_1 y_1(s) - \phi_1 y_2(s)\| \right. \\ \left. + \sup_{t\in[0,T]} a_3(t) \|\phi_3 y_1(s) - \phi_3 y_2(s)\| \right) ds + \frac{\|\psi_1\|_\infty}{\Gamma(\alpha)\left|\sum\limits_{i=1}^{m} \lambda_i \eta_i^{n-1} - T^{n-1}\right|} \sum\limits_{i=1}^{m} |\lambda_i| \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \\ \times \sup_{t\in[0,T]} a_1(t) \|y_1(s) - y_2(s)\| + \sup_{t\in[0,T]} a_2(t) \|\phi_1 y_1(s) - \phi_1 y_2(s)\| \\ \left. + \sup_{t\in[0,T]} a_3(t) \|\phi_3 y_1(s) - \phi_3 y_2(s)\| \right) ds \end{array} \tag{27}$$

Therefore,

$$
\begin{aligned}
&\left|R_1 y_1(t) - R_1 y_2(t)\right| \leq \\
&\frac{\|\psi_1\|_\infty}{\Gamma(\alpha+1)}\left(A_1 + A_2\delta_1 + A_3\delta_2\right)\|y_1 - y_2\| \\
&+\frac{\|\psi_1\|_\infty T^\alpha}{\Gamma(\alpha+1)\left|\sum\limits_{i=1}^m \lambda_i\eta_i^{n-1} - T^{n-1}\right|}\left(A_1 + A_2\delta_1 + A_3\delta_2\right)\|y_1 - y_2\| \\
&+\frac{\|\psi_1\|_\infty \sum\limits_{i=1}^m |\lambda_i|\eta_i^\alpha}{\Gamma(\alpha+1)\left|\sum\limits_{i=1}^m \lambda_i\eta_i^{n-1} - T^{n-1}\right|}\left(A_1 + A_2\delta_1 + A_3\delta_2\right)\|y_1 - y_2\| \\
&\leq \frac{\|\psi_1\|_\infty}{\Gamma(\alpha+1)}\left(1 + \frac{T^\alpha + \sum\limits_{i=1}^m |\lambda_i|\eta_i^\alpha}{\left|\sum\limits_{i=1}^m \lambda_i\eta_i^{n-1} - T^{n-1}\right|}\right)\left(A_1 + A_2\delta_1 + A_3\delta_2\right)\|y_1 - y_2\|
\end{aligned} \tag{28}
$$

and consequently we obtain

$$\|R_1(y_1) - R_1(y_2)\| \leq \theta_1\left(A_1 + A_2\delta_1 + A_3\delta_2\right)\|y_1 - y_2\|. \tag{29}$$

Similarly,

$$\|R_2(x_1) - R_2(x_2)\| \leq \theta_2\left(B_1 + B_2\gamma_1 + B_3\gamma_2\right)\|x_1 - x_2\|. \tag{30}$$

It follows from (29) and (30) that

$$
\begin{aligned}
&\|R(x_1, y_1) - R(x_2, y_2)\| \leq \\
&\left[\theta_1\left(A_1 + A_2\delta_1 + A_3\delta_2\right) + \theta_2\left(B_1 + B_2\gamma_1 + B_3\gamma_2\right)\right]\|(x_1 - x_2, y_1 - y_2)\|
\end{aligned} \tag{31}
$$

Thanks to (12), we conclude that R is contraction. As a consequence of Banach fixed point theorem, we deduce that R has a fixed point which is a solution of the problem (1).

The second main result is the following theorem :

Theorem 7 *Suppose that* $\sum\limits_{i=1}^m \lambda_i\eta_i^{n-1} \neq T^{n-1}$, $\sum\limits_{j=1}^m \mu_j\xi_j^{n-1} \neq T^{n-1}$ *and assume that the hypotheses* (H_1) *and* (H_3) *are satisfied.*

Then, the coupled system (1) has at least a solution on J.

Proof. We shall use Scheafer's fixed point theorem to prove that R has at least a fixed point on $X \times Y$. The continuity of f_1 and f_2 (hypothesis (H_1)) implies that the operator R is continuous on $X \times Y$.

[1∗ :] : We shall prove that R maps bounded sets into bounded sets in $X \times Y$: Taking $\rho > 0$, and $(x, y) \in B_\rho := \{(x, y) \in X \times Y; \|(x, y)\| \leq \rho\}$, then for each $t \in J$, we have

$$
\begin{aligned}
&\left|R_1 y(t)\right| \leq \\
&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \sup_{0\leq s\leq T}\left|\psi_1(s)\right|\left|f_1\left(s, y(s), (\phi_1 y)(s), (\phi_2 y)(s)\right)\right| ds \\
&+\left|x_0^*\right| + \frac{t^{n-1}}{\Gamma(\alpha)\left|\sum\limits_{i=1}^m \lambda_i\eta_i^{n-1} - T^{n-1}\right|}\int_0^T (T-s)^{\alpha-1} \\
&\times \sup_{0\leq s\leq T}\left|\psi_1(s)\right|\left|f_1\left(s, y(s), (\phi_1 y)(s), (\phi_2 y)(s)\right)\right| ds \\
&+\frac{t^{n-1}}{\Gamma(\alpha)\left|\sum\limits_{i=1}^m \lambda_i\eta_i^{n-1} - T^{n-1}\right|}\sum_{i=1}^m |\lambda_i|\int_0^{\eta_i} (\eta_i - s)^{\alpha-1} \\
&\times \sup_{0\leq s\leq T}\left|\psi_1(s)\right|\left|f_1\left(s, y(s), (\phi_1 y)(s), (\phi_2 y)(s)\right)\right| ds
\end{aligned} \tag{32}
$$

Thanks to $(H3)$, we can write

$$\begin{aligned}
|R_1 y(t)| &\leq \frac{\|\psi_1\|_\infty \sup\limits_{t\in J} l_1(t)}{\Gamma(\alpha+1)} + \frac{\|\psi_1\|_\infty T^\alpha \sup\limits_{t\in J} l_1(t)}{\Gamma(\alpha+1)\left|\sum\limits_{i=1}^{m}\lambda_i\eta_i^{n-1} - T^{n-1}\right|} \\
&\quad + \frac{\|\psi_1\|_\infty \sum\limits_{i=1}^{m}|\lambda_i|\,\eta_i^\alpha \sup\limits_{t\in J} l_1(t)}{\Gamma(\alpha+1)\left|\sum\limits_{i=1}^{m}\lambda_i\eta_i^{n-1} - T^{n-1}\right|} + |x_0^*| \qquad (33) \\
&\leq \sup_{t\in J} l_1(t)\left[\frac{\|\psi_1\|_\infty}{\Gamma(\alpha+1)}\left(1 + \frac{T^\alpha + \sum\limits_{i=1}^{m}|\lambda_i|\,\eta_i^\alpha}{\left|\sum\limits_{i=1}^{m}\lambda_i\eta_i^{n-1} - T^{n-1}\right|}\right)\right] + |x_0^*|.
\end{aligned}$$

Therefore,

$$|R_1 y(t)| \leq L_1\theta_1 + |x_0^*|, t\in[0,T]. \qquad (34)$$

Hence, we have

$$\|R_1(y)\| \leq L_1\theta_1 + |x_0^*|. \qquad (35)$$

Similarly, it can be shown that,

$$\|R_2(x)\| \leq L_2\theta_2 + |y_0^*|. \qquad (36)$$

It follows from (35) and (36) that

$$\|R(x,y)\| \leq L_1\theta_1 + L_2\theta_2 + |x_0^*| + |y_0^*|. \qquad (37)$$

Consequently

$$\|R(x,y)\| < \infty. \qquad (38)$$

[2∗ :] : Now, we will prove that R is equicontinuous on J : For $(x,y)\in B_\rho$, and $t_1, t_2 \in J$, such that $t_1 < t_2$. We have :

$$\begin{aligned}
&|R_1 y(t_2) - R_1 y(t_1)| \qquad (39) \\
&\leq \frac{\|\psi_1\|_\infty}{\Gamma(\alpha)}\int_0^{t_1}\left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right)|f_1(s,y(s),(\phi_1 y)(s),(\phi_2 y)(s))|\,ds \\
&\quad + \frac{\|\psi_1\|_\infty}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f_1(s,y(s),(\phi_1 y)(s),(\phi_2 y)(s))|\,ds \\
&\quad + \frac{\left(t_2^{n-1} - t_1^{n-1}\right)}{\Gamma(\alpha)\left|\sum\limits_{i=1}^{m}\lambda_i\eta_i^{n-1} - T^{n-1}\right|}\int_0^T (T-s)^{\alpha-1}|f_1(s,y(s),(\phi_1 y)(s),(\phi_2 y)(s))|\,ds \\
&\quad + \frac{\left(t_1^{n-1} - t_2^{n-1}\right)}{\Gamma(\alpha)\left|\sum\limits_{i=1}^{m}\lambda_i\eta_i^{n-1} - T^{n-1}\right|}\sum_{i=1}^{m}|\lambda_i|\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}|f_1(s,y(s),(\phi_1 y)(s),(\phi_2 y)(s))|\,ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\|R_1y\left(t_2\right)-R_1y\left(t_1\right)\right\| \\
\leq\ & \frac{\left\|\psi_1\right\|_\infty \sup\limits_{t\in J} l_1\left(t\right)}{\Gamma\left(\alpha+1\right)}\left(t_1^\alpha-t_2^\alpha\right)+\frac{2\left\|\psi_1\right\|_\infty \sup\limits_{t\in J} l_1\left(t\right)}{\Gamma\left(\alpha+1\right)}\left(t_2-t_1\right)^\alpha \\
&+\frac{\left\|\psi_1\right\|_\infty T^\alpha \sup\limits_{t\in J} l_1\left(t\right)}{\Gamma\left(\alpha+1\right)\left|\sum\limits_{i=1}^m \lambda_i\eta_i^{n-1}-T^{n-1}\right|}\left(t_2^{n-1}-t_1^{n-1}\right) \\
&+\frac{\left\|\psi_1\right\|_\infty \sum\limits_{i=1}^m\left|\lambda_i\right|\eta_i^\alpha \sup\limits_{t\in J} l_1\left(t\right)}{\Gamma\left(\alpha+1\right)\left|\sum\limits_{i=1}^m \lambda_i\eta_i^{n-1}-T^{n-1}\right|}\left(t_1^{n-1}-t_2^{n-1}\right) \\
\leq\ & \frac{L_1\left\|\psi_1\right\|_\infty}{\Gamma\left(\alpha+1\right)}\left(t_1^\alpha-t_2^\alpha\right)+\frac{2L_1\left\|\psi_1\right\|_\infty}{\Gamma\left(\alpha+1\right)}\left(t_2-t_1\right)^\alpha \\
&+\frac{L_1\left\|\psi_1\right\|_\infty T^\alpha}{\Gamma\left(\alpha+1\right)\left|\sum\limits_{i=1}^m \lambda_i\eta_i^{n-1}-T^{n-1}\right|}\left(t_2^{n-1}-t_1^{n-1}\right) \\
&+\frac{L_1\left\|\psi_1\right\|_\infty \sum\limits_{i=1}^m\left|\lambda_i\right|\eta_i^\alpha}{\Gamma\left(\alpha+1\right)\left|\sum\limits_{i=1}^m \lambda_i\eta_i^{n-1}-T^{n-1}\right|}\left(t_1^{n-1}-t_2^{n-1}\right).
\end{aligned} \tag{40}$$

With the same arguments as before, we get

$$\begin{aligned}
\left\|R_2x\left(t_2\right)-R_2x\left(t_1\right)\right\| \leq\ & \frac{L_2\left\|\psi_2\right\|_\infty}{\Gamma\left(\beta+1\right)}\left(t_1^\beta-t_2^\beta\right)+\frac{2L_2\left\|\psi_2\right\|_\infty}{\Gamma\left(\beta+1\right)}\left(t_2-t_1\right)^\beta \\
&+\frac{L_2\left\|\psi_2\right\|_\infty T^\beta}{\Gamma\left(\beta+1\right)\left|\sum\limits_{j=1}^m \mu_j\xi_j^{n-1}-T^{n-1}\right|}\left(t_2^{n-1}-t_1^{n-1}\right) \\
&+\frac{L_2\left\|\psi_2\right\|_\infty \sum\limits_{j=1}^m\left|\mu_j\right|\xi_j^\beta}{\Gamma\left(\beta+1\right)\left|\sum\limits_{j=1}^m \mu_j\xi_j^{n-1}-T^{n-1}\right|}\left(t_1^{n-1}-t_2^{n-1}\right).
\end{aligned} \tag{41}$$

Thanks to (40) and (41), we can state that $\left\|R\left(x,y\right)\left(t_2\right)-R\left(x,y\right)\left(t_1\right)\right\|\to 0$ as $t_2\to t_1$. Combining [1∗ :] and [2∗ :] and using Arzela-Ascoli theorem, we conclude that R is completely continuous operator.

[3∗ :] : Finally, we shall show that the set Ω defined by

$$\Omega=\left\{\left(x,y\right)\in X\times Y,\left(x,y\right)=\sigma R\left(x,y\right),0<\sigma<1\right\}, \tag{42}$$

is bounded :

Let $(x,y)\in\Omega$, then $(x,y)=\sigma R(x,y)$, for some $0<\sigma<1$. Thus, for each $t\in J$, we have :

$$x(t)=\sigma R_1 y(t), y(t)=\sigma R_2 x(t).$$

Then

$$\begin{aligned}\frac{1}{\sigma}|x(t)| \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sup_{0\leq s\leq T}|\psi_1(s)|\,|f_1(s,y(s),(\phi_1 y)(s),(\phi_2 y)(s))|\,ds \\ & +|x_0^*|+\frac{t^{n-1}}{\Gamma(\alpha)\left|\sum_{i=1}^m \lambda_i\eta_i^{n-1}-T^{n-1}\right|}\int_0^T (T-s)^{\alpha-1} \\ & \times\sup_{0\leq s\leq T}|\psi_1(s)|\,|f_1(s,y(s),(\phi_1 y)(s),(\phi_2 y)(s))|\,ds \\ & +\frac{t^{n-1}}{\Gamma(\alpha)\left|\sum_{i=1}^m \lambda_i\eta_i^{n-1}-T^{n-1}\right|}\sum_{i=1}^m|\lambda_i|\int_0^{\eta_i}(\eta_i-s)^{\alpha-1} \\ & \times\sup_{0\leq s\leq T}|\psi_1(s)|\,|f_1(s,y(s),(\phi_1 y)(s),(\phi_2 y)(s))|\,ds.\end{aligned} \tag{43}$$

Thanks to $(H3)$, we can write

$$\begin{aligned}\frac{1}{\sigma}|x(t)| \leq\ & \frac{\|\psi_1\|_\infty\sup_{t\in J}l_1(t)}{\Gamma(\alpha+1)}+\frac{\|\psi_1\|_\infty T^\alpha\sup_{t\in J}l_1(t)}{\Gamma(\alpha+1)\left|\sum_{i=1}^m \lambda_i\eta_i^{n-1}-T^{n-1}\right|} \\ & +\frac{\|\psi_1\|_\infty\sum_{i=1}^m|\lambda_i|\,\eta_i^\alpha\sup_{t\in J}l_1(t)}{\Gamma(\alpha+1)\left|\sum_{i=1}^m \lambda_i\eta_i^{n-1}-T^{n-1}\right|}+|x_0^*|.\end{aligned} \tag{44}$$

Therefore,

$$|x(t)|\leq\sigma\left(L_1\left[\frac{\|\psi_1\|_\infty}{\Gamma(\alpha+1)}\left(1+\frac{T^\alpha+\sum_{i=1}^m|\lambda_i|\,\eta_i^\alpha}{\left|\sum_{i=1}^m \lambda_i\eta_i^{n-1}-T^{n-1}\right|}\right)\right]+|x_0^*|\right). \tag{45}$$

Hence, we have

$$\|x\|\leq\sigma(L_1\theta_1+|x_0^*|), \tag{46}$$

Analogously, we can obtain

$$\|y\|\leq\sigma(L_2\theta_2+|y_0^*|). \tag{47}$$

It follows from (46) and (47) that

$$\|(x,y)\|\leq\sigma(L_1\theta_1+L_2\theta_2+|x_0^*|+|y_0^*|). \tag{48}$$

Hence,

$$\|R(x,y)\|<\infty. \tag{49}$$

This shows that the set Ω is bounded.

Thanks to $[1* :]$, $[2* :]$ and $[3* :]$, we deduce that R has at least one fixed point, which is a solution of the problem (1).

Corollary 8 *Suppose that* $\sum_{i=1}^{m}\lambda_i\eta_i^{n-1}\neq T^{n-1}$, $\sum_{j=1}^{m}\mu_j\xi_j^{n-1}\neq T^{n-1}$ *and and there exist non negative real numbers* k_1, k_2, *such that for all* $t\in[0,T]$ *and* $(x_1,y_1,z_1),(x_2,y_2,z_2)\in\mathbb{R}^3$, *we have*

$$\begin{aligned}\left|f_1(t,x_1,y_1,z_1)-f_1(t,x_2,y_2,z_2)\right| &\leq k_1\left(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\right),\\ \left|f_2(t,x_1,y_1,z_1)-f_1(t,x_2,y_2,z_2)\right| &\leq k_2\left(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\right).\end{aligned}$$

If

$$\theta_1k_1(1+\delta_1+\delta_2)+\theta_2k_2(1+\gamma_1+\gamma_2)<1,$$

then the fractional system (1) has a unique solution on J.

Corollary 9 *Assume that* $(H1)$ *holds and* $\sum_{i=1}^{m}\lambda_i\eta_i^{n-1}\neq T^{n-1}$, $\sum_{j=1}^{m}\mu_j\xi_j^{n-1}\neq T^{n-1}$. *If there exists two positive constants* N_1 *and* N_2 *such that*

$|f_1(t,x,y,z)|\leq N_1, |f_2(t,x,y,z)|\leq N_2$ for each $t\in J$ and all $x,y,z\in\mathbb{R}$,
then, the coupled system (1) has at least a solution on J.

4. Examples

Example 10 *Let us consider the following coupled system :*

$$\begin{cases} D^{\frac{5}{2}}x(t)=\frac{\ln(1+t)}{17(1+t^2)}\left(\frac{|y(t)|}{32+t^2(e^{-t}+|y(t)|)}+\frac{1}{18\ln(1+t)}\int_0^t\frac{e^{t-s}}{2}y(s)\,ds\right.\\ \qquad\left.+\frac{1}{16\left(e^{t^2}+1\right)}\int_0^t\frac{e^{\frac{t-s}{2}}}{3}y(s)\,ds\right),t\in[0,1],\\ D^{\frac{8}{5}}y(t)=\frac{e^{-\pi t}}{20\sqrt{1+\pi t^2}}\left(\frac{1}{20\pi+t^2}\sin|x(t)|+\frac{1}{16(\pi t^2+1)}\int_0^t\frac{(t-s)}{3}x(s)\,ds\right.\\ \qquad\left.+\frac{\pi e^{-t}}{\left(e^{t^2}+16\pi\right)}\int_0^t\frac{e^{t-s}}{5}x(s)\,ds\right),t\in[0,1],\\ x(0)=\sqrt{2},|x'(0)|=0,x(1)=\frac{4}{3}x\left(\frac{1}{3}\right)+2x\left(\frac{2}{5}\right)+\frac{5}{3}x\left(\frac{1}{2}\right),\\ y(0)=\sqrt{3},|y'(0)|=0,y(1)=y\left(\frac{1}{4}\right)+\frac{6}{5}y\left(\frac{5}{7}\right)+\frac{1}{3}y\left(\frac{4}{5}\right).\end{cases}\tag{50}$$

For this example, we have

$$\begin{aligned} f_1(t,x,\phi_1(x),\phi_2(x)) &= \frac{|x(t)|}{32+t^2\left(e^{-t}+|x(t)|\right)}+\frac{1}{18\ln(1+t)}\int_0^t\frac{e^{t-s}}{2}x(s)\,ds\\ +\frac{1}{16\left(e^{t^2}+1\right)}\int_0^t\frac{e^{\frac{t-s}{2}}}{3}x(s)\,ds,t &\in [0,1],x\in\mathbb{R},\\ f_2(t,x,\varphi_1(x),\varphi_2(x)) &= \frac{1}{20\pi+t^2}\sin|x(t)|+\frac{1}{16(\pi t^2+1)}\int_0^t\frac{(t-s)}{3}x(s)\,ds\\ +\frac{\pi e^{-t}}{\left(e^{t^2}+16\pi\right)}\int_0^t\frac{e^{t-s}}{5}x(s)\,ds,t &\in [0,1],x\in\mathbb{R}.\end{aligned}$$

Let $t \in [0,1]$ *and* $x, y \in \mathbb{R}$. *Then*

$$\begin{aligned}
& |f_1(t, x, \phi_1(x), \phi_2(x)) - f_1(t, y, \phi_1(y), \phi_2(y))| \\
\leq & \frac{1}{32+t^2}|x-y| + \frac{1}{18\ln(1+t)} \sup_{t\in[0,1]} \left| \int_0^t \frac{e^{t-s}}{2} ds \right| |x-y| \\
& + \frac{1}{16(e^{t^2}+1)} \sup_{t\in[0,1]} \left| \int_0^t \frac{e^{\frac{t-s}{2}}}{3} ds \right| |x-y|, \\
& |f_2(t, x, \varphi_1(x), \varphi_2(x)) - f_2(t, y, \varphi_1(y), \varphi_2(y))| \\
\leq & \frac{1}{20\pi+t^2}|x-y| + \frac{1}{16(\pi t^2+1)} \sup_{t\in[0,1]} \left| \int_0^t \frac{(t-s)}{3} ds \right| |x-y| \\
& + \frac{\pi e^{-t}}{(e^{t^2}+16\pi)} \sup_{t\in[0,1]} \left| \int_0^t \frac{e^{t-s}}{5} ds \right| |x-y|.
\end{aligned}$$

So, we have

$$\begin{aligned}
\psi_1(t) &= \frac{\ln(1+t)}{17(1+t^2)}, \psi_2(t) = \frac{e^{-\pi t}}{20\sqrt{1+\pi t^2}}, \\
\omega_1 &= \sup_{t\in[0,1]} \left| \int_0^t \frac{e^{t-s}}{2} ds \right|, \omega_2 = \sup_{t\in[0,1]} \left| \int_0^t \frac{e^{\frac{t-s}{2}}}{3} ds \right|, \\
\varpi_1 &= \sup_{t\in[0,1]} \left| \int_0^t \frac{(t-s)}{3} ds \right|, \varpi_2 = \sup_{t\in[0,1]} \left| \int_0^t \frac{e^{t-s}}{5} ds \right|, \\
a_1(t) &= \frac{1}{32+t^2}, a_2(t) = \frac{1}{18\ln(1+t)}, a_3(t) = \frac{1}{16(e^{t^2}+1)}, \\
b_1(t) &= \frac{1}{20\pi+t^2}, b_2(t) = \frac{1}{16(\pi t^2+1)}, b_3(t) = \frac{\pi e^{-t}}{(e^{t^2}+16\pi)}.
\end{aligned}$$

It follows then that

$$\begin{aligned}
A_1 &= \frac{1}{32}, A_2 = \frac{1}{18\ln 2}, A_3 = \frac{1}{32}, \omega_1 = 0.859\,1, \omega_2 = 0.2863, \\
B_1 &= \frac{1}{20\pi}, B_2 = \frac{1}{16(\pi+1)}, B_3 = \frac{\pi}{1+16\pi}, \varpi_1 = \frac{1}{2}, \varpi_2 = 0.6320 \\
\|\psi_1\|_\infty &= 0.0407, \|\psi_2\|_\infty = 0.0245, \theta_1 = 0.0307, \theta_2 = 0.0891.
\end{aligned}$$

and

$$[\theta_1(A_1 + A_2\omega_1 + A_3\omega_2) + \theta_2(B_1 + B_2\varpi_1 + B_3\varpi_2)] = 0.0108 < 1.$$

Hence by theorem 8 then the system (50) has a unique solution on $[0,1]$.

REFERENCES

[1] B. Ahmad : *On nonlocal boundary value problems for nonlinear integro-differential equations of arbitrary fractional order. Results. Math.* 63, pp. 183–194, 2013.

[2] C.Z. Bai and J.X. Fang : *The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations. Applied Mathematics and Computation.*150(3), pp. 611- 621, 2004.

[3] W. Coppel : Disconjugacy. Lecture Notes in Mathematics, vol. 220. *Springer, Berlin.* 1971.

[4] D. Delbosco, L. Rodino : *Existence and uniqueness for a nonlinear fractional differential equation. J. Math. Anal. Appl.* 204, 3(4), pp. 429-440, 1996.

[5] K. Diethelm, N.J. Ford : *Analysis of fractional differential equations. J. Math. Anal. Appl.* 265(2), pp. 229-248, 2002.

[6] Z. Du, X. Lin, W. Ge : *Nonlocal boundary value problem of higher order ordinary differential equations at resonance. Rocky Mt. J. Math.* 36, pp. 1471–1486, 2006.

[7] A.M.A. El-Sayed : *Nonlinear functional differential equations of arbitrary orders. Nonlinear Anal.* 33(2), pp. 181-186, 1998.

[8] P.W. Eloe, B. Ahmad : *Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions. Appl. Math. Lett.* 18, pp. 521–527, 2005.

[9] V. Gafiychuk, B. Datsko, and V. Meleshko : *Mathematical modeling of time fractional reaction-diffusion systems. Journal of Computational and Applied Mathematics.* 220(1-2), pp. 215-225, 2008.

[10] J.R. Graef, B. Yang : *Positive solutions of a third order nonlocal boundary value problem. Discret. Contin. Dyn. Syst.* 1, 89-97, 2008.

[11] J. Henderson and R. Luca : *Existence and multiplicity for positive solutions of a system of higher-order multi-point boundary value problems. Nonlinear Differ. Equ. Appl.* 20, pp. 1035-1054, 2013.

[12] M. Houas, Z. Dahmani : *New results for a coupled system of fractional differential equations. Facta Universitatis (NIS) Ser. Math. Inform.* 28, (2), pp. 133-150, 2013.

[13] M. Houas, Z. Dahmani : New results for a system of two fractional differential equations involving n Caputo derivatives. *Kragujevac Journal of Mathematics.* 38(2), pp. 283-301, 2014.

[14] M. Houas, Z. Dahmani : A coupled system of integro-differential equations involving Riemann-liouville integral and Caputo derivative. *Acta Universitatis Apulensis.* 40, pp. 67-84, 2015.

[15] A.A. Kilbas, S.A. Marzan : *Nonlinear differential equation with the Caputo fraction derivative in the space of continuously differentiable functions*, *Differ. Equ.* 41(1), pp. 84-89, 2005.

[16] N. Kosmatov : *Integral equations and initial value problems for nonlinear differential equations of fractional order. Nonlinear Analysis Theory, Methods & Applications.* 70, (7), pp. 2521- 2529, 2009.

[17] V. Lakshmikantham, A.S. Vatsala : *Basic theory of fractional differential equations. Nonlinear Anal.* 69(8), pp. 2677-2682, 2008.

[18] J. Liang,Z. Liu, X. Wang : *Solvability for a coupled system of nonlinear fractional differential equations in a Banach space. Fractional Calculus and Applied Analysis.* 16(1), pp. 51-63, 2013.

[19] R. Liu, C. Kou, and X. Xie : *Existence results for a coupled system of nonlinear fractional boundary value problems at resonance. Mathematical Problems in Engineering.* Article ID 267386, 9 pages, 2013.

[20] M. Muslim, C. Conca, and A. K. Nandakumaran : *Approximation of solutions to fractional integral equation. Computers & Mathematics with Applications.* 59, (3), pp. 1236 1244, 2010.

[21] K. S.Miller and B. Ross : *An introduction to the fractional calculus and fractional differential equations. A Wiley-Interscience Publication, JohnWiley & Sons, New York, NY, USA*. 1993.

[22] F. Mainardi, Fractional calculus : *Some basic problem in continuum and statistical mechanics. Fractals and fractional calculus in continuum mechanics*, *Springer, Vienna.* 1997.

[23] I. Podlubny, I. Petras, B.M. Vinagre, P. O'leary, L. Dorcak : *Analogue realizations of fractional-order controllers. Fractional order calculus and its applications. Nonlinear Dynam.* 29(4), pp. 281-296, 2002.

[24] M. ur Rehman and R. A. Khan : *Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations. Applied Mathematics Letters.* 23, (9), pp. 1038-1044, 2010.

[25] X. Su : *Boundary value problem for a coupled system of nonlinear fractional differential equations. Applied Mathematics Letters.* 22(1), pp. 64-69, 2009.

[26] J. Wang, H. Xiang, Z. Liu : *Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations. International Journal of Differential Equations.* Article ID 186928, 12 pages, 2010.

[27] G. Wang, W. Liu, S. Zhu and T. Zheng : Existence results for a coupled system of nonlinear fractional 2m-point boundary value problems at resonance. *Advances in Difference Equations.* Vol 2011, pp. 1-17, 2011.

[28] J.R.L. Webb : *Nonlocal conjugate type boundary value problems of higher order. Nonlinear Anal.* 71, pp. 1933-1940, 2009.

[29] J.R.L. Webb : Infante, G : *Nonlocal boundary value problems of arbitrary order. J. Lond. Math. Soc.* 79, pp. 238-258, 2009.

[30] W. Yang : *Positive solutions for a coupled system of nonlinear fractional differential equations with integral boundary conditions. Comput. Math. Appl.* 63, pp. 288-297, 2012.

[31] Y. Zhang, Z. Bai, T. Feng : *Existence results for a coupled system of nonlinear fractional three-point boundary value problems at resonance. Comput. Math. Appl.* 61, pp. 1032-1047, 2011.