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On three phase lags thermodiffusion theory in micropolar porous circular plate

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ABSTRACT

The present work examines a two dimensional axisymmetric problem of micropolar porous thermodiffusion circular plate due to thermal and chemical potential sources. The governing equations are solved by using the potential function. The expressions of displacements, microrotation, volume fraction field, temperature distribution, concentration and stresses are obtained in the transformed domain by using Laplace and Hankel transforms. The inversion of transforms using Fourier expansion techniques has been applied to obtain the results in the physical domain. The numerical results for resulting quantities are obtained and depicted graphically to show the influence of porosity, relaxation time, phase lags, with and without energy dissipation on the resulting quantities. Some particular cases are also deduced.

1. Introduction

Nowacki (1966) and Eringen (1970) developed the theory of micropolar thermoelasticity by extending the micropolar elasticity theory including thermal effects. Tauchert, Claus and Ariman (1968) examined the linear theory of micropolar coupled thermoelasticity. Boschi and Iesan (1973) extended the linear theory of generalized thermoelasticity for the case of a homogeneous micropolar continuum with two relaxation times. Using Green Lindsay theory (1972), the theory of generalized micropolar thermoelasticity was investigated by Dost and Tabarrok (1978). The theory of micropolar thermoelasticity that includes heat flux among constitutive variables was developed by Chandrasekharaiah (1986).

Iesan (1985) developed the linear theory of micropolar materials with voids and studied the propagation of waves in micropolar elastic medium with voids. Marin (1996a,

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1996b) presented the generalized solutions for boundary value problems in micropolar elastic bodies with voids. Othman and Atwa (2012) studied the deformation of micropolar thermoelalstic solid with voids under the influence of various sources by employing the Green Naghdi theory of thermoelasticity. Sharma and Kumar (2013) constructed the fundamental solution of the system of differential equations in thermoviscoelastic medium with voids and also studied the propagation of plane waves. Marin Abd-Alla, Raducanu and Abo-Dahab (2015) presented the solution of mixed initial boundary value problem for micropolar porous materials depending continuously on coefficients which couple the micropolar deformation equations with the equations that model the evolution of voids. Yong Ai and Wu (2016) introduced the precise integrationsolution with thermal diffusivity and permeability for thermal consolidation problems of a multilayered porous thermoelastic medium due to a heat source.

The theory of thermoelastic diffusion using coupled thermoelastic model was developed by Nowacki (1974a, b, c, 1976). Aouadi (2007) proved the uniqueness and reciprocity theorems in generalized thermoelastic diffusion medium based on the Lord Shulman theory (1967). Sharma (2013) studied the deformation in homogeneous and isotropic thermodiffusion elastic half space with normal and tangential loads. Kumar (2015) studied the propagation of plane waves in a microstretch thermoelastic diffusion solid. Tripathi, Kedar and Deshmukh (2015) studied the effect of axisymmetric heat supply on the phenomena of diffusion in a thermoelastic thick circular plate in the context of the theory of generalized thermoelastic diffusion with relaxation time. The deformation due to inclined load in micropolar thermoelastic diffusion medium subjected to thermal laser pulse investigated by Kumar and Kumar (2016). Using generalized thermoelasticity theory with two time delays and kernel functions, the constitutive equations for thermoelastic diffusion in anisotropic and isotropic solids are derived by El-Karamany and Ezzat (2016).

Roychoudhuri (2007) introduced a three phase lag model by taking the heat conduction law that includes thermal displacement gradient and temperature gradient among constitutive variables in the theory of coupled thermoelasticity. This model is an extension of the thermoelastic models proposed by Lord Shulman (1967) and Tzou (1995a, 1995b). Kumar and Chawla (2013) studied the propagation of longitudinal and transverse waves at the interface between uniform elastic solid half space and thermoelastic solid with three phase lag model. El-Karamany and Ezzat (2013) presented the uniqueness, reciprocal theorems and variational principle for the micropolar thermoelasticity theory with three phase lag model. Akbarzadeh, Fu and Chen (2014) studied the heat conduction in a functionally graded, infinitely long hollow cylinder based on the three phase lag model. Zenkour (2016) presented the generalized thermoelasticity theory based on the dual phase lags theory to study the problem of a thick walled simply supported beam with different applied heat source and mechanical loads. Kartoshov (2016) presented a mathematical theory for boundary value problems of nonstationary heat conduction by using dual phase lag and also presented the features of analytical solutions of such heat problems.

In the work, we investigate an axisymmetric problem of micropolar porous circular plate with mass diffusion in the context of three phase lag theory of thermoelasticity due to thermal and chemical potential sources. The potential functions and Laplace and Hankel transforms are proposed to solve the problem. Inversion of transforms is applied to obtain the results in the physical domain. Effect of porosity, relaxation time, phase lag, with and without energy dissipation are presented on the resulting quantities.

2. Basic equations

Following Kumar and Partap (2008), Roychoudhuri (2007) and Kumar and Kansal (2008), the field equations and the constitutive relations for a micropolar porous thermodiffusion medium with three phase lag model in the absence of body forces, body couples, heat sources and extrinsic equilibrated body force are taken as

$$(\lambda+2\mu+k)\nabla(\nabla.\vec{u})-(\mu+K)\nabla\times\nabla\times\vec{u}+K\nabla\times\vec{\phi}+b\nabla\phi^{*}-\beta_1\nabla T-\beta_2\nabla C=\rho\frac{\partial^2\vec{u}}{\partial t^2} \quad (1)$$

$$(\alpha+\beta+\gamma)\nabla\left(\nabla.\vec{\phi}\right)-\gamma\nabla\times\left(\nabla\times\vec{\phi}\right)+\mathrm{K}\nabla\times\vec{u}-2K\vec{\phi}=\rho j\frac{\partial^2\vec{\phi}}{\partial t^2} \quad (2)$$

$$\begin{gathered}\left[K^{*}\left(1+\tau_\nu\frac{\partial}{\partial t}\right)+K_1^{*}\frac{\partial}{\partial t}\left(1+\tau_t\frac{\partial}{\partial t}\right)\right]\nabla^2 T=\\ \left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2}{\partial t^2}\left[\rho C^{*}T+\beta_1\mathrm{T}_0(\nabla.\vec{u})+\nu_1\mathrm{T}_0\phi^{*}+a_0\mathrm{T}_0\mathrm{C}\right]\end{gathered} \quad (3)$$

$$D\beta_2\nabla^2(\nabla.\vec{u})+\mathrm{D}\nu_2\nabla^2\phi^{*}+Da_0\nabla^2\mathrm{T}+\dot{\mathrm{C}}-\mathrm{D}b_0\nabla^2\mathrm{C}=0 \quad (4)$$

$$P=-\beta_2(\nabla.\vec{u})+b_0C-a_0T \quad (5)$$

$$t_{ij}=\lambda u_{r,r}\delta_{ij}+\mu(u_{i,j}+u_{j,i})+K(u_{j,i}-\varepsilon_{ijk}\phi_k)-\beta_1T\delta_{ij}-\beta_2C\delta_{ij}+b\delta_{ij}\phi^{*} \quad (6)$$

$$m_{ij}=\alpha\phi_{r,r}\delta_{ij}+\beta\phi_{i,j}+\gamma\phi_{j,i} \quad (7)$$

where $\vec{u}$ is the displacement vector, $\vec{\phi}$ is the microrotation vector, ρ is the density, j is the micro inertia, λ, μ, K, α, β, γ are micropolar constants, α_1, b, ξ_1, ω_0, m and $\varkappa$ are the elastic constants due to the presence of voids, ϕ^{*} is the change in volume fraction field, K_1^{*} is the coefficient of thermal conductivity, T is the change in temperature of the medium at any time, C^{*} is the specific heat at constant strain, C is the concentration of the diffusion material in the body, D is the thermoelastic diffusion constant, a_0, b_0 are respectively, coefficients describing the measure of thermodiffusion and of mass diffusion effects, $\beta_1=(3\lambda+2\mu+k)\alpha_{t1}$, $\beta_2=(3\lambda+2\mu+k)\alpha_{c1}$, $\nu_1=(3\lambda+2\mu+k)\alpha_{t2}$, $\nu_2=(3\lambda+2\mu+k)\alpha_{c2}$, α_{t1}, α_{t2} are coefficients of linear thermal expension and α_{c1}, α_{c2} are the coefficients of linear diffusion expansion, τ_t, τ_q and τ_v respectively, the phase lag of the temperature gradient, the phase lag of the heat flux and the phase lag of the thermal displacement. t_{ij}, m_{ij} are the stress tensor and couple stress tensor, δ_{ij} is the kroneckor delta and $\nabla^2=\frac{\partial^2}{\partial \mathrm{r}^2}+\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}+\frac{1}{\mathrm{r}^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial \mathrm{z}^2}$ is the Laplacian operator.

3. Formulation of the problem

A homogeneous and isotropic micropolar porous thermodiffusion elastic circular plate of thickness $2d$ is considered and the region $0\le r\le\infty$, $-d\le z\le d$ is occupied by the plate. We consider cylindrical polar coordinate system $(r,\ \theta,\ z)$ with symmetry about z-axis. The origin of the coordinate system $(r,\ \theta,\ z)$ is taken as the middle surface of the plate. We assume that the z-axis is normal to the plate along its thickness. T_0 is the initial temperature of the thick circular plate taken as a constant temperature.

For two dimensional case, the displacement and microrotation vectors as

$$\vec{u} = (u_r,\quad 0,\quad u_z)\,, \qquad \vec{\phi} = (0,\quad \phi_\theta,\quad 0) \tag{8}$$

The following non-dimensional variables are defined as

$$\begin{array}{l} r\prime = \frac{\omega^* r}{c_1},\quad z\prime = \frac{\omega^* z}{c_1},\quad u'_r = \frac{\rho c_1 \omega^* u_r}{\beta_1 T_0},\quad u'_z = \frac{\rho c_1 \omega^* u_z}{\beta_1 T_0},\quad \phi'_\theta = \frac{\rho c_1^2 \phi_\theta}{\beta_1 T_0},\ \phi^*\prime = \frac{\rho c_1^2 \phi^*}{\beta_1 T_0} \\ T\prime = \frac{T}{T_0},\quad C\prime = \frac{\beta_2 C}{\beta_1 T_0},\quad t\prime = \omega^* t,\quad \tau'_t = \omega^* \tau_t,\quad \tau'_q = \omega^* \tau_q,\ \tau'_\nu = \omega^* \tau_\nu,\ P\prime = \frac{P}{\beta_2} \\ t'_{ij} = \frac{t_{ij}}{\beta_1 T_0},\qquad m'_{ij} = \frac{\omega^*}{c_1 \beta_1 T_0} m_{ij} \end{array} \tag{9}$$

where

$$c_1^2 = \frac{\lambda + 2\mu + K}{\rho},\ \omega^* = \frac{K}{\rho j}$$

With the aid of expression relating displacement components u_r and u_z to scalar potentials ϕ and ψ as

$$u_r = \frac{\partial \phi}{\partial r} + \frac{\partial \psi}{\partial z} \tag{10}$$

$$u_z = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial r} - \frac{\psi}{r} \tag{11}$$

The Laplace and Hankel transforms are given by

$$\bar{f}(r,z,s) = L\left\{\bar{f}(r,z,t)\right\} = \int_0^\infty f(r,z,t)\, e^{-st} dt \tag{12}$$

$$\tilde{f}(\xi,z,s) = H\left\{\bar{f}(x,z,s)\right\} = \int_0^\infty r\bar{f}(x,z,s)\, J_n(\xi r)\, dr \tag{13}$$

Making use of (9) and (10) in (1)-(5) and with the use of (11), (12) and also applying Laplace and Hankel transforms defined by (13) and (14) on resulting expressions, after simplification, we obtain

$$\left(\tilde{\phi},\ \widetilde{\phi^*},\ \tilde{T}, \tilde{C}\right) = \sum_{i=1}^{4} (1,\ a_i,\ b_i,\ d_i)\, A_i coshm_i z \tag{14}$$

$$\left(\tilde{\psi},\quad \widetilde{\phi_\theta}\right) = \sum_{i=5}^{6} (1,\ e_i)\, B_i sinhm_i z \tag{15}$$

where $a_i,\ b_i,\ d_i,\ e_i$ and H are given in appendix I, also

$$c_2^2 = \frac{\mu + K}{\rho}, \quad \delta^2 = \frac{c_2^2}{c_1^2}, \quad p = \frac{K}{\rho c_1^2}, \quad p_0 = \frac{b}{\rho c_1^2}, \quad \delta^{*2} = \frac{K c_1^2}{\gamma \omega^{*2}}, \ \delta_1^2 = \frac{c_3^2}{c_1^2}$$

$$c_3^2 = \frac{\gamma}{\rho j}, \quad \delta_1^* = \frac{\rho c_1^4}{\alpha_1 \omega^{*2}}, \quad \bar{\beta}_1 = \frac{\nu_1}{\beta_1}, \ \bar{\beta}_2 = \frac{\nu_2}{\beta_2}, \quad p_1 = \frac{\xi_1}{\rho c_1^2}, \quad \delta_2^* = \frac{\omega_0 c_1^2}{\alpha_1 \omega^*}, \quad \delta_3^* = \frac{\rho \chi c_1^2}{\alpha_1}$$

$$Q^* = \frac{\rho C^* c_1^2}{K_1^* \omega^*}, \quad = \frac{\beta_1^2 T_0}{\rho K_1^* \omega^*}, \ S^* = \frac{a_0 \beta_1 T_0 c_1^2}{\beta_2 K_1^* \omega^*}, \ A^* = \frac{a_0 \rho c_1^2}{\beta_1 \beta_2}, \ B^* = \frac{\rho c_1^4}{D \omega^* \beta_2^2}, \ D^* = \frac{b_0 \rho c_1^2}{\beta_2^2}$$

$$E^* = \frac{\beta_1 T_0}{\rho c_1^2}, \quad F^* = \frac{b_0 \beta_1 T_0}{\beta_2^2}, \ G^* = \frac{a_0 T_0}{\beta_2}, \ R_1 = \left(Z^* \left(1 + \tau_\nu s\right) + s\left(1 + \tau_t s\right)\right)$$

$$R_2 = \left(1 + \tau_q s + \frac{\tau_q^2}{2} s^2\right) s^2$$

$m_i, \ (i = 1, \ 2, \ 3, \ 4)$ and $m_i, \ (i = 5, \ 6)$ are respectively roots of

$$\left[D^8 + P_1 D^6 + P_2 D^4 + P_3 D^2 + P_4\right] = 0 \tag{16}$$

$$\left[D^4 + Q_1 D^2 + Q_2\right] = 0 \tag{17}$$

where $P_1, \ P_2, \ P_3, \ P_4, \ Q_1$ and Q_2 are given on appendix II.

4. Boundary conditions

The boundary conditions may be defined at the surface $z = \pm d$ of the plate as

$$\frac{dT}{dz} = \pm g_0 F\left(r, \ z\right) \tag{18}$$

$$t_{zz} = 0 \tag{19}$$

$$t_{zr} = 0 \tag{20}$$

$$m_{z\theta} = 0 \tag{21}$$

$$\frac{d\phi^*}{dz} = 0 \tag{22}$$

$$P = p_0 \frac{\delta\left(r\right) H\left(t\right)}{2\pi r} \tag{23}$$

where $F\left(r, \ z\right) = z^2 e^{-\omega r}, \quad \omega > 0.$

$\delta()$ is the Dirac delta function and $H()$is the Heavyside unit step function.

With the use of (6)-(16), (19)-(24), the expressions of displacements, microrotation, volume fraction field, temperature distribution, concentration and stresses are obtained in the transformed domain as

$$\widetilde{u_r} = -\frac{1}{\Delta}\sum_{i=1}^{4}\xi\Delta_i coshm_i z + \frac{1}{\Delta}\sum_{i=5}^{6} m_{i\Delta i} coshm_i z \tag{24}$$

$$\widetilde{u_z} = \frac{1}{\Delta}\sum_{i=1}^{4} m_{i\Delta i} sinhm_i z - \frac{1}{\Delta}\sum_{i=5}^{6}\xi\Delta_i sinhm_i z \tag{25}$$

$$\left(\widetilde{\phi^*},\ \tilde{T}, \tilde{C}\right) = \frac{1}{\Delta}\sum_{i=1}^{4}(a_i,\ b_i,\ d_i)\,\Delta_i coshm_i z \tag{26}$$

$$\widetilde{\phi_\theta} = \frac{1}{\Delta}\sum_{i=5}^{6} e_{i\Delta i} sinhm_i z \tag{27}$$

$$\widetilde{t_{zz}} = \frac{1}{\Delta}\sum_{i=1}^{4} L_{i\Delta i} coshm_i z + \frac{1}{\Delta}\sum_{i=5}^{6} M_i\Delta_i coshm_i z \tag{28}$$

$$\widetilde{t_{zr}} = \frac{1}{\Delta}\sum_{i=1}^{4} N_i\Delta_i sinhm_i z + \frac{1}{\Delta}\sum_{i=5}^{6} S_i\Delta_i sinhm_i z \tag{29}$$

$$\widetilde{m_{z\theta}} = \frac{1}{\Delta}\sum_{i=5}^{6} T_i\Delta_i coshm_i z \tag{30}$$

where

$$\Delta = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 & 0 & 0 \\ V_1 & V_2 & V_3 & V_4 & Y_5 & Y_6 \\ W_1 & W_2 & W_3 & W_4 & Z_5 & Z_6 \\ 0 & 0 & 0 & 0 & O_5 & O_6 \\ X_1 & X_2 & X_3 & X_4 & 0 & 0 \\ P_1 & P_2 & P_3 & P_4 & 0 & 0 \end{bmatrix}$$

and $\Delta_i\,(i = 1, 2, 3, 4, 5,\ 6)$ are obtained from Δ by replacing ith column of Δ with $|Q,\ 0,\ 0,\ 0,\ R|^{tr}$, also

$$U_i = b_i m_i sinhm_i d,\ V_i = L_i coshm_i d,\ W_i = N_i sinhm_i d,\ X_i = a_i m_i sinhm_i d,\ i = 1,\ 2,\ 3,\ 4$$

$$O_i = T_i coshm_i d,\quad Y_i = M_i coshm_i d,\quad Z_i = S_i sinhm_i d,\quad P_i = K_i coshm_i d,\quad i = 5,\ 6$$

$$Q = \pm g_0 \frac{z^2 \omega}{(z^2 + \omega^2)^{3/2}}, \qquad R = \frac{p_0}{2\pi s}$$

$$L_i = m_i^2 - \frac{\lambda \xi^2}{\rho c_1^2} + p_0 a_i - b_i - d_i,\ i = 1,\ 2,\ 3, 4$$

$$M_i = \xi \left(\frac{\lambda}{\rho c_1^2} - 1 \right) m_i,\ i = 5,\ 6$$

$$N_i = -\left(\frac{2\mu}{\rho c_1^2} + p \right) \xi m_i,\ i = 1,\ 2,\ 3,\ 4$$

$$S_i = \frac{(\mu + K)\, m_i^2 + \mu \xi^2}{\rho c_1^2} - p d_i,\ i = 5,\ 6$$

$$T_i = \frac{\gamma \omega^{*2}}{\rho c_1^4} d_i m_i,\ i = 5,\ 6$$

5. Particular Cases

1. If we take $K^* = 0$, in equations (25)-(31), then we obtain the corresponding results for micropolar porous thermodiffusion with dual phase lag model.
2. If we take $C = 0$, in equations (25)-(31), then we obtain the corresponding results for micropolar porous thermoiffusion with three phase lag model.
3. If we take, $K^* = 0$, $\tau_t = \tau_q^2 = 0$ and $\tau_q = \tau_0$, in equations (25)-(31), then we obtain the corresponding results for micropolar porous thermodiffusion with one relaxation time.
4. If we take $\tau_v = K_1^* = \tau_q = \tau_q^2 = 0$, in equations (25)-(31), then we obtain the corresponding results for micropolar porous thermodiffusion with energy dissipation.
5. If we take, $\tau_v = \tau_t = \tau_q = \tau_q^2 = 0$, in equations (25)-(31), then we obtain the corresponding results for micropolar porous thermodiffusion without energy dissipation.
6. Neglecting the porous effect i.e., α_1, b, ξ_1, ω_0, $\varkappa$ and ϕ^* tend to zero. Then, the boundary conditions for two temperature micropolar thermoelastic solid with three phase lag model are given by

$$\frac{dT}{dz} = \pm g_0 F\,(r,\ z)$$

$$t_{zz} = 0$$

$$t_{zr} = 0$$

$$m_{z\theta} = 0$$

$$P = \delta(r)\,\delta(t)$$

and the corresponding expressions are given by

$$\widetilde{u_r} = -\frac{1}{\Delta}\sum_{i=1}^{3}\xi\Delta_i coshm_i z + \frac{1}{\Delta}\sum_{i=4}^{5} m_i\Delta_i coshm_i z$$

$$\widetilde{u_z} = \frac{1}{\Delta}\sum_{i=1}^{3} m_i\Delta_i sinhm_i z - \frac{1}{\Delta}\sum_{i=4}^{5}\xi\Delta_i sinhm_i z$$

$$\left(\tilde{T},\ \tilde{C}\right) = \frac{1}{\Delta}\sum_{i=1}^{3} b_i\Delta_i coshm_i z$$

$$\widetilde{\phi_\theta} = \frac{1}{\Delta}\sum_{i=4}^{5} d_i\Delta_i sinhm_i z$$

$$\widetilde{t_{zz}} = \frac{1}{\Delta}\sum_{i=1}^{3} L_i\Delta_i coshm_i z + \frac{1}{\Delta}\sum_{i=4}^{5} M_i\Delta_i coshm_i z$$

$$\widetilde{t_{zr}} = \frac{1}{\Delta}\sum_{i=1}^{3} N_i\Delta_i sinhm_i z + \frac{1}{\Delta}\sum_{i=4}^{5} S_i\Delta_i sinhm_i z$$

$$\widetilde{m_{z\theta}} = \frac{1}{\Delta}\sum_{i=4}^{5} T_i\Delta_i coshm_i z$$

where

$$\Delta^* = \begin{bmatrix} U_1^{**} & U_2^{**} & U_3^{**} & 0 & 0 \\ V_1^{**} & V_2^{**} & V_3^{**} & Y_4^{**} & Y_5^{**} \\ W_1^{**} & W_2^{**} & W_3^{**} & Z_4^{**} & Z_5^{**} \\ 0 & 0 & 0 & O_4^{**} & O_5^{**} \\ P_1^{**} & P_2^{**} & P_3^{**} & 0 & 0 \end{bmatrix}$$

and $\Delta_i^{**}\,(i = 1, 2, 3, 4, 5)$ are obtained from Δ^{**} by replacing ith column of Δ^{**} with $|\ Q,\ 0,\ 0,\ 0,\ R|^{tr}$, also

$$U_1^{**} = b_i m_i sinhm_i d,\ V_i^{**} = L_i^{**} coshm_i d,\ W_i^{**} = N_i^{**} sinhm_i d, P_i^{**} = K_i^{**} coshm_i d, i = 1,\ 2,\ 3$$

$$O_i^{**} = T_i^{**} coshm_i d,\quad Y_i^{**} = M_i^{**} coshm_i d,\quad Z_i^{**} = S_i^{**} sinhm_i d,\quad i = 4,\ 5$$

$$L_i^{**} = m_i^2 - \frac{\lambda\xi^2}{\rho c_1^2} - b_i - d_i,\ i = 1,\ 2,\ 3$$

$$M_i^{**} = \xi \left(\frac{\lambda}{\rho c_1^2} - 1 \right) m_i, \ i = 4, \ 5$$

$$N_i^{**} = -\left(\frac{2\mu}{\rho c_1^2} + p \right) \xi m_i, \ i = 1, \ 2, \ 3$$

$$S_i^{**} = \frac{(\mu + K)\, m_i^2 + \mu \xi^2}{\rho c_1^2} - p d_i, \quad i = 4, \ 5$$

$$T_i^{**} = \frac{\gamma \omega^{*2}}{\rho c_1^4} d_i m_i, \ i = 4, \ 5$$

$$b_i = \left[\left(m_i^2 - \xi^2 - s^2 \right) \left\{ B^* s - D^* \left(m_i^2 - \xi^2 \right) \right\} + \left(m_i^2 - \xi^2 \right)^2 \right] / \mathcal{H}$$

$$d_i = \left[-\left(m_i^2 - \xi^2 \right)^2 - A^* \left(m_i^2 - \xi^2 \right) \left(m_i^2 - \xi^2 - s^2 \right) \right] / \mathcal{H}$$

$$\mathcal{H} = B^* s - (D^* + A^*) \left(m_i^2 - \xi^2 \right)$$

6. Inversion of Transforms

We have to obtain the transformed displacements, microrotation, volume fraction field, temperature distribution, concentration and stresses in the physical domain, so, we invert the transforms in the resulting expressions (25)-(31). All these expressions are functions of the form $\tilde{f}(\xi, z, s)$. Therefore, we get the function $f(r, z, t)$ by using the inversion of the Hankel and Laplace transforms are defined by

$$\tilde{f}(\xi, z, s) = \int_0^\infty \xi \bar{f}(\xi, z, s)\, J_n(\xi r)\, d\xi \tag{31}$$

$$f(r, z, t) = \frac{1}{2\iota\pi} \int_{c-\iota\infty}^{c+\iota\infty} \bar{f}(r, z, s)\, e^{-st} ds \tag{32}$$

where c is an arbitrary constant greater than all real parts of the singularities of $\bar{f}(r, z, t)$.

7. Numerical Results and Discussions

Following Eringen (1984), the values of micropolar parameters is taken as

$$\lambda = 9.4 \times 10^{10} Nm^{-2}, \quad \mu = 4.0 \times 10^{10} Nm^{-2}, \quad K = 1.0 \times 10^{10} Nm^{-2}$$

$$\rho = 1.74 \times 10^3 Kgm^{-3}, \quad j = 0.2 \times 10^{-19} m^2, \ \gamma = 0.779 \times 10^{-9} N$$

Following Dhaliwal and Singh (1980), the values of thermal parameters are given by

$$C^* = 1.04 \times 10^3 JKg^{-1}K^{-1}, \quad K_1^* = 1.7 \times 10^6 Jm^{-1}s^{-1}K^{-1}, \quad \alpha_t = 2.33 \times 10^{-5}K^{-1}$$

$$\tau_t = 0.1s \times 10^{-13}sec, \quad \tau_q = 0.2s \times 10^{-13}sec, \quad \tau_0 = 6.131 \times 10^{-13}sec$$

$$\tau_\nu = 8.765 \times 10^{-13}sec, \quad T_0 = 0.298 \times 10^3 K, \quad m = 1.13849 \times 10^{10} N/m^2 K$$

The diffusion parameters are given by

$$\alpha_{t1} = 2.33 \times 10^{-5}K^{-1}, \; \alpha_{t2} = 2.48 \times 10^{-5}K^{-1}, \quad \alpha_{c1} = 2.65 \times 10^{-4}m^3Kg^{-1}$$

$$\alpha_{c2} = 2.83 \times 10^{-4}m^3Kg^{-1}, \; a_0 = 2.9 \times 10^4 m^2s^{-2}K^{-1}, \quad b_0 = 3.2 \times 10^5 Kg^{-1}m^5s^{-2}$$

$$D = 0.85 \times 10^{-8} Kgm^{-3}s$$

The values of void parameters are given as

$$\alpha_1 = 3.688 \times 10^{-9}N, \quad b = 1.138494 \times 10^{10} N/m^2, \quad \xi_1 = 1.1475 \times 10^{10} N/m^2$$

$$\chi = 1.1753 \times 10^{-19}m^2, \quad \omega_0 = 0.0787 \times 10^{-1} N \times sec/m^2$$

We have determined the variations of normal stress, shear stress, couple shear stress, volume fraction field, temperature distribution and concentration with distance r in case of micropolar thermodiffusion porous with three phase lag (MDPT), micropolar thermodiffusion with three phase lag (MDT), micropolar thermoelastic porous with three phase lag (MPT), micropolar thermodiffusion porous with dual phase lag model (MDPD), micropolar thermodiffusion porous with Lord Shulman theory (MDPL), micropolar thermodiffusion porous with energy dissipation (MDPII) and micropolar thermodiffusion porous without energy dissipation (MDPIII). In all these figures, MDPT, MPT, MDPD, MDT, MDPL, MDD, MDPII and MDPIII corresponding to solid line, small dash line,dash line, small dash line with centered symbol, small dash line with star, small dash line with cross symbol and small dash line with zero sympol respectively.

Figure 1 displays that the values of t_{zz} decay sharply for MDPII for $1 \leq r \leq 1.5$, increase for $1.5 \leq r \leq 3.4$ in comparison to MDPT, MPT and MDPIII and then decrease for $3.4 \leq r \leq 4$. Its values for MDT, MDPD and MDPL, initially increased and then decreased. It is also seen that for $1 \leq r \leq 2.3$, the values are higher for MDT and smaller for MDPII whereas for $2.8 \leq r \leq 3.9$, the values are higher for MDPII and smaller for MDT. All the quantities have similar values away from the source.

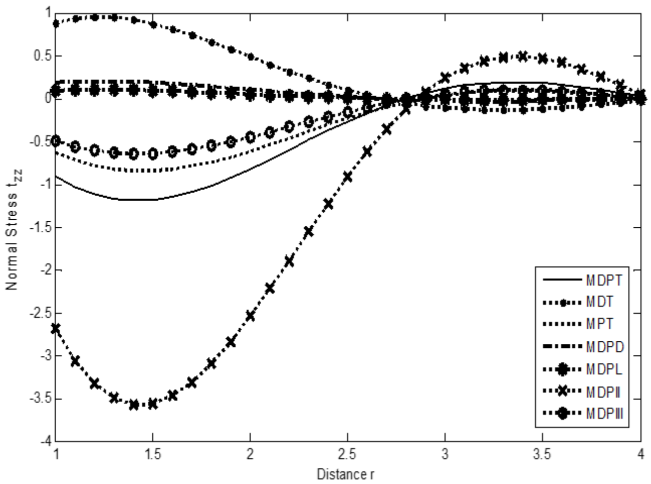

Fig. 1 – Variations of normal stress t_{zz}.

Figure 2 describes that the values of t_{zr} initially increase for $1 \leq r \leq 1.5$, decrease for $1.5 \leq r \leq 3.5$ and the values are stationary for $3.5 \leq r \leq 4$ for MDPT and MDPD. The small variation are exhibited in the case of MDPL. The value is initially increased and then decreased for MPT. Its value for MDT is stationary in the beginning and then increased and further gets decreased slowly. For MDPII and MDPIII, its values initially decreased rapidly and then increased gradually for the further range. The values are maximum for MDPIII and minimum for MDT near the application of the source.

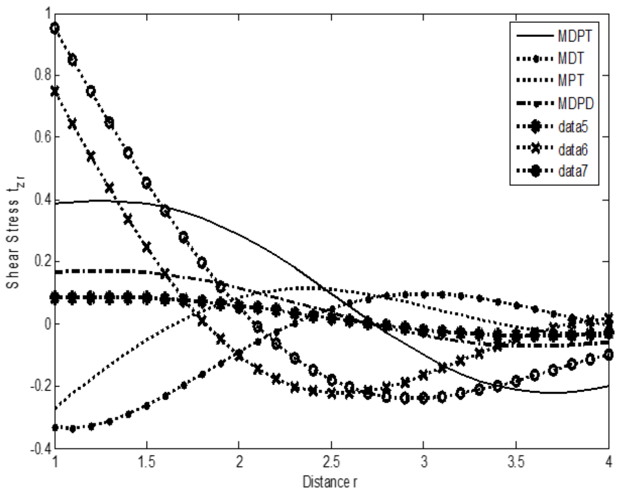

Fig. 2 – Variations of shear stress t_{zr}.

Figure 3 shows that the values of $m_{z\theta}$ increase for $1 \leq r \leq 2.5$, decrease for $2.5 \leq r \leq 3.6$ and then its values become stationary for $3.6 \leq r \leq 4$ for MDPT, MDT, MDPD, MDPL, MDPII and MDPIII. Its value for MPT initially decreases for $1 \leq r \leq 2.3$, increases for $2.3 \leq r \leq 3.7$ and then becomes stationary for the remaining range. The maximum magnitude is occurred in the case of MDPII.

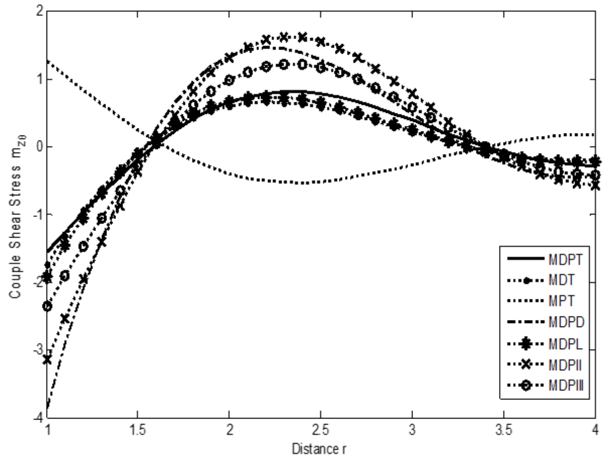

Fig. 3 – Variations of couple shear stress $m_{z\theta}$.

Figure 4 shows that the value of ϕ^* initially increases for $1 \le r \le 1.3$, decreases for $1.3 \le r \le 3.5$ and again increases for $3.5 \le r \le 4$ for MDPT. Its values decrease for $1 \le r \le 1.3$, increase for $1.3 \le r \le 3.3$ and again decrease for $3.3 \le r \le 4$ for MPT, MDPD, MDPL and MDPII. Its values are decreasing in the case of MDPIII over the whole range. ϕ^* takes large value in the case of MDPT and small value in the case of MDPL for $1 \le r \le 2.5$. The values are the same for MDPT, MPT, MDPD, MDPL and MDPII away from the source.

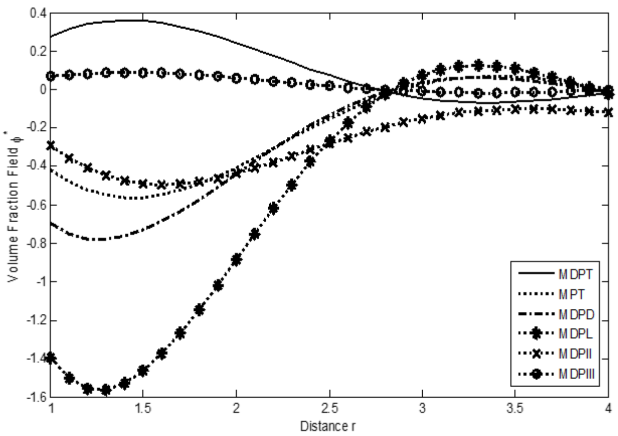

Fig. 4 – Variations of volume fraction field ϕ^*.

Figure 5 exhibits that the value of T increased at the beginning, then decreased and then again increased away from the source. The maximum value is exhibited in the case of MDPT. Near the application of the source, the values are similar for MDPT, MDPL and MDPIII. The values are also similar for MPT, MDPIII and MDT, MDPD, MDPL, MDPII for the ranges $3.4 \le r \le 3.8$, $3.2 \le r \le 3.7$ respectively.

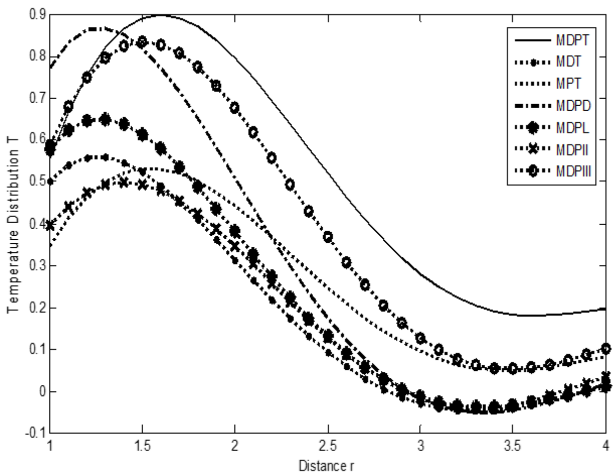

Fig. 5 – Variations of temperature distribution T.

Figure 6 represents that the values of C decrease at the beginning for $1 \leq r \leq 2.7$ and then increase for $2.7 \leq r \leq 4$ for MDPT, MDPL and MDPIII. Its value is increased for $1 \leq r \leq 2.5$ and decreased for $2.5 \leq r \leq 4$ for MDPD and MDPII. The value of C is decreased and attains its minimum value for $1 \leq r \leq 1.3$, rapidly increased for $1.3 \leq r \leq 3.4$ and then again decreased for $3.4 \leq r \leq 4$ for MDT. All the curve obtained have a different starting point. The values are coinciding for MDPL and MDPIII for $1.7 \leq r \leq 3.8$.

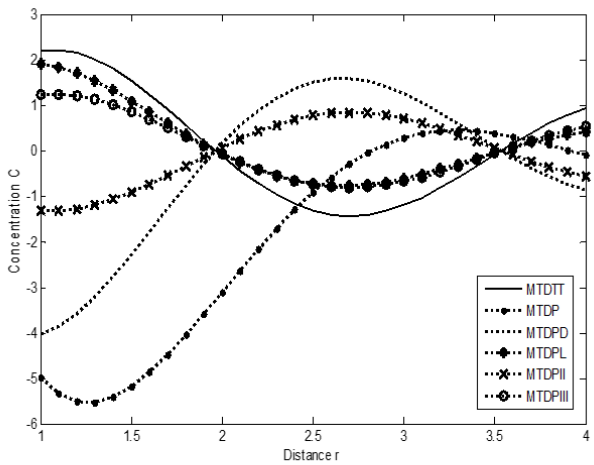

Fig. 6 – Variations of concentration C.

8. Conclusion

In the paper, a two dimensional axisymmetric problem of micropolar porous thermodiffusion circular plate with thermal and chemical potential sources has been investigated. The potential functions and Laplace and Hankel transforms are used to solve the problem. We have presented the effect of porosity, relaxation time, phase lags, with and without energy dissipation on the resulting quantities.

For couple shear stress, all the quantities have similar behavior except MPT. Volume fraction field has also similar behavior for MPT, MDPD, MDPL and which is opposite to MDPT. The decreasing behavior is observed in the case of MDPIII for volume fraction field. Similarity is also exhibited in the cases of MDPT, MPT, MDPII and MDPIII whereas reserved behavior is observed in the case of MDT for normal stress. The values are decreased in the cases of MDPD and MDPL for normal stress. For shear stress, it is observed that the variations are small in the cases of MDPD and MDPL. For MDPII and MDPIII, the variations are similar. Due to void and diffusion effect, the values are initially increased and then decreased. For concentration, the variation are similar for MDPD and MDPII which is opposite to MDPT, MDPL and MDPTIII. For temperature distribution, the variations remain almost same in all the cases. Thus, we have observed that most of the quantities are similar in nature which concludes that the effect of void, diffusion, phase lags, thermal relaxation time have an important role in micropolar thermoelasticity.

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Appendix I

$$\begin{aligned} a_i &= [\{\left(m_i^2-\xi^2\right)^3(1-D^*)+\Big(s\left(m_i^2-\xi^2\right)^2(B^*+D^*s)-\left(m_i^2-\xi^2\right)s^3B^*\Big)\}R_1 \\ &\quad +\{\left(m_i^2-\xi^2\right)^2((A^*+D^*)+Q^*(D^*-1)+S^*(A^*+1))-\left(m_i^2-\xi^2\right)\{B^*s(+Q^*) \\ &\quad +s^2(D^*Q^*+A^*S^*)\}+s^3B^*Q^*\}]R_2/\mathcal{H} \\ i &= 1,\ 2,\ 3,\ 4 \end{aligned}$$

$$\begin{aligned} b_i &= [\left(m_i^2-\xi^2\right)^2\left\{(1-D^*)\bar{\beta}_1-(+S^*)\bar{\beta}_2+D^*p_0+p_0S^*\right\} \\ &\quad +\left(m_i^2-\xi^2\right)\left\{sB^*\left(\bar{\beta}_1-p_0\right)+s^2\left(D^*\bar{\beta}_1+S^*\bar{\beta}_2\right)\right\}-s^3B^*\bar{\beta}_1]R_2/\mathcal{H} \\ i &= 1,\ 2,\ 3,\ 4 \end{aligned}$$

$$\begin{aligned} d_i &= [\{\left(m_i^2-\xi^2\right)^3\left(p_0-\bar{\beta}_2\right)+\left(m_i^2-\xi^2\right)^2\bar{\beta}_2s^2\}R_1+\{\left(m_i^2-\xi^2\right)^2(-(1+A^*)\bar{\beta}_1 \\ &\quad +(+Q^*)\bar{\beta}_2+p_0A^*-p_0Q^*)+\left(m_i^2-\xi^2\right)s^2\left(A^*\bar{\beta}_1-\bar{\beta}_2Q^*\right)\}R_2]/\mathcal{H} \\ i &= 1,\ 2,\ 3,\ 4 \end{aligned}$$

$$e_i=\frac{\delta^2}{p}\left(m_i^2-\xi^2-\frac{s^2}{\delta^2}\right),\quad i=5,\ 6$$

$$\begin{aligned} H &= \left(m_i^2-\xi^2\right)[\{\left(m_i^2-\xi^2\right)\left(D^*p_0-\bar{\beta}_2\right)-sB^*p_0\}R_1-\{\bar{\beta}_1(A^*+D^*)+Q^*\bar{\beta}_2 \\ &\quad -p_0D^*Q^*-\bar{\beta}_2S^*-A^*S^*p_0\}R_2]+B^*s\left(\bar{\beta}_1+p_0Q^*\right)R_2 \\ i &= 1,\ 2,\ 3,\ 4 \end{aligned}$$

Appendix I

$$\begin{aligned} P_1 &= [f^{11}(D^*-1)+R_1\left(f^{13}-f^{12}\right)-2\xi^2R_1+(S^*+D^*)R_2 \\ &\quad +\bar{\beta}_2\delta_1^*R_1\left(2p_0-\bar{\beta}_2\right)+(+S^*)A^*R_2-p_0^2\delta_1^*R_1D^*+R_1D^*\left(f^{12}+\xi^2+s^2\right)]/\mathcal{G} \end{aligned}$$

$$
\begin{aligned}
P_2 = {} & [f^{11}f^{12} + 2\xi^2\left(R_1f^{12} + f^{11}\right) + R_1\xi^4 + \delta_1^*R_2\left(\bar{\beta}_1^2 + \bar{\beta}_2^2\right) - \bar{\beta}_1\bar{\beta}_2R_2 \\
& + \bar{\beta}_1\bar{\beta}_2\delta_1^*R_2\left(-S^* - A^*\right) + \left(\bar{\beta}_1 + \bar{\beta}_2\right)\left(p_0\delta_1^*S^*R_2 + p_0\delta_1^*A^*R_2\right) + \delta_1^*R_2 \\
& \left(p_0^2\delta_1^*A^* + 2p_0\bar{\beta}_1D^* - \bar{\beta}_1^2D^*\right) - \left(f^{11} + 2\xi^2R_1\right)p_0\bar{\beta}_2\delta_1^* - S^*R_2\left(f^{12} + 2\xi^2\right) \\
& - p_0\bar{\beta}_2\delta_1^*\left(f^{11} + 2\xi^2R_1\right) + \bar{\beta}_2^2\delta_1^*\left(f^{11} + \left(2\xi^2 + s^2\right)R_1\right) - A^*R_2\left(f^{12} + 2\xi^2\right) \\
& - S^*A^*R_2\left(f^{12} + 2\xi^2 + s^2\right) - R_2\left(f^{13} + \left(f^{12} + \xi^2\right)D^*\right) \\
& + p_0^2\delta_1^*\left(D^*f^{11} + \left(f^{13} + D^*\xi^2\right)R_1\right) - f^{12}\left(\xi^2 + s^2\right)R_1D^* \\
& - \left(f^{12} + \xi^2 + s^2\right)\left(R_1f^{13} + f^{11}D^*\right)]/\mathcal{G}
\end{aligned}
$$

$$
\begin{aligned}
P_3 = {} & [-2\xi^2f^{11}f^{12} - \left(R_1f^{12} + f^{14}\right)\xi^4 + S^*R_2\xi^2\left(2f^{12} + \xi^2\right) \\
& - \bar{\beta}_2^2\delta_1^*\left(f^{11}\xi^2 + \left(\xi^2 + s^2\right)\left(f^{11} + \xi^2R_1\right)\right) + A^*R_2\xi^2\left(2f^{12} + \xi^2\right) \\
& - 2\delta_1^*R_2\xi^2\left(\bar{\beta}_1^2 + \bar{\beta}_2^2\right) + \bar{\beta}_1\bar{\beta}_2\delta_1^*R_2\left\{4\xi^2 + \left(S^* + A^*\right)\left(2\xi^2 + s^2\right)\right\} \\
& - 2p_0\delta_1^*R_2\xi^2\left(\bar{\beta}_1 + \bar{\beta}_2\right)\left(S^* + A^*\right) + 2p_0\delta_1^*\bar{\beta}_2\xi^2\left(2f^{11} + R_1\xi^2\right) - 2p_0^2\delta_1^*S^*A^*R_2 \\
& + S^*A^*R_2\left\{\xi^2\left(\xi^2 + s^2\right) + f^{12}\left(2\xi^2 + s^2\right)\right\} + R_2\left\{\left(f^{12} + \xi^2\right)f^{13} + f^{12}\xi^2D^*\right\} \\
& - 2p_0\bar{\beta}_1\delta_1^*R_2\left(f^{13} + D^*\xi^2\right) + \bar{\beta}_1^2\delta_1^*R_2\left(f^{13} + D^*\left(\xi^2 + s^2\right)\right) \\
& - p_0^2\delta_1^*\left\{f^{11}f^{13} + \left(D^*f^{11} + R_1f^{13}\right)\xi^2\right\} + f^{12}\left(\xi^2 + s^2\right)\left(D^*f^{11} + R_1f^{13}\right) \\
& + f^{11}f^{13}\left(f^{12} + \xi^2 + s^2\right)]/\mathcal{G}
\end{aligned}
$$

$$
\begin{aligned}
P_4 = {} & [f^{11}f^{12}\xi^4 - p_0\bar{\beta}_2\delta_1^*f^{11}\xi^4 - S^*\xi^4R_2f^{12} + \bar{\beta}_2^2\delta_1^*f^{11}\xi^2\left(\xi^2 + s^2\right) + p_0\delta_1^*\bar{\beta}_2S^*R_2\xi^4 \\
& - A^*f^{12}\xi^4R_2 + \delta_1^*\xi^4R_2\left(\bar{\beta}_1^2 + \bar{\beta}_2^2\right) - \bar{\beta}_1\bar{\beta}_2\delta_1^*R_2\xi^2\left\{2\xi^2 + \left(S^* + A^*\right)\left(\xi^2 + s^2\right)\right\} \\
& + p_0\delta_1^*A^*R_2\xi^4\left(\bar{\beta}_1 + \bar{\beta}_2\right) + p_0\delta_1^*\xi^4\left(\bar{\beta}_1S^*R_2 - \bar{\beta}_2f^{11}\right) + p_0\delta_1^*R_2\xi^2\left(p_0S^*A^*\xi^2 + 2\bar{\beta}_1f^{13}\right) \\
& - S^*A^*R_2\xi^2\left(\xi^2 + s^2\right)f^{12} - f^{12}f^{13}\xi^2R_2 + \bar{\beta}_1^2\delta_1^*R_2f^{13}\left(\xi^2 + s^2\right) + p_0^2\delta_1^*\xi^2f^{11}f^{13} \\
& - f^{11}f^{12}f^{13}\left(\xi^2 + s^2\right)]/\mathcal{G}
\end{aligned}
$$

$$Q_1 = -\left(2\xi^2 + \frac{s^2}{\delta^2} + \frac{s^2}{\delta_1^2} + 2\delta^{*2} - \frac{p\delta^{*2}}{\delta^2}\right)$$

$$Q_2 = \left\{\frac{2\xi^2s^2}{\delta^2} + 2\xi^2\delta^{*2} + \frac{s^2}{\delta^2}\left(\frac{s^2}{\delta_1^2} + 2\delta^{*2}\right) - \frac{p\delta^{*2}\xi^2}{\delta^2}\right\}$$

$$f^{11} = \left(R_1\xi^2 + R_2Q^*\right), \quad f^{12} = \left(\xi^2 + p_1\delta_1^* + \delta_2^*s + \delta_3^*s^2\right), \; f^{13} = B^*s + D^*\xi^2$$

$$f^{14} = R_1f^{12} + f^{11}, \qquad \mathcal{G} = \left(1 - D^*\right)R_1$$