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Numerical Approximation to Spherical Functions by Regularization method

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Abstract:

In this paper, we apply Alternating Direction Method of Multipliers(ADMM) to solve a l_1-regularization optimization problem over the unit sphere. For different functions, we set up proper regularization operators. In particular, we consider approximation to Wendland function and cone function, with or without the presence of data errors. Based on choosing nodes as well conditioned spherical t-design, numerical experiments demonstrate approximation quality vividly.

Keywords **—spherical polynomial approximation, ADMM, l_1- regularization.**

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I. INTRODUCTION

On the unit sphere, [1] introduces a spherical discrete least squares model with rotationally invariant regularization operators. This model includes a series of least squares model, such as spherical polynomial interpolation, hyperinterpolation and filtered hyperinterpolation.

In this paper, we consider a class of spherical l_1-regularization least squares approximation model over the two-dimensional unit sphere $\mathbb{S}^2 := \{\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$:

$$\min_{p\in \mathbb{P}_L} \left\{ \sum_{j=1}^{N} \left(p(\mathbf{x}_j) - f(\mathbf{x}_j) \right)^2 + \lambda \sum_{j=1}^{N} \left| \mathrm{R}_L p(\mathbf{x}_j) \right| \right\}, \quad (1)$$

where f is a given continuous function with values (possibly noisy) given at N points $\mathrm{X}_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{S}^2$. $\mathbb{P}_L := \mathbb{P}_L(\mathbb{S}^2)$ is the linear space of spherical polynomial of degree $\le L$. Regularazation operator R_L is a linear operator which can be chosen in different ways, and $\lambda > 0$ is a parameter. Many different approximations are included in (1) through the freedom to vary the point sets X_N and the regularization operator R_L.

To simplify the model (1), we choose a basis for $\mathbb{P}_L$. We take a basis of orthonormal spherical harmonics[2]

$$\{ Y_{\ell,k} \mid \ell = 0, \ldots, L, k = 1, \ldots, 2\ell+1 \}.$$

This orthogonal basis is normalized so that $Y_{0,1} = \frac{1}{\sqrt{4\pi}}$. Its dimension is $\dim(\mathbb{P}_L) = \sum_{\ell=0}^{L}(2\ell+1) = (L+1)^2$.

The spherical harmonics $Y_{\ell,k}$ with fixed ℓ forms a basis for the $2\ell+1$-dimensional space H_ℓ of homogeneous, harmonic polynomials of degree ℓ. The orthonormality is with respect to the L_2 inner product

$$(f, g)_{\mathrm{L}_2} := \int_{\mathbb{S}_2} f(\mathbf{x}) g(\mathbf{x}) \, d\omega(\mathbf{x}),$$

which induces the norm $\|f\|_{\mathrm{L}_2} := (f, f)_{\mathrm{L}_2}^{1/2}$. Then for arbitrary $p \in \mathbb{P}_L$, there is a unique vector $\boldsymbol{\alpha} = (\alpha_{\ell,k}) \in \mathbb{R}^{(L+1)^2}$ such that

$$p(\mathbf{x}) = \sum_{\ell=0}^{L} \sum_{k=1}^{2\ell+1} \alpha_{\ell,k} Y_{\ell,k}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^2.$$

Given a continuous function f, let $\mathbf{f} := \mathbf{f}(\mathrm{X}_N)$ be the column vector

$$\mathbf{f} = [f(\mathbf{x}_1), \ldots, f(\mathbf{x}_N)]^T \in \mathbb{R}^N.$$

Let $\mathbf{Y}_L := \mathbf{Y}_L(\mathrm{X}_N) \in \mathbb{R}^{(L+1)^2 \times N}$ be a matrix of spherical harmonics evaluated at the points of X_N with elements

$$Y_{\ell,k}(\mathbf{x}_j), \quad \ell = 0, \ldots, L, k = 1, \ldots, 2\ell+1; \; j = 1, \ldots, N.$$

For regularization operator, we consider the following two cases:

1) The first regularization operator R_L is defined in its most general rotationally invariant form by its action on $p \in \mathrm{P}_L$,

$$\mathrm{R}_L p(\mathbf{x}) = \sum_{\ell=0}^{L} \beta_\ell \sum_{k=1}^{2\ell+1} Y_{\ell,k}(\mathbf{x})(Y_{\ell,k}, p)_{\mathrm{L}_2} = \sum_{\ell=0}^{L} \beta_\ell \int_{\mathrm{S}^2} \frac{2\ell+1}{4\pi} P_\ell(\mathbf{x}\cdot\mathbf{y}) p(\mathbf{y})\, d\omega(\mathbf{y}),$$

where $\mathbf{x}\cdot\mathbf{y}$ denotes the euclidean inner product and P_ℓ denotes the orthogonal Legendre polynomials of degree ℓ which satisfies $P_\ell(1)=1$. In the last step we used the addition theorem for spherical harmonics[2]:

$$\sum_{k=1}^{2\ell+1} Y_{\ell,k}(\mathbf{x}) Y_{\ell,k}(\mathbf{y}) = \frac{2\ell+1}{4\pi} P_\ell(\mathbf{x}\cdot\mathbf{y}) \quad \forall\, \mathbf{x},\mathbf{y} \in \mathrm{S}^2. \quad (2)$$

So, we have

$$\mathrm{R}_L p(\mathbf{x}) = (\mathbf{B}_L \mathbf{Y}_L)^{\mathrm{T}} \boldsymbol{\alpha} = \mathbf{R}_L^T \boldsymbol{\alpha}.$$

where $\mathbf{B}_L \in \mathbb{R}^{(L+1)^2 \times (L+1)^2}$ is a positive semidefinite diagonal matrix defined by

$$\mathbf{B}_L := \mathrm{diag}\Big(\beta_0, \underbrace{\beta_1, \beta_1, \beta_1}_{3}, \ldots, \underbrace{\beta_L, \ldots, \beta_L}_{2L+1}\Big).$$

The problem (1) can be reformulated as the following least squares problem:

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^{(L+1)^2}} \left\|\mathbf{Y}_L^T \boldsymbol{\alpha} - \mathbf{f}\right\|_2^2 + \lambda \left\|\mathbf{R}_L^T \boldsymbol{\alpha}\right\|_1, \quad \lambda > 0. \quad (3)$$

2) We consider a special case of (3): the regularization operator acts directly on the coefficients $\boldsymbol{\alpha}$, i.e. $\mathbf{R}_L^T \boldsymbol{\alpha} = \mathbf{B}_L^T \boldsymbol{\alpha}$.

The problem (1) can be reformulated as the following least squares problem:

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^{(L+1)^2}} \left\|\mathbf{Y}_L^T \boldsymbol{\alpha} - \mathbf{f}\right\|_2^2 + \lambda \left\|\mathbf{B}_L^T \boldsymbol{\alpha}\right\|_1, \quad \lambda > 0. \quad (4)$$

In the sequel, we introduce Alternating Direction Method of Multipliers to solve problem (3). The discussion of the choice of regularization operator is given in Section III. Section IV considers point set on sphere, testing functions, and numerical experiments on Wendland function and Cone function on the unit sphere.

II. ADMM METHOD

Now, we illustrate Alternating Direction Method of Multipliers(ADMM)[3] to solve l_1-regularization optimization problem (3). First, let $\boldsymbol{\zeta} = \mathbf{R}_L^T \boldsymbol{\alpha}$. (3) can be transformed into the following constraint optimization problem

$$\begin{aligned} \min_{\boldsymbol{\alpha} \in \mathbb{R}^{(L+1)^2}} \quad & \left\|\mathbf{Y}_L^T \boldsymbol{\alpha} - \mathbf{f}\right\|_2^2 + \lambda \left\|\boldsymbol{\zeta}\right\|_1 \\ \text{s.t.} \quad & \mathbf{R}_L^T \boldsymbol{\alpha} - \boldsymbol{\zeta} = \mathbf{0}. \end{aligned} \quad (5)$$

Then we form the augmented Lagrangian of (5)

$$L_\rho(\boldsymbol{\alpha}, \boldsymbol{\zeta}, \mathbf{y}) = \left\|\mathbf{Y}_L^T \boldsymbol{\alpha} - \mathbf{f}\right\|_2^2 + \lambda \left\|\boldsymbol{\zeta}\right\|_1 + \mathbf{y}^T \left(\mathbf{R}_L^T \boldsymbol{\alpha} - \boldsymbol{\zeta}\right) + \rho \left\|\mathbf{R}_L^T \boldsymbol{\alpha} - \boldsymbol{\zeta}\right\|_2^2,$$

where $\rho > 0$ is the penalty parameter. So, ADMM consists of the iterations

$$\boldsymbol{\alpha}^{k+1} := \operatorname*{argmin}_{\boldsymbol{\alpha}} \left(\left\|\mathbf{Y}_L^T \boldsymbol{\alpha} - \mathbf{f}\right\|_2^2 + \frac{\rho}{2} \left\|\mathbf{R}_L^T \boldsymbol{\alpha} - \zeta^k + \mathbf{u}^k\right\|_2^2 \right), \quad (6)$$

$$\zeta^{k+1} := \operatorname*{argmin}_{\zeta} \left(\lambda \left\|\zeta\right\|_1 + \frac{\rho}{2} \left\|\mathbf{R}_L^T \boldsymbol{\alpha}^{k+1} - \zeta + \mathbf{u}^k\right\|_2^2 \right), \quad (7)$$

$$\mathbf{u}^{k+1} := \mathbf{u}^k + \mathbf{R}_L^T \boldsymbol{\alpha}^{k+1} - \zeta^{k+1}. \quad (8)$$

where $\mathbf{u} = \frac{1}{\rho}\mathbf{y}$. In each iteration, we need to solve two sub-problems (6)(7).

By the first order optimality conditions of (6), we obtain that its optimal solution $\boldsymbol{\alpha}$ satisfies

$$\left(2\mathbf{Y}_L \mathbf{Y}_L^T + \rho \mathbf{R}_L \mathbf{R}_L^T\right) \boldsymbol{\alpha} - \left(2\mathbf{Y}_L \mathbf{f} + \rho \mathbf{R}_L \left(\zeta^k - \mathbf{u}^k\right)\right) = \mathbf{0}.$$

By solving the system of linear equations, we have

$$\boldsymbol{\alpha}^{k+1} = \left(\mathbf{Y}_L \mathbf{Y}_L^T + \tfrac{\rho}{2} \mathbf{R}_L \mathbf{R}_L^T\right)^{-1} \left(\mathbf{Y}_L \mathbf{f} + \tfrac{\rho}{2} \mathbf{R}_L \left(\zeta^k - \mathbf{u}^k\right)\right). \quad (9)$$

For sub-problem (7), let $\mathbf{v}^k = \mathbf{R}_L^T \boldsymbol{\alpha}^{k+1} + \mathbf{u}^k$. Since (7) is separable, we have

$$\zeta_i^{k+1} = \operatorname*{argmin}_{\zeta_i} \left(\lambda |\zeta_i| + \tfrac{\rho}{2} (\mathbf{v}_i^k - \zeta_i)^2 \right),$$

where the first term $\lambda|\zeta_i|$ is not differentiable. By using the theory of subdifferential calculus[3], we can compute a closed-form solution, that is

$$\zeta_i^{k+1} := S_{\lambda/\rho}\left(\mathbf{v}_i^k\right),$$

where $S_k(a) = \max(0, a-k) + \min(0, a+k)$. is the soft thresholding operator:

In summary, using ADMM algorithm for solving (5) is equivalent to solving a system of linear equations and using the soft thresholding operator alternately. Therefore, the iterations can be reformulated as follows:

$$\begin{aligned} \boldsymbol{\alpha}^{k+1} &:= \left(\mathbf{Y}_L \mathbf{Y}_L^T + \tfrac{\rho}{2} \mathbf{R}_L \mathbf{R}_L^T\right)^{-1} \left(\mathbf{Y}_L \mathbf{f} + \tfrac{\rho}{2} \mathbf{R}_L \left(\zeta^k - \mathbf{u}^k\right)\right), \\ \zeta_i^{k+1} &:= S_{\lambda/\rho}(\mathbf{v}_i^k), \\ \mathbf{u}^{k+1} &:= \mathbf{u}^k + \mathbf{R}_L^T \boldsymbol{\alpha}^{k+1} - \zeta^{k+1}. \end{aligned}$$

(4) can be solved by ADMM similarly.

III. REGULARIZATION OPERATOR

The regularization operator R_L is determined by the choice of the diagonal matrix $\mathbf{B}_L$ with diagonal elements β_ℓ. In the following, we present some interesting examples.

*1) **Filtered Regularization Operator:*** The diagonal element of the corresponding diagonal matrix $\mathbf{B}_L$ of this operator is defined as follows:

$$\beta_\ell = \sqrt{\frac{1}{h(\ell/L)} - 1}, \quad \ell = 0,\ldots,L-1, \tag{10}$$

where $h(x)$ is filter function. In this paper, we consider the following two C^∞ exponential filter function[4]:

- $h_1(x) = \begin{cases} 1, & x \in [0,1/2], \\ \exp\left(\frac{\exp(-2/(2x-1))}{x-1}\right), & x \in (1/2,1), \\ 0, & x \in [1,\infty). \end{cases}$
- $h_2(x) = \begin{cases} 1, & x \in [0,1/2], \\ 1-\exp\left(\frac{2\exp(1/(x-1))}{1-2x}\right), & x \in (1/2,1), \\ 0, & x \in [1,\infty). \end{cases}$

In (10), we have excluded $\ell = L$ because if $\ell = L$ were allowed we would have $\beta_L = \infty$ and hence $\alpha_{L,k} = 0$.

*2) **Differential operator:*** The Laplace–Beltrami operator Δ^* [2] is

$$\Delta^* := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}.$$

The spherical harmonics have an intrinsic characterization as the eigenfunctions of the Laplace-Beltrami operator Δ^*, that is,

$$\Delta^* Y_{\ell,k}(\mathbf{x}) = -\ell(\ell+1)\ Y_{\ell,k}(\mathbf{x}).$$

It follows that $-\Delta^*$ is a semipositive operator[2], and for any $s > 0$ we may define $(-\Delta^*)^s$ by

$$(-\Delta^*)^s Y_{\ell,k}(\mathbf{x}) = [\ell(\ell+1)]^s Y_{\ell,k}(\mathbf{x}).$$

We used $(-\Delta^*)^s$ as a regularization operator. The corresponding matrix $\mathbf{B}_L$ is then

$$\mathbf{B}_L = \mathrm{diag}\left(0^s, \underbrace{2^s, 2^s, 2^s}_{3}, \ldots, \underbrace{[L(L+1)]^s, \ldots, [L(L+1)]^s}_{2L+1}\right).$$

This operator can recover the function with noise[5].

IV. NUMERICAL EXPERIMENT

In this section we investigate spherical l_1-regularization least squares approximation model to approximate some test functions over the sphere.

In this paper, we choose the spherical t-design with properly degree t as the point set X_N, which definition is as follows:

Definition 1[6] A point set $X_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset S^2$ is a spherical t-design, if it satisfies

$$\frac{1}{N}\sum_{j=1}^{N} p(\mathbf{x}_j) = \frac{1}{4\pi}\int_{S^2} p(\mathbf{x})d\omega(\mathbf{x}) \quad \forall p \in P_t, \tag{11}$$

where $d\omega(\mathbf{x})$ denotes area measure on the unit sphere. That is, X_N is a spherical t-design if a properly scaled equal-weight quadrature rule with nodes at the points of X_N integrates all (spherical) polynomial up to degree t exactly.

In the following experiments, we assume X_N is well condition spherical t-design with $t \ge 2L$ and $N = (t+1)^2$.

We use the following two test functions. The first one is Wendland function[7]:

$$f_1(\mathbf{x}) = \sum_{i=1}^{6} \phi_k\left(|\mathbf{z}_i - \mathbf{x}|\right)$$

where $\phi_k(r) = \tilde{\phi}_k\left(\frac{r}{\delta_k}\right)$ is normalized Wendland function, $\delta_k = \frac{(3k+3)\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}$, $k \ge 0$ and $\mathbf{z}_1 = (1, 0, 0)^T$, $\mathbf{z}_2 = (-1, 0, 0)^T$, $\mathbf{z}_3 = (0, 1, 0)^T$, $\mathbf{z}_4 = (0, -1, 0)^T$, $\mathbf{z}_5 = (0, 0, 1)^T$, $\mathbf{z}_6 = (0, 0, -1)^T$. In the following experiments, we set $k = 2$ and the corresponding original Wendland function is $\tilde{\phi}_k(r) = (1-r)_+^6(35r^2 + 18r + 3)/3$ where $(r)_+ = \max\{r, 0\}$. The second function is the cone function f_2 [8]:

$$f_2 = f_{\text{cone}} = \begin{cases} 2\left(1 - \frac{\arccos(\mathbf{x}_c \cdot \mathbf{x})}{r}\right), & \mathbf{x} \in C(\mathbf{x}_c, r) \\ 0, & \text{otherwise,} \end{cases}$$

where $C(\mathbf{x}_c, r) := \{\mathbf{x} \in S^2 \mid \arccos(\mathbf{x}, \mathbf{x}_c) \le r\}$ is a spherical cap with center $\mathbf{x}_c$ and radius r. This function is continuous on S^2 but not differentiable on the boundary of the spherical cap $C(\mathbf{x}_c, r)$ and the center $\mathbf{x}_c$. In our numerical experiments, we set $\mathbf{x}_c = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)^T$ and $r = \frac{1}{2}$.

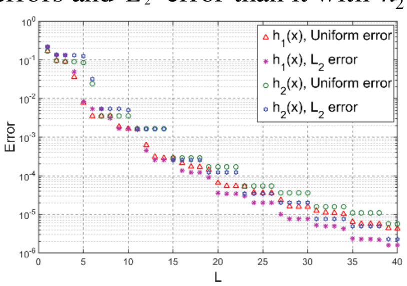
In order to measure the approximating quality, both uniform error and the L_2-error are used:

- The uniform error of approximation is estimated by
$$\|f-p_L\|_{C(\mathbb{S}^2)} = \sup_{x\in\mathbb{S}^2}|f(x)-p_L(x)| \approx \max_{x_i\in X_N}|f(x_i)-p_L(x_i)|,$$
where X_N is a finite but large set of well distributed points over the sphere. In the following experiment, we choose X_N to be an equal area partitioning point set with $N=50000$ points[9].
- The L_2-norm of approximation error is estimated by
$$\|f-p_L\|_{L_2} := \left(\int_{\mathbb{S}^2}|f(\mathbf{x})-p_L(\mathbf{x})|^2\, d\omega(\mathbf{x})\right)^{1/2} \approx \left(\frac{4\pi}{m}\sum_{j=1}^{m}|f(\mathbf{x}_j)-p_L(\mathbf{x}_j)|^2\right)^{1/2},$$
The set $\{\mathbf{x}_1,\ldots,\mathbf{x}_m\}$ is the nodes of the spherical 160-design with $m=25921$.

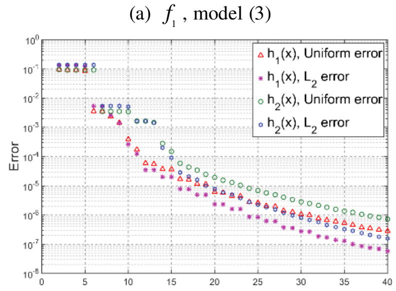
A. *Filtered regularization operator for exact data*

In this subsection, we report numerical results to compare different filter function. For a given L, we consider $t=2L$ and set $N=(t+1)^2$. We use both filter function $h_1(x)$ and $h_2(x)$ with β_ℓ given by (10) and $\lambda=1$

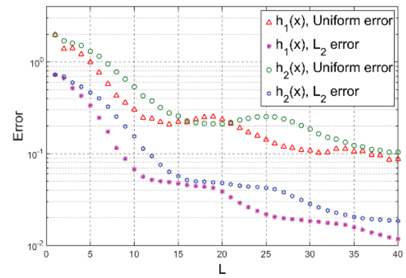
Fig. 1 reports the uniform error and L_2-error of approximations for the functions f_1 and f_2 with $L=1,\ldots,40$. Fig. 1 shows that model with filtered regularization operator with $h_1(x)$ has smaller uniform errors and L_2-error than it with $h_2(x)$.



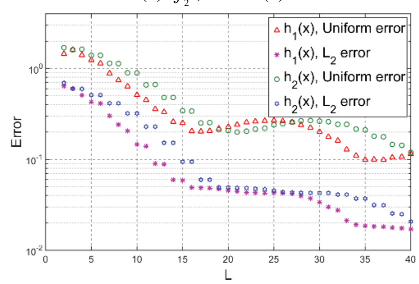
(a) f_1, model (3)



(b) f_1, model (4)



(c) f_2, model (3)



(d) f_2, model (4)

Fig. 1 Errors of model (3) and (4) with filtered regularization operator

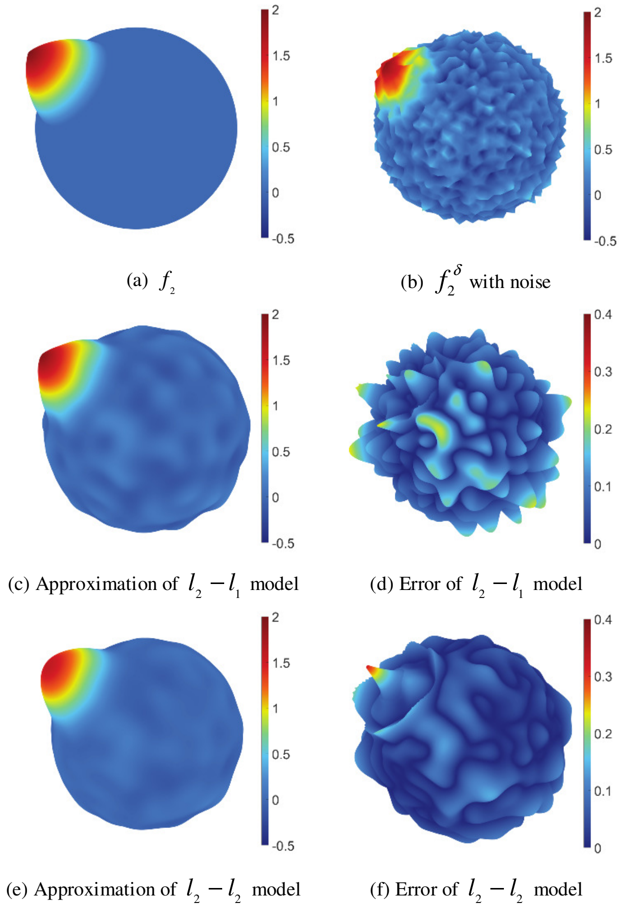
B. *Laplace–Beltrami regularization operator for contaminated data*

In this subsection, we report numerical results for reconstructing the nonsmooth function f_2 when the data has been contaminated with noise. We use model (4) with differential operator ($s=2$) and different values of λ.

Fig. 2(a) illustrates the function f_2 while Fig. 2(b) shows the contaminated function $f_2^\delta(\mathbf{x}) = f_2+\delta(\mathbf{x})$, where for each $\mathbf{x}$, $\delta(\mathbf{x})$ is a sample of a normal random variable with mean $\mu=0$ and standard deviation $\sigma=0.2$. In this experiment, the choice of the regularization parameter λ is critical. We set the λ that achieve the minimal uniform error as the optimal λ.

The well condition spherical 50-design with $N=2601$ is used to recover the data with noise. As a comparison, we choose the least squares model with l_2-regularization term[1]:
$$\min_{p\in\mathbb{P}_L}\left\{\sum_{j=1}^{N}\left(p(\mathbf{x}_j)-f(\mathbf{x}_j)\right)^2+\lambda\sum_{j=1}^{N}\left(\mathrm{R}_L p(\mathbf{x}_j)\right)^2\right\},\ \lambda>0.$$
Fig. 2(c)-(f) show the approximation and error for two models. From the subplots (c) and (e), it can be seen that both two model can recover the image of f_2 well. From Fig. 2(e)(f), restoration by l_2-l_1

model is not as smooth as restoration by $l_2 - l_2$ model. But, $l_2 - l_1$ model recovers the non-smooth edges of the spherical cap more accurately than $l_2 - l_2$ model. At last, Fig. 3 reports the uniform and L_2- errors for recovering the function f_2 from contaminated data.



(a) f_2 (b) f_2^{δ} with noise

(c) Approximation of $l_2 - l_1$ model (d) Error of $l_2 - l_1$ model

(e) Approximation of $l_2 - l_2$ model (f) Error of $l_2 - l_2$ model

Fig. 2 Differential operator to recover f_2 from contaminated data.

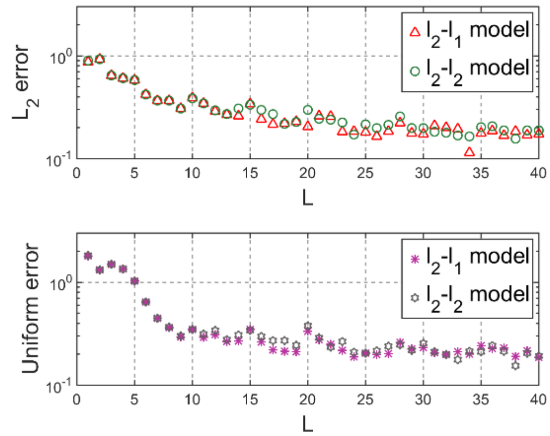


Fig. 3 Errors of model (4) with differential operator

V. CONCLUSIONS

In this paper, we study the l_1- regularization optimization problem over the unit sphere. Based on variant regularization operators, we set up a class of spherical regularization least squares approximation model. We illustrate the algorithm, includes ADMM, to solve this approximation problem by using well conditioned spherical t-design as sampling point set. Finally, numerical experiments demonstrate the theoretical results can provide satisfactory approximation on the sphere, with or without errors on data. The results show that this model can approximate the smooth and non-smooth spherical functions well, especially at the non-smooth edge.

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