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On solvability of mixed boundary value problem for Laplace equation set on bad domain

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ABSTRACT

This note is devoted to the study of Laplace equation set on a planar cusp domain. A mixed boundary conditions are imposed to our problem. The study is performed in the little Hölder spaces framework.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^2$ a bounded cusp domain defined by

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < a,\ \varphi_2(x) < y < \varphi_1(x)\right\}, \tag{1}$$

where $a > 0$ is small enough and φ_1, φ_2 are two real-valued functions defined on $[0,a]$ and satisfying the following conditions

1. $\varphi = \varphi_1 - \varphi_2 \in C^2([0,a])$ and $\varphi(0) \neq 0$.
2. φ_1, φ_2 are strictly monotone functions on $[0,a]$.
3. $\varphi_i(0) = \varphi_i'(0) = 0$, for $i = 1, 2$.

We suppose that $\partial\Omega$ the boundary of Ω is defined by

$$\begin{aligned}
\Gamma_1 &= \{(x, \varphi_1(x)) : 0 < x < a\}, \\
\Gamma_2 &= \{(x, \varphi_2(x)) : 0 < x < a\}, \\
\Gamma_3 &= \{(a, y) : \varphi_2(a) < y < \varphi_1(a)\},
\end{aligned}$$

We wish to study the following model elliptic problem

$$\Delta u = h, \quad \text{in } \Omega, \tag{2}$$

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subject to the following conditions

$$\partial_n u = 0, \quad \text{on } \Gamma_2, \tag{3}$$

and

$$u = 0, \quad \text{on } \Gamma_1, \quad u = 0, \quad \text{on } \Gamma_3, \tag{4}$$

where $\Delta u = \partial_x^2 u + \partial_y^2 u$ and $\partial_n u$ is the outward normal derivative.
We are concerned with the solvability of Problem (2)∼(4) when the right-hand side h is taken in the little Hölder space $h^{2\nu}(\Omega)$, with $0 < 2\nu < 1$ consisting of functions f such that $f \in C^{2\nu}(\Omega)$ and

$$\lim_{\varepsilon \to 0^+} \sup_{0<\|(x,y)-(x',y')\|_2 \leq \varepsilon} \frac{|f(x,y) - f(x',y')|}{\|(x,y) - (x',y')\|_2^{2\nu}} = 0.$$

We assume also that

$$h(0,0) = (a, \varphi_1(a)) = (a, \varphi_2(a)) = 0. \tag{5}$$

To the best of our knowledge, Comparatively with the L^p-theory, there is a few results available in the published literature concerning the Hölder continuous regularity for elliptic problems with mixed boundary conditions set on cusp domains. Most of these studies focused on some particular domains containing a conical singularities, see [1], [8], [11]. This communication can be viewed as a continuation of [2], [4], [5], [6] in where the solvability of Dirichlet problem for Laplace equation in little Hölder space was discussed. The paper is organized as follows. In the next section, we show that our problem can be transformed by natural changes of variables into an abstract second order differential of elliptic type. Section 3 is devoted to the study of the abstract version of the transformed problem. In the last section, we go back to our first problem in the cusp domain and we will prove our main result expressed by the following theorem

Theorem 1 *Let $h \in h^{2\nu}(\Omega)$, with $0 < 2\nu < 1$. Assume that*

$$h(0,0) = (a, \varphi_1(a)) = (a, \varphi_2(a)) = 0.$$

Then, the problem

$$\partial_x^2 u + \partial_y^2 u = h, \qquad \text{in } \Omega,$$

$$\partial_n u = 0, \quad \text{on } \Gamma_2,$$
$$u = 0, \quad \text{on } \Gamma_1, \quad u = 0, \quad \text{on } \Gamma_3,$$

has a unique strict solution $u \in C^2(\overline{\Omega})$ such that

$$(\varphi(x))^{2\nu} \partial_x^2 u \text{ and } (\varphi(x))^{2\nu} \partial_y^2 u \in h^{2\nu}(\overline{\Omega}).$$

2. Change of variables

Consider the following change of variables

$$T \ : \Omega \to \Pi,$$
$$(x,y) \mapsto (\xi, \eta) = \left(-\int_a^x \frac{d\sigma}{\varphi(\sigma)}, \frac{y - \varphi_2(x)}{\varphi(x)} \right),$$

where Π is the semi infinite domain

$$\Pi = \left]0, +\infty\right] \times \left]0, 1\right[.$$

Then

$$\begin{aligned} \partial_x^2 u + \partial_y^2 u &= \frac{1}{\varphi^2}\left\{\partial_\xi^2 v + \partial_\eta^2 v\right\} + \frac{\varphi'}{\varphi^2}\partial_\xi v \\ &+ \frac{1}{\varphi^2}\left\{\eta^2\left(\varphi_2' + \eta\varphi'\right)^2 \partial_\eta^2 v + 2\left(\varphi_2' + \eta\varphi'\right)\partial_{\eta\xi}^2 v\right\} \\ &+ \frac{1}{\varphi^2}\left\{2\varphi'\varphi_2' - \varphi\varphi_2'' - \eta\left(\varphi\varphi'' - 2\left(\varphi'\right)^2\right)\right\}\partial_\eta v. \end{aligned}$$

whith

$$\begin{cases} v(\xi,\eta) := (v \circ T)(x,y) = u(x,y), \\ g(\xi,\eta) := (g \circ T)(x,y) = h(x,y). \end{cases} \tag{6}$$

Consequently, Problem (2) becomes

$$\partial_\xi^2 v + \partial_\eta^2 v + \mathcal{L}v = f, \quad \text{in } \Pi, \tag{7}$$

where

$$f(\xi,\eta) = \varphi^2(\xi)\, g(\xi,\eta), \quad (\xi,\eta) \in \Pi, \tag{8}$$

and $\mathcal{L}$ is the second differential operator with C^∞-bounded coefficients on Π given by

$$\begin{aligned} \mathcal{L}v &= \eta^2\left(\varphi_2' + \eta\varphi'\right)^2 \partial_\eta^2 v + 2\left(\varphi_2' + \eta\varphi'\right)\partial_{\eta\xi}^2 v + \varphi'\partial_\xi v \\ &+ \left(2\varphi'\varphi_2' - \varphi\varphi_2'' - \eta\left(\varphi\varphi'' - 2\left(\varphi'\right)^2\right)\right)\partial_\eta v. \end{aligned}$$

It is easy to see that that condition (3) means that

$$\partial_\eta v(\xi,0) - \varrho(\xi)\,\partial_\xi v(\xi,0) = 0, \tag{9}$$

where

$$\varrho(\xi) = -\frac{\varphi_2'(\xi)}{1 + \left(\varphi_2'(\xi)\right)^2}.$$

On the other hand, condition (4) implies that

$$\begin{array}{ll} v(1,\eta) = 0, & 0 \le \eta \le 1, \\ v(+\infty,\eta) = 0, & 0 \le \eta \le 1, \\ v(\xi,1), & \xi \ge 0. \end{array} \tag{10}$$

The following lemma give us some informations about the regularity of the new hand term f given by (8)

Lemma 2 *Let $0 < 2\nu < 1$. Then*

1. $$h \in h^{2\nu}(\Omega) \Rightarrow f \in h^{2\nu}(\Pi).$$

2. $$f \in h^{2\nu}(\Pi) \Rightarrow (\varphi)^{2\nu} h \in h^{2\nu}(\Omega).$$

Proof. The result is easily obtained by reiterating the same technique used in Proposition 3.1 in [3].

Remark 3 *It is necessary to recall that any function of $h^{2\nu}(\Omega)$ can be extended to a function of $h^{2\nu}\left(\overline{\Omega}\right)$. This is why we shall write in the sequel $h^{2\nu}\left(\overline{\Omega}\right)$ or $h^{2\nu}(\Omega)$.*

As in [3] and [5], our strategy is based on the study of problem

$$\partial_\xi^2 v + \partial_\eta^2 v = f, \quad \text{in } \Pi, \tag{11}$$

associated with the following boundary conditions

$$\begin{array}{ll} v(1,\eta) = 0, v(+\infty,\eta) = 0, & 0 \leq \eta \leq 1, \\ \partial_\eta v(\xi,0) - \varrho(\xi)\,\partial_\xi v(\xi,0) = 0, v(\xi,0) = 0, & \xi \geq 0. \end{array} \tag{12}$$

3. Statement and study of the abstract version of Problem (11)-(12).

In order to study the Problem (11)-(12) with respect to the variable η, we introduce the following vector-valued functions :

$$\begin{array}{lll} v & : & [0,1] \to C_b([0,+\infty[)\,;\ \eta \longrightarrow v(\eta); \quad v(\eta)(\xi) = v(\xi,\eta), \\ f & : & [0,1] \to C_b([0,+\infty[)\,;\ \eta \longrightarrow f(\eta); \quad f(\eta)(\xi) = f(\xi,\eta). \end{array}$$

where

$$C_b([0,+\infty[) = \left\{\psi \in C([0,+\infty[) : \lim_{\xi\to+\infty} \psi(\xi) = 0\right\}.$$

Problem (11)-(12) can be written as an abstract differential equation

$$\left\{\begin{array}{l} v''(\eta) + Av(\eta) = f(\eta) \qquad 0 \leq \eta \leq 1, \\ \\ v'(0) - Bv(0) = 0, v(0) = 0. \end{array}\right. \tag{13}$$

where

$$\left\{\begin{array}{ll} D(A) & = \left\{w \in C_b^2([0,+\infty[) : w(0) = 0\right\}, \\ (Aw)(\eta) & = \partial_\xi^2 w(\xi). \end{array}\right. \tag{14}$$

and

$$\left\{\begin{array}{ll} D(B) & = \left\{w \in C_b^1([0,+\infty[) : w(0) = 0\right\}, \\ (Bw)(\xi) & = \varrho(\xi)\,\partial_\xi v(\xi). \end{array}\right. \tag{15}$$

The complete study of the abstract version of Problem (11)-(12) needs some informations about the spectral properties of (14) and (15)

Lemma 4 *The non densely defined operators $(A, D(A))$ enjoys the following propriety :*

$$]0,+\infty[\subset \rho(A) \text{ and } \exists\ C_A > 0 : \forall\lambda > 0 \quad \left\|(A-\lambda I)^{-1}\right\|_{L(E)} \leqslant \frac{C_A}{\lambda}, \tag{16}$$

Proof. First, observe that

$$\overline{D(A)} = \left\{w \in C_b([0,+\infty[) : w(0) = 0\right\}$$

On the hand, for $\phi \in D(A)$, one has

$$(A-\lambda I)^{-1}\phi = \int_0^{+\infty} k(\xi,s)\phi(s)ds,$$

where

$$k(\xi,s)=\begin{cases}\dfrac{e^{-\sqrt{\lambda}\xi}\sinh\sqrt{\lambda}s}{\sqrt{-\lambda}} & 0\leqslant s\leqslant\xi,\\[2ex] \dfrac{e^{-\sqrt{\lambda}s}\sinh\sqrt{\lambda}\xi}{\sqrt{\lambda}} & s\geqslant\xi,\end{cases}$$

with

$$\operatorname{Re}\sqrt{\lambda}>0.$$

Then

$$\begin{aligned}&\left\|\int_0^{+\infty}k(\xi,s)\phi(s)ds\right\|_E\\ \leqslant\ &\frac{e^{-(\operatorname{Re}\sqrt{\lambda})\xi}}{|\lambda|^{1/2}\operatorname{Re}\sqrt{\lambda}}\left(\sinh(\operatorname{Re}\sqrt{\lambda}\xi)+\cosh(\operatorname{Re}\sqrt{\lambda}\xi)\right)\|\phi\|_E\\ \leqslant\ &\frac{1}{|\lambda|^{1/2}\operatorname{Re}\sqrt{\lambda}}\|\phi\|_E\\ \leqslant\ &\frac{C_A}{|\lambda|}\|\phi\|_E\,.\end{aligned}$$

By the same way, one has

Lemma 5 *The non densely defined operators $(B,D(B))$ enjoys the following propriety :*

$$]-\infty,0[\subset\rho(B),\ \exists\ C_B>0:\forall\lambda>0\quad\left\|(B+\lambda I)^{-1}\right\|_{L(E)}\leqslant\frac{C_B}{\lambda}. \tag{17}$$

Remark 6 *Now, we provide some auxiliary materials which are used to state our main Theorem :*

1. Propriety (16) implies that

$$Q=-\sqrt{-A},$$

is well defined and it is the infinitesimal generator of a generalized analytic semigroup $\left(e^{\eta Q}\right)_{0<\eta\leq1}$, see E. Sinestrari [12].

2. Observe that due to (5) and (16), the real Banach interpolation spaces $D_A(\nu)$ are given by

$$D_A(\nu)=\left\{\phi\in h^{2\nu}([0,1]):\phi(0)=\phi(1)=0\right\}.$$

for more details, see A. Lunardi [10]

3. Thanks to (5) and taking into account Lemma 3 in [2], we deduce that

$$h^{2\sigma}\left(\overline{\Pi}\right)=L^{\infty}(]0,1[\,;D_A(\nu))\cap h^{2\sigma}([0,1]\,;E). \tag{18}$$

From [13], we know that the unique solution of (13) is given by

$$e^{\eta Q}b_1+e^{(1-\eta)Q}b_2+\frac{Q^{-1}}{2}\int_0^{\eta}e^{(\eta-s)Q}f(s)ds+\frac{Q^{-1}}{2}\int_{\eta}^{1}e^{(s-\eta)Q}f(s)ds$$

where b_1 and b_2 belong to E. These constants are uniquely determined by the boundary conditions of problem. A formal computation gives

$$\begin{aligned} u(\eta) &= \frac{\Delta^{-1}_{(Q,B)}}{2}(Q+B)\,e^{\eta Q}\int_0^1 e^{sQ}Q^{-1}f(s)ds \\ &-\frac{\Delta^{-1}_{(Q,B)}}{2}(Q+B)\,e^{(\eta+1)Q}\int_0^1 e^{(1-s)Q}Q^{-1}f(s)ds \\ &-\frac{\Delta^{-1}_{(Q,B)}}{2}(Q+B)\,e^{(2-\eta)Q}\int_0^1 e^{sQ}Q^{-1}f(s)ds \\ &-\frac{\Delta^{-1}_{(Q,B)}}{2}(I-Q-B)\,e^{(3-\eta)Q}\int_0^1 e^{sQ}Q^{-1}f(s)ds \\ &+\frac{1}{2}\int_0^\eta e^{(\eta-s)Q}Q^{-1}f(s)ds+\frac{1}{2}\int_\eta^1 Q^{-1}e^{(s-\eta)Q}f(s)ds \end{aligned}$$

where

$$\Delta_{(Q,B)}=\left(I-e^{2Q}\right)(Q+B)+2Qe^{2Q}$$

Remark 7 *It is necessary here to indicate that the existence of* $\Delta^{-1}_{(Q,B)}$ *is justified by*
1. The use of the sum's operator theory which allows us to prove that the bounded operator $(Q+B)^{-1}$ *is well-defined, see [7],[5]*
2. The use of the results of [10] p.60, which affirm that the operator $I-e^{2Q}$ *is well-defined, boundedly invertible.*

Due to the regularizing term $\Delta^{-1}_{(Q,B)}$ and by adapting the same techniques used in [**3**] and [**9**], we obtain the following useful results

Proposition 8 *Let* $f\in h^{2\nu}([0,1];E)$ *with* $0<2\nu<1$ *such that*

$$f(0)=f(1)=0.$$

Then, Problem (13) has a unique strict solution

$$v\in C^2([0,1];E)\cap C([0,1];D(A))$$

satisfying the maximal regularity property

$$v'' \text{ and } Av\in h^{2\nu}([0,1];E).$$

Proposition 9 *Let* $f\in L^\infty(]0,1[;D_A(\nu))\cap C([0,1];E)$, *with* $0<2\nu<1$ *such that*

$$f(0)=f(1)=0.$$

Then, Problem (13) has a unique strict solution

$$v\in W^{2,\infty}(]0,1[.;E)\cap L^\infty(]0,1[;D(A)),$$

satisfying the maximal regularity property

$$v'' \text{ and } Av\in L^\infty(];0,1[;D_A(\nu)).$$

Now, we are in position to summarize all results concerning our Problem (11)-(12)

Proposition 10 *Let $f \in h^{2\nu}(\overline{\Pi})$ such that*
$$f(0.1) = f(0,0) = f(+\infty,.) = 0.$$

Then, there exists a unique strict solution $v \in C^2(\overline{\Pi})$ of Problem (11) such that
$$\frac{\partial^2 v}{\partial \xi^2}, \frac{\partial^2 v}{\partial \eta^2} \in h^{2\nu}(\overline{\Pi}).$$

4. Resolution of the original problem

Recall the complete equation of our initial problem (7) in Π
$$\partial_\xi^2 v + \partial_\eta^2 v + \mathcal{L}v = f, \quad \text{in } \Pi,$$

where
$$f(\xi,\eta) = \varphi^2(\xi)\, g(\xi,\eta), \quad (\xi,\eta) \in \Pi,$$

and $\mathcal{L}$ is the second differential operator with C^∞-bounded coefficients on Π given by

$$\begin{aligned}\mathcal{L}v \; = & \\ & \eta^2 (\varphi_2' + \eta\varphi')^2 \partial_\eta^2 v + 2(\varphi_2' + \eta\varphi')\partial_{\eta\xi}^2 v \\ & + \varphi' \partial_\xi v \\ & + \left(2\varphi'\varphi_2' - \varphi\varphi_2'' - \eta\left(\varphi\varphi'' - 2(\varphi')^2\right)\right)\partial_\eta v.\end{aligned}$$

Consider now the following Banach spaces

$$\begin{aligned}& X_d \\ = \; & \left\{v \in h^{2+2\nu}(\overline{\Pi}) : v(1,.) = 0, v(+\infty,.) = 0, \partial_\eta v(.,0) - \varrho(.)\,\partial_\xi v(.,0) = 0, v(.,0) = 0\right\},\end{aligned}$$

and
$$X_a = \left\{f \in h^{2\nu}(\overline{\Pi}) : f(0,1) = f(0,0) = f(+\infty,.) = 0\right\}.$$

Therefore, we deduce that the Laplace operator Δ is an isomorphism from X_d into X_a. We will denote its inverse by Δ^{-1} meaning
$$\begin{array}{lll}\Delta & X_d & \to & X_a, \\ \Delta^{-1} & X_a & \to & X_d.\end{array}$$

For a fixed $\xi_0 > 0$ large enough, we introduce the linear operator
$$\begin{aligned}M : \; & h^{2\nu}(\overline{\Pi}) \to h^{2\nu}(\overline{\Pi}), \\ & f \mapsto Mf = k(\xi)\, f,\end{aligned} \tag{19}$$

where k is a real function of class C^∞ defined as follows
$$k(\xi) = \begin{cases} 0 & 0 < \xi < 1, \\ \xi - \xi_1 & 1 < \xi < 2, \\ 1 & \xi > 2. \end{cases}$$

Proposition 11 *The linear operator*
$$\begin{aligned}M : \; & h^{2\nu}(\overline{\Pi}) \to h^{2\nu}(\overline{\Pi}), \\ & f \mapsto Mf = k(\xi)\, f,\end{aligned}$$

is continuous and one has

$$\|Mf\|_{h^{2\nu}(\overline{\Pi})} \leq 2\|f\|_{h^{2\nu}(\overline{\Pi})}.$$

Proof. Since $0 < k < 1$, then

$$\|Mf\|_{L^\infty(\Pi)} \leq \|f\|_{L^\infty(\Pi)}.$$

Given a small $\delta > 0$. Let $(\xi_2, \eta_2), (\xi_1, \eta_1) \in \overline{\Pi}$ such that

$$(\xi_2, \eta_2) \neq (\xi_1, \eta_1),$$

and $\|(\xi_2, \eta_2) - (\xi_1, \eta_1)\| \leq \delta$. Assume, for instance that $\xi_1 \leq \xi_2$, $\eta_1 \leq \eta_2$
It is easy to see that

$$\begin{aligned}
& Mf(\xi_2, \eta_2) - Mf(\xi_1, \eta_1) \\
= \ & k(\xi_2) f(\xi_2, \eta_2) - k(\xi_1) f(\xi_1, \eta_1) \\
= \ & k(\xi_2) f(\xi_2, \eta_2) - k(\xi_2) f(\xi_1, \eta_1) + k(\xi_2) f(\xi_1, \eta_1) - k(\xi_1) f(\xi_1, \eta_1) \\
= \ & k(\xi_2)[f(\xi_2, \eta_2) - f(\xi_1, \eta_1)] + f(\xi_1, \eta_1)[k(\xi_2) - k(\xi_1)].
\end{aligned}$$

Observe that

$$|k(\xi_2) - k(\xi_1)| \leq |\xi_2 - \xi_1|^{2\nu}.$$

On the other hand, one has

$$\begin{aligned}
& \frac{|Mf(\xi_2, \eta_2) - Mf(\xi_1, \eta_1)|}{\|(\xi_2 - \xi_1, \eta_2 - \eta_1)\|^{2\nu}} \\
\leq \ & \frac{|f(\xi_2, \eta_2) - f(\xi_1, \eta_1)|}{\|(\xi_2 - \xi_1, \eta_2 - \eta_1)\|^{2\nu}} + \|f\|_{L^\infty(\Pi)} \frac{|\xi_2 - \xi_1|}{|\xi_2 - \xi_1|^{2\nu}} \\
\leq \ & \frac{|f(\xi_2, \eta_2) - f(\xi_1, \eta_1)|}{\|(\xi_2 - \xi_1, \eta_2 - \eta_1)\|^{2\nu}} + \|f\|_{L^\infty(\Pi)},
\end{aligned}$$

so

$$\begin{aligned}
& \|Mf\|_{L^\infty(\Pi)} + \sup_{\|(\xi_2 - \xi_1, \eta_2 - \eta_1)\| \leq \delta} \frac{|Mf(\xi_2, \eta_2) - Mf(\xi_1, \eta_1)|}{\|(\xi_2 - \xi_1, \eta_2 - \eta_1)\|^{2\nu}} \\
\leq \ & 2\|f\|_{L^\infty(\Pi)} + \sup_{\|(\xi_2 - \xi_1, \eta_2 - \eta_1)\| \leq \delta} \frac{|f(\xi_2, \eta_2) - f(\xi_1, \eta_1)|}{\|(\xi_2 - \xi_1, \eta_2 - \eta_1)\|^{2\nu}}.
\end{aligned}$$

this implies that

$$\|Mf\|_{C^{2\nu}(\overline{\Pi})} \leq 2\|f\|_{C^{2\nu}(\overline{\Pi})}.$$

Moreover

$$\begin{aligned}
& \lim_{\delta \to 0} \sup_{\|(\xi_2 - \xi_1, \eta_2 - \eta_1)\| \leq \delta} \frac{|Mf(\xi_2, \eta_2) - Mf(\xi_1, \eta_1)|}{\|(\xi_2 - \xi_1, \eta_2 - \eta_1)\|^{2\nu}} \\
\leq \ & \lim_{\delta \to 0} \sup_{\|(\xi_2 - \xi_1, \eta_2 - \eta_1)\| \leq \delta} \frac{|f(\xi_2, \eta_2) - f(\xi_1, \eta_1)|}{\|(\xi_2 - \xi_1, \eta_2 - \eta_1)\|^{2\nu}} + \|f\|_{L^\infty(\Pi)} \frac{|\xi_2 - \xi_1|}{|\xi_2 - \xi_1|^{2\nu}} \\
\leq \ & \lim_{\delta \to 0} \delta^{1-2\nu} = 0.
\end{aligned}$$

Set

$$\overline{\mathcal{L}} := M \circ \mathcal{L}.$$

It is easy to see that

$$\begin{aligned} \overline{\mathcal{L}} : \ & h^{2+2\nu}(\overline{\Pi}) \to h^{2\nu}(\overline{\Pi}), \\ & f \mapsto \overline{\mathcal{L}}f = k\,(\xi)\, f, \end{aligned}$$

is continuous and one has

$$\left\|\overline{\mathcal{L}}f\right\|_{h^{2\sigma}(\overline{\Pi})} \leq 2\left\|L\right\|_{h^{2\sigma}(\overline{\Pi})}.$$

Consequently we deduce that for a suitable ξ large enough, the operator

$$\Delta + \overline{\mathcal{L}},$$

is an isomorphism from X_d into X_a. This, alllows us say that the following equation

$$v + \left[\Delta^{-1}L\right] v = \Delta^{-1}f. \tag{20}$$

is well defined. Therefore one can inverse equation (20) in the space X_d. Consequently one obtains the following representation

$$v = \left(I + \Delta^{-1}L\right)^{-1}\Delta^{-1}f \tag{21}$$

At this level, to justify our main result it suffices Just to follow the same steps as in [3] and [6].

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