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On some generalized integral inequalities for Hadamard fractional integrals

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ABSTRACT

In this paper, we use the Hadamard fractional integral operator to establish some new weighted integral inequalities involving a family of n, $(n \geq 1)$ positive functions.

1. Introduction

The fractional integral inequalities play a fundamental role in the theory of defferential equations and applied sciences. These inequalities have various applications in applied fields such as transform theory, numerical quadrature, probability, and statistical problems. Recently, by applying the fractional integral operators, many researchers have obtained a lot of fractional integral inequalities and applications. For details, we refer to [1, 3, 4, 5, 8, 9, 10, 11, 13, 14, 15, 16, 17] and the references therein. Z. Dahmani et al. [6], Z. Dahmani [8], and A. Anber et al. [1], established some new fractional integral inequalities by using the Riemann-Liouville fractional integral operators. Also, V. Chinchane et al. [4], W. Yang [19] and Z. Dahmani et al. [7] derived some fractional integral inequalities involving Hadamard fractional integral operators, Saigo fractional integral operators and fractional $q-$integral operators. Motivated by the results presented in [6, 8], we prove some new weighted fractional integral inequalities using Hadamard fractional integral operator.

2. Hadamard fractional calcul

In the following we will give some necessary definitions and mathematical preliminaries of Hadamard fractional calculus which are used further in this paper. More details, one

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can consult [2, 12, 18].

Definition 1 *The Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of a function $f(t)$, for all $t > 1$ is defined as*

$$ {}_HJ_1^\alpha [f(t)] := \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{x}\right)^{\alpha-1} \frac{f(x)}{x} dx \ , \tag{2.1} $$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

Proposition 2 *If $0 < \alpha < 1$, the following equality holds*

$$ {}_HJ_1^\alpha (\log t)^\mu := \frac{\Gamma(\mu)}{\Gamma(\alpha-\mu)} (\log t)^{\alpha+\mu-1} \ . \tag{2.2} $$

For the convenience of establishing the results, we give the following properties

$$ {}_HJ_1^\alpha \ {}_HJ_1^\beta [f(t)] := \ {}_HJ_1^{\alpha+\beta} [f(t)] \ , \tag{2.3} $$

and

$$ {}_HJ_1^\alpha \ {}_HJ_1^\beta [f(t)] := \ {}_HJ_1^\beta \ {}_HJ_1^\alpha [f(t)] \ . \tag{2.4} $$

3. Hadamard fractional integral inequalities

In this section, we prove some new weighted fractional integral inequalities concerning the Hadamard fractional integral.

Theorem 3 *Let f and h be tow positive and continuous functions on $[1, \infty)$ and let $w : [1, \infty) \to \mathbb{R}^+$ be a positive continuous function. Then for all $t > 1$, we have*

$$ \begin{aligned} & {}_HJ_1^\alpha \left[w(t) f^{\delta+\sigma}(t)\right] \ {}_HJ_1^\alpha \left[w(t) h^\sigma(t) f^\theta(t)\right] \\ \geq & \ {}_HJ_1^\alpha \left[w(t) f^{\sigma+\theta}(t)\right] \ {}_HJ_1^\alpha \left[w(t) h^\sigma(t) f^\delta(t)\right], \end{aligned} \tag{3.1} $$

where $\alpha > 0$, $\delta \geq \theta > 0$, $\sigma > 0$.

Proof. Consider

$$ F(t,x) := \frac{1}{\Gamma(\alpha)} \left(\log \frac{t}{x}\right)^{\alpha-1} \frac{w(x) f^\theta(x)}{x}, \ \alpha > 0, \ \theta > 0, \ x \in (1,t) ; \ t > 1 \ , \tag{3.2} $$

Since f and h are tow positive and continuous functions on $[1, \infty)$, then for all $x, y \in (1, t)$; $t > 1$ and for any $\sigma > 0$, $\delta \geq \theta > 0$, we have

$$ \left(h^\sigma(y) f^\sigma(x) - h^\sigma(x) f^\sigma(y)\right) \left(f^{\delta-\theta}(x) - f^{\delta-\theta}(y)\right) \geq 0 \ , \tag{3.3} $$

By (3.3), we write

$$ \begin{aligned} & h^\sigma(y) f^{\delta+\sigma-\theta}(x) + h^\sigma(x) f^{\delta+\sigma-\theta}(y) \\ \geq & \ h^\sigma(y) f^\sigma(x) f^{\delta-\theta}(y) + h^\sigma(x) f^\sigma(y) f^{\delta-\theta}(x) . \end{aligned} \tag{3.4} $$

We observe that the function $F(t,x)$ remains positive, for all $x \in (1,t)$; $t > 1$. Multiplying both sides of (3.4) by $F(t,x)$ and integrating the reslting inequality with respect to x from 1 to t, we obtain

$$ \begin{aligned} & h^\sigma(y) \ {}_HJ_1^\alpha \left[w(t) f^{\delta+\sigma}(t)\right] + f^{\delta+\sigma-\theta}(y) \ {}_HJ_1^\alpha \left[w(t) h^\sigma(t) f^\theta(t)\right] \\ \geq & \ h^\sigma(y) f^{\delta-\theta}(y) \ {}_HJ_1^\alpha \left[w(t) f^{\sigma+\theta}(t)\right] + f^\sigma(y) \ {}_HJ_1^\alpha \left[w(t) h^\sigma(t) f^\delta(t)\right] . \end{aligned} \tag{3.5} $$

Now, multiplying both sides of (3.5) by $F(t,y)$ and integrating the reslting inequality with respect to y from 1 to t, we have

$$\begin{aligned} & {}_HJ_1^{\alpha}\left[w(t)\,f^{\delta+\sigma}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f^{\theta}(t)\right] \\ & +\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f^{\theta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f^{\delta+\sigma}(t)\right] \\ & \geq\ {}_HJ_1^{\alpha}\left[w(t)\,f^{\sigma+\theta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f^{\delta}(t)\right] \\ & +\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f^{\delta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f^{\sigma+\theta}(t)\right]. \end{aligned} \tag{3.6}$$

which implies 3.1.

We also present the Hadamard fractional result using two fractional parameters :

Theorem 4 *Let f and h be tow positive and continuous functions on $[1,\infty)$ and let $w:[1,\infty)\to\mathbb{R}^{+}$ be a positive continuous function. Then for all $t>1$, we have*

$$\begin{aligned} & {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\,f^{\theta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f^{\delta+\sigma}(t)\right] \\ & +\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f^{\theta}(t)\right]\ {}_HJ_1^{\beta}\left[w(t)\,f^{\delta+\sigma}(t)\right] \\ & \geq\ {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\,f^{\delta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f^{\sigma+\theta}(t)\right] \\ & +\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f^{\delta}(t)\right]\ {}_HJ_1^{\beta}\left[w(t)\,f^{\sigma+\theta}(t)\right], \end{aligned} \tag{3.7}$$

where $\alpha>0$, $\beta>0$, $\delta\geq\theta>0$, $\sigma>0$.

Proof. Multiplying the inequality (3.4) by $G(t,y)$, $y\in(1,t)$; $t>1$, where

$$G(t,x) := \frac{1}{\Gamma(\beta)}\left(\log\frac{t}{y}\right)^{\beta-1}\frac{w(y)f^{\theta}(y)}{y},\ \beta>0,\ \theta>0,\ y\in(1,t)\,;\ t>1\ . \tag{3.8}$$

Integrating the reslting inequality obtained with respect to y from 1 to t, it yields that

$$\begin{aligned} & f^{\delta+\sigma-\theta}(x)\ {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\,f^{\theta}(t)\right] + h^{\sigma}(x)\ {}_HJ_1^{\beta}\left[w(t)\,f^{\delta+\sigma}(t)\right] \\ & \geq f^{\sigma}(x)\ {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\,f^{\delta}(t)\right] + h^{\sigma}(x)\,f^{\delta-\theta}(x)\ {}_HJ_1^{\beta}\left[w(t)\,f^{\sigma+\theta}(t)\right]. \end{aligned} \tag{3.9}$$

Multiplying both sides of (3.9) by $F(t,x)$ and integrating the reslting inequality with respect to x from 1 to t, we observe that

$$\begin{aligned} & {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\,f^{\theta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f^{\delta+\sigma}(t)\right] \\ & +\ {}_HJ_1^{\beta}\left[w(t)\,f^{\delta+\sigma}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f^{\theta}(t)\right] \\ & \geq\ {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\,f^{\delta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f^{\sigma+\theta}(t)\right] \\ & +\ {}_HJ_1^{\beta}\left[w(t)\,f^{\sigma+\theta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f^{\delta}(t)\right]. \end{aligned} \tag{3.10}$$

which implies (3.7).

Remark 5 *If we take for $\alpha=\beta$, in Theorem 4, we obtain Theorem 3.*

We give also the following Hadamard fractional integral inequality :

Theorem 6 *Let f and h are tow positive and continuous functions on $[1,\infty)$ such that f is decreasing and h is increasing on $[1,\infty)$, and let $w:[1,\infty)\to\mathbb{R}^{+}$ be a positive continuous function. Then for all $t>1$, the following Hadamard fractional integral inequality holds*

$$\begin{aligned} & {}_HJ_1^{\alpha}\left[w(t)f^{\delta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right] \\ \geq\ & {}_HJ_1^{\alpha}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)f^{\theta}(t)\right], \end{aligned} \tag{3.11}$$

for all $\alpha>0$, $\delta\geq\theta>0$, $\sigma>0$.

Proof. Since f and h are tow positive and continuous functions on $[1,\infty)$ such that f is decreasing and h is increasing on $[1,\infty)$, then for all $\delta\geq\theta>0$, $\sigma>0$, $x,y\in(1,t)$; $t>1$, we have

$$\left(h^{\sigma}(y)-h^{\sigma}(x)\right)\left(f^{\delta-\theta}(x)-f^{\delta-\theta}(y)\right)\geq 0\ , \tag{3.12}$$

which implies that,

$$h^{\sigma}(y)f^{\delta-\theta}(x)+h^{\sigma}(x)f^{\delta-\theta}(y)\geq h^{\sigma}(y)f^{\delta-\theta}(y)+h^{\sigma}(x)f^{\delta-\theta}(x)\ . \tag{3.13}$$

Multiplying both sides of (3.13) by $F(t,x)$ and integrating the reslting inequality with respect to x from 1 to t, we get

$$\begin{aligned} & h^{\sigma}(y)\ {}_HJ_1^{\alpha}\left[w(t)f^{\delta}(t)\right]+f^{\delta-\theta}(y)\ {}_HJ_1^{\alpha}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right] \\ \geq\ & h^{\sigma}(y)f^{\delta-\theta}(y)\ {}_HJ_1^{\alpha}\left[w(t)f^{\theta}(t)\right]+\ {}_HJ_1^{\alpha}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right]. \end{aligned} \tag{3.14}$$

Then we can write

$$\begin{aligned} & {}_HJ_1^{\alpha}\left[w(t)f^{\delta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right] \\ +\ & {}_HJ_1^{\alpha}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)f^{\delta}(t)\right] \\ \geq\ & {}_HJ_1^{\alpha}\left[w(t)f^{\theta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right] \\ +\ & {}_HJ_1^{\alpha}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)f^{\theta}(t)\right], \end{aligned} \tag{3.15}$$

which implies (3.11).

The previous result can be generalized to the following :

Theorem 7 *Let f and h are tow positive and continuous functions on $[1,\infty)$ such that f is decreasing and h is increasing on $[1,\infty)$ and let $w:[1,\infty)\to\mathbb{R}^{+}$ be a positive continuous function. Then for all $t>1$, we have*

$$\begin{aligned} & {}_HJ_1^{\beta}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)f^{\delta}(t)\right] \\ +\ & {}_HJ_1^{\alpha}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right]\ {}_HJ_1^{\beta}\left[w(t)f^{\delta}(t)\right] \\ \geq\ & {}_HJ_1^{\beta}\left[w(t)f^{\theta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right] \\ +\ & {}_HJ_1^{\alpha}\left[w(t)f^{\theta}(t)\right]\ {}_HJ_1^{\beta}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right], \end{aligned} \tag{3.16}$$

where $\alpha > 0,\ \beta > 0,\ \delta \geq \theta > 0,\ \sigma > 0.$

Proof. Multiplying the inequality (3.13) by $G(t, y)$, we can write

$$\begin{aligned} & f^{\delta-\theta}(x)\ {}_HJ_1^{\beta}\left[w(t)\, h^{\sigma}(t)\, f^{\theta}(t)\right] + h^{\sigma}(x)\ {}_HJ_1^{\beta}\left[w(t)\, f^{\delta}(t)\right] \\ & \geq h^{\sigma}(y)\, f^{\delta-\theta}(y)\ {}_HJ_1^{\beta}\left[w(t)\, f^{\theta}(t)\right] + {}_HJ_1^{\beta}\left[w(t)\, h^{\sigma}(t)\, f^{\delta}(t)\right]. \end{aligned} \tag{3.17}$$

Consequently,

$$\begin{aligned} & {}_HJ_1^{\beta}\left[w(t)\, h^{\sigma}(t)\, f^{\theta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\, f^{\delta}(t)\right] \\ & + {}_HJ_1^{\beta}\left[w(t)\, f^{\delta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\, h^{\sigma}(t)\, f^{\theta}(t)\right] \\ & \geq {}_HJ_1^{\beta}\left[w(t)\, f^{\theta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\, h^{\sigma}(t)\, f^{\delta}(t)\right] \\ & + {}_HJ_1^{\beta}\left[w(t)\, h^{\sigma}(t)\, f^{\delta}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\, f^{\theta}(t)\right], \end{aligned} \tag{3.18}$$

which implies (3.16).

Remark 8 *Taking* $\alpha = \beta$, *in Theorem 7, we obtain Theorem 6.*

Now, we estalish some fractional results using a family of n positive functions defined on $[1, \infty)$.

Theorem 9 *Let* $f_i,\ i = 1, ..., n,$ *and* h *be positive continuous functions on* $[1, \infty)$ *and* $w : [1, \infty) \to \mathbb{R}^{+}$*is positive continuous function. Then for all* $t > 1,\ \alpha > 0,$ *the fractional integral inequality*

$$\begin{aligned} & {}_HJ_1^{\alpha}\left[w(t)\, f_k^{\delta+\sigma}(t) \prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\, h^{\sigma}(t) \prod_{i=1}^{n} f^{\theta_i}(t)\right] \\ & \geq {}_HJ_1^{\alpha}\left[w(t)\, h^{\sigma}(t)\, f_k^{\delta}(t) \prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\, f_k^{\sigma}(t) \prod_{i=1}^{n} f^{\theta_i}(t)\right], \end{aligned} \tag{3.19}$$

is valid, for all $\sigma > 0, \delta \geq \theta_k > 0,\ k \in \{1, ..., n\}$.

Proof. Let us consider

$$F^{*}(t, x) := \frac{1}{\Gamma(\alpha)}\left(\log \frac{t}{x}\right)^{\alpha-1} \frac{w(x) \prod_{i=1}^{n} f_i^{\theta_i}(x)}{x},\ \theta_i > 0, i \in \{1, ..., n\},\ x \in (1, t);\ t > 1\ \cdot \tag{3.20}$$

Let $x, y \in (1, t);\ t > 1$ for any $\sigma > 0,\ \delta \geq \theta_k > 0, k \in \{1, ..., n\}$. Then we have

$$\left(h^{\sigma}(y)\, f_k^{\sigma}(x) - h^{\sigma}(x)\, f_k^{\sigma}(y)\right)\left(f_k^{\delta-\theta_k}(x) - f_k^{\delta-\theta_k}(y)\right) \geq 0\ , \tag{3.21}$$

It follows that

$$\begin{aligned} & h^{\sigma}(y)\, f_k^{\delta+\sigma-\theta_k}(x) + h^{\sigma}(x)\, f_k^{\delta+\sigma-\theta_k}(y) \\ & \geq h^{\sigma}(y)\, f_k^{\sigma}(x)\, f_k^{\delta-\theta_k}(y) + h^{\sigma}(x)\, f_k^{\sigma}(y)\, f_k^{\delta-\theta_k}(x). \end{aligned} \tag{3.22}$$

Multiplying both sides of (3.22) by $F^{*}(t,x)$ and integrating the reslting inequality with respect to x from 1 to t, we obtain

$$\begin{aligned}
&h^{\sigma}(y)\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\delta+\sigma}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]\\
&+f_k^{\delta+\sigma-\theta_k}(y)\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n}f^{\theta_i}(t)\right]\\
&\geq h^{\sigma}(y)\,f_k^{\delta-\theta_k}(y)\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\sigma}(t)\prod_{i=1}^{n}f^{\theta_i}(t)\right]\\
&+f_k^{\sigma}(y)\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right].
\end{aligned}\tag{3.23}$$

The integration of (3.23) gives

$$\begin{aligned}
&{}_HJ_1^{\alpha}\left[w(t)\,f_k^{\delta+\sigma}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n}f^{\theta_i}(t)\right]\\
&+\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n}f^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\delta+\sigma}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]\\
&\geq\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\sigma}(t)\prod_{i=1}^{n}f^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]\\
&+\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\sigma}(t)\prod_{i=1}^{n}f^{\theta_i}(t)\right].
\end{aligned}\tag{3.24}$$

This ends the proof.

We shall futher generalize Theorem 9 by considering two fractional positive parameters :

Theorem 10 *Let f_i, $i=1,...,n$, and h be positive continuous functions on $[1,\infty)$ and let $w:[1,\infty)\to\mathbb{R}^{+}$ is a positive continuous function. Then for any $t>1$ and $\alpha>0$, $\beta>0$, we have*

$$\begin{aligned}
&{}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\delta+\sigma}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]\\
&{}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]+\ {}_HJ_1^{\beta}\left[w(t)\,f_k^{\delta+\sigma}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]\\
&\geq\ {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\
&{}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]+\ {}_HJ_1^{\beta}\left[w(t)\,f_k^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right],
\end{aligned}\tag{3.25}$$

where $\sigma>0$, $\delta\geq\theta_k>0$, $k\in\{1,...,n\}$.

Proof. We multiply the inequality (3.22) by $G^{*}(t,y)$, $y\in(1,t)$; $t>1$, where

$$G^{*}(t,y):=\frac{1}{\Gamma(\beta)}\left(\log\frac{t}{y}\right)^{\beta-1}\frac{w(y)\prod_{i=1}^{n}f_i^{\theta_i}(y)}{y},\ \ \theta_i>0, i\in\{1,...,n\},\ \ y\in(1,t);\ \ t>1\ ,\tag{3.26}$$

and integrating the reslting inequality obtained with respect to y on $(1,t)$, we can write

$$\begin{aligned} & f_k^{\delta+\sigma-\theta_k}(x)\ {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right] \\ & +h^{\sigma}(x)\ {}_HJ_1^{\beta}\left[w(t)\,f_k^{\delta+\sigma}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right] \\ & \geq f_k^{\sigma}(x)\ {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right] \\ & +h^{\sigma}(x)\,f_k^{\delta-\theta_k}(x)\ {}_HJ_1^{\beta}\left[w(t)\,f_k^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]. \end{aligned} \tag{3.27}$$

Now, multiplying both sides of (3.27) by $F^*(t,x)$, $x\in(1,t)$, $t>1$, and integrating the reslting inequality with respect to x from 1 to t, we obtain

$$\begin{aligned} & {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\delta+\sigma}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right] \\ & +\ {}_HJ_1^{\beta}\left[w(t)\,f_k^{\delta+\sigma}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right] \\ & \geq\ {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right] \\ & +\ {}_HJ_1^{\beta}\left[w(t)\,f_k^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]. \end{aligned} \tag{3.28}$$

The proof is completed.

Remark 11 *For $\alpha=\beta$, Theorem 10 immediately reduce to Theorem 9.*

Another generalization is the following fractional inaquality :

Theorem 12 *Let f_i, $i=1,...,n$ and h be positive continuous functions on $[1,\infty)$, such that h is increasing and f_i, $i=1,...,n$ are decreasing on $[1,\infty)$ and $w:[1,\infty)\to\mathbb{R}^+$. Then for all $t>1$ and $\alpha>0$, we have*

$$\begin{aligned} & {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n}f^{\theta_i}(t)\right] \\ & \geq\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\prod_{i=1}^{n}f^{\theta_i}(t)\right], \end{aligned} \tag{3.29}$$

where $\sigma>0$, $\delta\geq\theta_k>0$, $k\in\{1,...,n\}$.

Proof. Using the conditions of Theorem 9, we can write

$$\left(h^{\sigma}(y)-h^{\sigma}(x)\right)\left(f_k^{\delta-\theta_k}(x)-f_k^{\delta-\theta_k}(y)\right)\geq 0\ , \tag{3.30}$$

for any $x,y\in[1,t]$; $t>1$, $\sigma>0$, $\delta\geq\theta_k>0$, $k\in\{1,...,n\}$.

This implies that

$$h^{\sigma}(y)\,f_k^{\delta-\theta_k}(x)+h^{\sigma}(x)\,f_k^{\delta-\theta_k}(y)\geq h^{\sigma}(y)\,f_k^{\delta-\theta_k}(y)+h^{\sigma}(x)\,f_k^{\delta-\theta_k}(x)\ . \tag{3.31}$$

We multiply (3.31) by $F^*(t,x)$ then integrate the reslting inequality with respect to x on $(1,t)$, we have

$$\begin{aligned} & h^{\sigma}(y)\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] \\ & +f_k^{\delta-\theta_k}(y)\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right] \\ & \geq h^{\sigma}(y)\,f_k^{\delta-\theta_k}(y)\ {}_HJ_1^{\alpha}\left[w(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right] \\ & +\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f^{\theta_i}(t)\right]. \end{aligned} \tag{3.32}$$

The inequality (3.32) implies that

$$\begin{aligned} & {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right] \\ & +\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] \\ & \geq\ {}_HJ_1^{\alpha}\left[w(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f^{\theta_i}(t)\right] \\ & +\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right]. \end{aligned} \tag{3.33}$$

The ends the proof.

We also present the following result for the Hadamard fractional integral with two parameters :

Theorem 13 *Let f_i, $i=1,...,n$ and h be positive continuous functions on $[1,\infty)$, such that h is increasing and f_i, $i=1,...,n$ are decreasing on $[1,\infty)$ and $w:[1,\infty)\to\mathbb{R}^+$. Then we have*

$$\begin{aligned} & {}_HJ_1^{\beta}\left[w(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right] \\ & +\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right] \\ & \geq\ {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right] \\ & +\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f^{\theta_i}(t)\right]\ {}_HJ_1^{\beta}\left[w(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right], \end{aligned} \tag{3.34}$$

for all $t>1$, $\alpha>0$, $\beta>0$, $\sigma>0$, $\delta\geq\theta_k>0$, $k\in\{1,...,n\}$.

Proof. Using (3.26) and (3.31), we can write

$$\begin{aligned} & f_k^{\delta-\theta_k}(x)\ {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right] \\ & +h^{\sigma}(x)\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] \\ & \geq\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f^{\theta_i}(t)\right] \\ & +h^{\sigma}(x)\,f_k^{\delta-\theta_k}(x)\ {}_HJ_1^{\alpha}\left[w(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right], \end{aligned} \tag{3.35}$$

By (3.20) and (3.35), we have

$$\begin{aligned} & {}_HJ_1^{\beta}\left[w(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right] \\ & +\ {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] \\ & \geq\ {}_HJ_1^{\beta}\left[w(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f^{\theta_i}(t)\right] \\ & +\ {}_HJ_1^{\beta}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f^{\theta_i}(t)\right]\ {}_HJ_1^{\alpha}\left[w(t)\prod_{i=1}^{n} f^{\theta_i}(t)\right], \end{aligned} \tag{3.36}$$

This completes the proof of Theorem 13.

Remark 14 *If we take $\alpha = \beta$, in Theorem 13, we obtain Theorem 12.*

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