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Multiplicity solutions for a class of quasilinear critical problems in $\mathbb{R}^N$involving sign-changing weight function

Multiplicidade de soluções para uma classe de problemas quaselineares criticos em $\mathbb{R}^N$ envolvendo função peso com mudança de sinal

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Abstract

In this paper, existence and multiplicity results to the following quasilinear critical problem

$$\begin{cases} -\Delta_p u = \lambda f(x)|u|^{q-1} + |u|^{p^*-1}, & \textit{in} \quad \mathbb{R}^N \\ 0 \lneq u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \end{cases}$$

are established, where $\lambda > 0$, $1 < q < p$, *with* $2 \leq p < N$, $p^* = \frac{Np}{N-p}$ *and the weight function* f, *among other conditions, can possibly change sign in* $\mathbb{R}^N$. *The study is based on comparison of Palais-Smale critical levels in Nehari manifold.*

Keywords: *Quasilinear elliptic equations,unbounded domains, multiple solutions, critical Sobolev exponent, sign-changing weight function.*

Abstract

Neste artigo, resultados de existência e multiplicidade de soluções para o seguinte problema quaselinear crítico

$$\begin{cases} -\Delta_p u = \lambda f(x)|u|^{q-1} + |u|^{p^*-1}, & \textit{em} \quad \mathbb{R}^N \\ 0 \lneq u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \end{cases}$$

serão estabelecidos, onde $\lambda > 0$, $1 < q < p$, *com* $2 \leq p < N$, $p^* = \frac{Np}{N-p}$ *e a função peso* f, *além de outras condições, pode possivelmente mudar de sinal em* $\mathbb{R}^N$. *O estudo é baseado na comparação dos níveis críticos de Palais-Smale na variedade de Nehari.*

Palavras-chave: *Equações elíticas quaselineares, domínios ilimitados, multiplicidade de soluções, expoente crítico de Sobolev, função peso com mudança de sinal.*

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1 Introduction

In this paper, we are concerned with the existence and multiplicity of solutions for the following quasilinear critical problem:

$$\begin{cases} -\Delta_p u &= \lambda f(x)|u|^{q-1} + |u|^{p^*-1}, \quad \text{in} \quad \mathbb{R}^N \\ 0 &\lneqq u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \end{cases} \tag{1}$$

where $\lambda > 0$, $1 < q < p$ with $2 \leq p < N$, $p^* = \frac{Np}{N-p}$ and the weight function f satisfies the following conditions:

(H) $f \doteq f_+ + f_-$ ($f_+ = \max\{f,0\}$, $f_- = \min\{f,0\}$) is a measurable function, locally bounded on $\mathbb{R}^N \backslash \{0\}$, with $0 \not\equiv f_+ \in C(\mathbb{R}^N \backslash \{0\})$ and

$$f(x) = \begin{cases} o(|x|^b), & \text{as} \quad |x| \to 0 \\ o(|x|^a), & \text{as} \quad |x| \to \infty, \end{cases}$$

for any a, b verifying

$$a < \frac{N}{p^*}(q - p^*) < b.$$

Similar assumption was already used by Egnell (1988), Noussair et al. (1993) and by Szulkin e Willem (1998). The necessity of such growth conditions for a class of quasilinear elliptic problems was established in order to get a compactness condition. On the other hand, elliptic problems involving critical Sobolev exponents were studied, first, by Brezis e Nirenberg (1983).

Existence and multiplicity of solutions for quasilinear elliptic equations with nonlinearities concave convex in bounded domains are widely studied. For example, the problem

$$\begin{cases} -\Delta_p u &= \lambda |u|^{q-1} + |u|^{s-1}, & \text{in} \quad \Omega \\ u &> 0, & \text{in} \quad \Omega \\ u &= 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{$E_{\lambda,p}$}$$

where $1 < q < p < s \leq p^*$ has been studied, for instance, by Ambrosetti et al. (1994). In that paper the authors have proved the existence of $\lambda_0 > 0$ such that problem $(E_{\lambda,2})$, this is, problem $(E_{\lambda,p})$ with $p = 2$, admits at least two positive solutions for $\lambda \in (0,\lambda_0)$, has one positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. Garcia Azorero e Peral Alonso (1994) studied $(E_{\lambda,p})$ considering $\frac{2N}{N+2} < p < N$ and $s = p^*$. They proved, in case $\frac{2N}{N+2} < p < 3$ and $1 < q < p$, or $p \geq 3$ and $p^* - \frac{2}{p-1} < q < p$, the existence of $\lambda_0 > 0$ such that problem $(E_{\lambda,p})$ admits at least two positive solutions for $\lambda \in (0,\lambda_0)$. Huang (1998) extended, only for $2 \leq p < N$, the results of Garcia Azorero e Peral Alonso (1994) in the sense that its results are valid for any $1 < q < p$. Wu (2008), considered problem (1) with $p = 2$ in a bounded domain $\Omega \subset \mathbb{R}^N$, under the assumption that $f \in C(\overline{\Omega})$, with $f_+ \not\equiv 0$. He obtained, using variational methods on the Nehari manifold, the existence of $\lambda_0 > 0$ such that problem (1), with $p = 2$, admits at least two positive solutions for $\lambda \in (0,\lambda_0)$. For more general results in bounded domains see e.g. the papers by Ambrosetti et al. (1996); Birindelli e Demengel (2004); Pacella et al. (1997) de Figueiredo et al. (2006); Silva e Xavier (2003); Azorero et al. (2000) and their references.

In whole space, Ambrosetti et al. (2000), among other results, have also proved via variational methods, the existence of a positive constant λ_0 such that problem (1), with $p = 2$, admits at least two non-negative solutions for $\lambda \in (0,\lambda_0)$, under the hypothesis that the function $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $f_+ \not\equiv 0$. Silva e Soares (2001), under assumption that $f \in C(\mathbb{R}^N)$ and $f_+ \in L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)$ with $f_+ \not\equiv 0$, have established, beyond others results, the existence of positive constant λ_0 such that problem (1) admits at least a nontrivial solution for $\lambda \in (0,\lambda_0)$, for $1 < p^2 < N$ and such that $\max\{p^* - \frac{p}{p-1}\} < q \leq p$. Alves (1997) has considered problem (1) under the assumption that f is a non-negative function with $f \in L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)$. Actually, using the Ekeland variational principle and the mountain pass Theorem, he showed the existence of $\lambda_0 > 0$ such that problem (1) admits at least two solutions for all $\lambda \in (0,\lambda_0)$. For more general related results for unbounded domains we would like to mention the papers Gonçalves e Miyagaki (1998); Cerami et al. (2007); Miyagaki (2005); Drábek e Huang (1997) and their references.

Our result is the following:

Theorem 1.1. *Suppose that f is a measurable function in $\mathbb{R}^N$ satisfying (H). Then there exists a positive constant $\Lambda = \Lambda(q,p,f,N)$ such that for all $\lambda \in (0, \Lambda)$ problem (1) has at least two nontrivial solutions.*

Remark 1.1. *Our result still holds replacing the hypothesis (H) by a following more general behavior on f:*

(H') *Suppose that f is a measurable function in $\mathbb{R}^N$ satisfies: $f \doteq f_+ + f_-$ is locally bounded on $\mathbb{R}^N \backslash \overline{E}$, with $0 \not\equiv f_+ \in C(\mathbb{R}^N \backslash \overline{E})$, where $E = \{y_n\} \subset \mathbb{R}^N$ and*

$$f(x) = \begin{cases} o(|x|^{b_n}), & as \quad |x| \to y_n \\ o(|x|^a), & as \quad |x| \to \infty, \end{cases}$$

for any a, b_n verifying

$$a < \frac{N}{2^*}(q - 2^*) < b_n, \; n = 1, 2 \ldots$$

Then, the same conclusion as in Theorem 1.1 holds.

The aim of the our work is to extend the results mentioned above for whole space and/or to a class of elliptic problems involving weights functions that can possibly change sign. For example, the function $f(x) = -|x|^{-p}\chi_A(|x|) + |x|^{-N}\chi_B(|x|)$, where $A = (0,1)$ and $B = (2,\infty)$ satisfies the hypothesis (H), but neither $f \geq 0$ nor $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ or $f \in C(\mathbb{R}^N)$, conditions these had been assumed in Alves (1997), Ambrosetti et al. (2000) and Wu (2008) respectively. The author in Miotto (2010) treated problem (1) with $p = 2$, combining techniques used by Tarantello (1992) and Ambrosetti et al. (2000) (see also Brown e Zhang (2003) and Wu (2008)). For $p > 2$, the same arguments used above in Miotto (2010) does not work any longer in direct way, mainly in the proof of the estimates of the critical level, due to lack of regularity of solutions. In order to get such estimative, we will show that the solutions of problem (1) belong in $C^{1,\gamma}_{loc}(\mathbb{R}^N\backslash\{0\})$. Then, in this situation, adapting same arguments used in Jianfu (1995) (see also Huang (1998)) we were able to overcome these difficulties.

This paper is organized as follows. In Section 2, we give some notation and technical results. In Section 3, we establish the existence of nontrivial solution of (1), as well as its regularity. In Section 4, we prove some estimates concerning the energy levels in order to ensure the existence of a second solution for problem (1).

2 Preliminary Results

We will consider $\mathcal{D} = \mathcal{D}^{1,p}(\mathbb{R}^N)$, the closure of C_0^∞ with respect to the norm given by

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

Our notation for the norm in Lebesgue space $L^r(\mathbb{R}^N)$ is

$$\|u\|_{L^r}^r = \int_{\mathbb{R}^N} |u|^r dx,\ 1 \leq r < \infty.$$

We also put,

$$S = \inf\left\{\frac{\|u\|^p}{\|u\|_{L^{p^*}}^p} : u \in \mathcal{D}\backslash\{0\}\right\}.$$

Since the proof of our result is based on variational methods, we consider the functional associated to (1) namely,

$$I_\lambda(u) = \frac{1}{p}\|u\|^p - \frac{\lambda}{q}\int_{\mathbb{R}^N} f|u|^q dx - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} dx,$$

for all $u \in \mathcal{D}$. It follows from (H) that $I_\lambda \in C^1(\mathcal{D},\mathbb{R})$ with Gateaux derivative $I'_\lambda(u)$ at each $u \in \mathcal{D}$ given by

$$\begin{aligned} < I'_\lambda(u),\varphi >= & \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi\, dx \\ & -\lambda\int_{\mathbb{R}^N} f|u|^{q-2}u\varphi dx - \int_{\mathbb{R}^N} |u|^{p^*-2}u\varphi dx, \end{aligned}$$

for all $\varphi \in \mathcal{D}$. Therefore, the critical points of I_λ are precisely the (weak) solutions of (1). Still from hypothesis (H), there exists $C_f > 0$ such that

$$\left|\int_{\mathbb{R}^N} f|u|^q dx\right| \leq C_f\|u\|^q, \tag{2}$$

for all $u \in \mathcal{D}$.

Now we will cite some relevant facts for the study of problem (1).

Remark 2.1. *i) Let $c \in \mathbb{R}$. A sequence $(u_n) \subset \mathcal{D}$ such that $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$ is called $(PS)_c$−sequence for I_λ. A number $c \in \mathbb{R}$ is a (PS)−value for I_λ if there exists a (PS)−sequence for I_λ.*

ii) Any $(PS)_c$−sequence for I_λ, (u_n), is bounded.

Indeed,

$$\begin{aligned} \frac{p}{N-p}\|u_n\|^p = & \ p^* I_\lambda(u_n) - \langle I'_\lambda(u_n),u_n\rangle \\ & -\lambda\tfrac{(p^*-q)}{q}\int_{\mathbb{R}^N} f|u_n|^q dx, \end{aligned}$$

then, we have for (2)

$$\frac{p}{N-p}\|u_n\|^p \leq C(1 + \|u_n\| + \|u_n\|^q),$$

for some $C > 0$, showing that (u_n) is bounded.

iii) If (u_n) is a $(PS)_c$−sequence for I_λ then $(|u_n|)$ is also $(PS)_c$−sequence for I_λ.

Indeed, we have that $< I'_\lambda(|u_n|),\varphi >=< I'_\lambda(u_n),\varphi >$, for all $\varphi \in \mathcal{D}$ and $I_\lambda(|u_n|) = I_\lambda(u_n) \to c$.

iv) In what follows, we will assume, by eventually passing to a subsequences if necessary, that the $(PS)_c$−sequence for I_λ, (u_n) satisfies the following conditions

$$u_n \geq 0 \ \text{a.e in } \mathbb{R}^N, \quad u_n \rightharpoonup u \ \text{weakly in } \mathcal{D},$$

$$u_n \to u \ \text{a.e in } \mathbb{R}^N, \quad u \geq 0 \ \text{a.e in } \mathbb{R}^N.$$

Since the functional I_λ is not bounded from bellow in $\mathcal{D}$, for $\lambda > 0$, we consider the Nehari manifold

$$M_\lambda = \{u \in \mathcal{D}\backslash\{0\} :< I'_\lambda(u),u >= 0\},$$

and

$$\alpha_\lambda = \inf_{u\in M_\lambda}\{I_\lambda(u)\}.$$

We recall that any nonzero solution of problem (1) belongs to M_λ. Moreover, by definition, we have that $u \in M_\lambda$ if, and only if,

$$\|u\| \neq 0 \text{ and } \|u\|^p = \lambda\int_{\mathbb{R}^N} f|u|^q dx + \int_{\mathbb{R}^N} |u|^{p^*} dx. \tag{3}$$

We will show that $-\infty < \alpha_\lambda$. In fact, let $u \in M_\lambda$ be arbitrary, then by (2) and (3) we get

$$I_\lambda(u) \geq \frac{1}{N}\|u\|^p - \left(\frac{1}{q} - \frac{1}{p^*}\right)\lambda C_f \|u\|^q.$$

Since $q < p$, it follows that I_λ is bounded from below on M_λ.

For $\lambda > 0$, define the functional $\psi_\lambda : \mathcal{D} \to \mathbb{R}$, by $\psi_\lambda(u) =< I'_\lambda(u),u >$, this is,

$$\psi_\lambda(u) = \|u\|^p - \lambda \int_{\mathbb{R}^N} f|u|^q dx - \int_{\mathbb{R}^N} |u|^{p^*} dx.$$

It is easily seen that ψ_λ is of class C^1 with

$$< \psi'_\lambda(u),u >= p\|u\|^p - q\lambda \int_{\mathbb{R}^N} f|u|^q dx - p^* \int_{\mathbb{R}^N} |u|^{p^*} dx.$$

Furthermore, if $u \in M_\lambda$, then by (3) we have that

$$< \psi'_\lambda(u),u > =(p-q)\|u\|^p - (p^*-q)\int_{\mathbb{R}^N}|u|^{p^*}dx \quad (4)$$
$$=(p-p^*)\|u\|^p - (q-p^*)\lambda\int_{\mathbb{R}^N} f|u|^q dx \quad (5)$$

As in Tarantello (1992), we divide M_λ in three sets;

$$M_\lambda^+ = \{u \in M_\lambda :< \psi'_\lambda(u),u >> 0\},$$
$$M_\lambda^0 = \{u \in M_\lambda :< \psi'_\lambda(u),u >= 0\},$$
$$M_\lambda^- = \{u \in M_\lambda :< \psi'_\lambda(u),u >< 0\}.$$

The following result shows that minimizers on M_λ are the "usual" critical points for I_λ.

Lemma 2.1. *Suppose that u_0 is a local minimizer for I_λ on M_λ and $u_0 \notin M_\lambda^0$, then $I'_\lambda(u_0) = 0$ in $(\mathcal{D})^*$.*

Proof. 1. *We have that there exists a neighborhood U of u_0 in $\mathcal{D}$, where*

$$I_\lambda(u_0) = \min_{u \in U \cap M_\lambda} I_\lambda(u) = \min_{\substack{u \in U\setminus\{0\} \\ \psi_\lambda(u)=0}} I_\lambda(u).$$

Furthermore, by the Lagrange multipliers Theorem, there exists $\rho \in \mathbb{R}$, such that $I'_\lambda(u_0) = \rho\psi'_\lambda(u_0)$. Then, since $u_0 \in M_\lambda$ we get that

$$0 =< I'_\lambda(u_0),u_0 >= \rho < \psi'_\lambda(u_0),u_0 > .$$

Now $u_0 \notin M_\lambda^0$, then $\rho = 0$ and consequently $I'_\lambda(u_0) = 0$ in $\mathcal{D}^*$. □

Motivated by the above result, we will get conditions for $M_\lambda^0 = \emptyset$.

Lemma 2.2. *There exists $\Lambda = \Lambda(q,p,f,N) > 0$, such that $M_\lambda^0 = \emptyset$ if*

$$0 < \lambda < \Lambda.$$

Proof. 2. *Suppose by absurd that $M_\lambda^0 \neq \emptyset$ for any $\lambda > 0$ small. Let $u \in M_\lambda^0$ be arbitrary, then we have by relations (3), (4) and (5) that*

$$0 < \|u\|^p = \frac{p^*-q}{p-q}\int_{\mathbb{R}^N} |u|^{p^*} dx, \quad (6)$$

$$0 < \lambda \int_{\mathbb{R}^N} f|u|^q dx = \frac{p^*-p}{p-q}\int_{\mathbb{R}^N} |u|^{p^*} dx. \quad (7)$$

Thus, for any $u \in M_\lambda^0$, by relations (2), (6) and (7) we get

$$\|u\| \leq \left(\frac{p^*-q}{p^*-p}\right)^{\frac{1}{p-q}} (\lambda C_f)^{\frac{1}{p-q}}. \quad (8)$$

For $\lambda > 0$, define $F_\lambda : \mathcal{D}\setminus\{0\} \to \mathbb{R}$, by

$$F_\lambda(u) = k\left(\frac{\|u\|^{p^*}}{\int_{\mathbb{R}^N} |u|^{p^*} dx}\right)^{\frac{N}{p}-1} - \lambda \int_{\mathbb{R}^N} f|u|^q dx,$$

with $k = k(q,p,N) = \left(\frac{p^*-p}{p-q}\right)\left(\frac{p-q}{p^*-q}\right)^{\frac{N}{p}}$. Notice that F_λ is defined on M_λ^0, for all $\lambda > 0$. Also, by (6) and (7) follow that

$$F_\lambda(u) = 0, \quad (9)$$

for all $u \in M_\lambda^0$. Now, if $u \in M_\lambda^0$, by relations (2), (8) and the definition of S, we infer that

$$F_\lambda(u) \geq kS^{\frac{N}{p}} - \lambda C_f \|u\|^q$$
$$\geq \|u\|^q C_f \lambda^{\frac{-q}{p-q}}\left(C_1 - \lambda^{\frac{p}{p-q}}\right), \quad (10)$$

where $C_1 = kS^{\frac{N}{p}} C_f^{-\frac{p}{p-q}} \left(\frac{p^*-p}{p^*-q}\right)^{\frac{q}{p-q}}$.

For the future estimates we need some constants, namely $C_2 = C_f^{-1}kS^{\frac{Np-Nq}{p^2}}$, $C_3 = \frac{N-p}{2N}C_f^{-1}\left(S^{\frac{p^*}{p}}\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}}$, $C_4 = kS^{\frac{N}{p}}C_f^{-1}(NC_f)^{-\frac{q}{p-q}}$, $C_5 = \frac{C_f^{-1}}{4^{p-q}}\frac{p^*-p}{p^*-q}\left(S^{\frac{p^*}{p}}\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}}$. Defining

$$0 < \Lambda = \Lambda(q,p,f,N) < \min\{C_1^{\frac{p-q}{p}}, C_2, C_3, C_4^{\frac{p-q}{p}}, C_5\}, \quad (11)$$

we obtain, if $0 < \lambda < \Lambda$, a contradiction between the relations (9) and (10), which concludes the proof of lemma. □

It follows by Lemma 2.1 that if $\lambda \in (0,\Lambda)$ and $u_0 \in M_\lambda$ is local minimum of I_λ on M_λ, then u_0 is a solution of (1). Thus, our purpose is to find local minimum of functional I_λ on M_λ. For this, for each λ we define

$$\alpha_\lambda^+ = \inf_{u \in M_\lambda^+}\{I_\lambda(u)\} \quad \text{and} \quad \alpha_\lambda^- = \inf_{u \in M_\lambda^-}\{I_\lambda(u)\}.$$

Also, for each $u \in \mathcal{D}\backslash\{0\}$ consider

$$0 < t_{max} = t_{max}(u) = \left[\left(\frac{p-q}{p^*-q}\right)\frac{\|u\|^p}{\int_{\mathbb{R}^N}|u|^{p^*}dx}\right]^{\frac{1}{p^*-p}}.$$

The following results establish some properties on the sets $M_\lambda^{\pm}$, as well as, the values $\alpha_\lambda^{\pm}$.

Lemma 2.3. *For any $\lambda \in (0,\Lambda)$ and $u \in \mathcal{D}\backslash\{0\}$, we have:*

i) if $\lambda\int_{\mathbb{R}^N} f|u|^q dx \leq 0$, there exists a unique positive constant $t^-(u) = t^- > t_{max}$, such that $t^-u \in M_\lambda^-$ and

$$I_\lambda(t^-u) = \sup_{t\geq 0} I_\lambda(tu) > 0.$$

ii) if $\lambda\int_{\mathbb{R}^N} f|u|^q dx > 0$, then there exists unique positive constants $t^+(u) = t^+ < t_{max} < t^- = t^-(u)$, such that $t^+u \in M_\lambda^+$, $t^-u \in M_\lambda^-$ and also

$$I_\lambda(t^+u) = \inf_{0\leq t\leq t_{max}} I_\lambda(tu) < 0,$$

$$I_\lambda(t^-u) = \sup_{t\geq t_{max}} I_\lambda(tu) > 0.$$

iii) $M_\lambda^- = \left\{u \in \mathcal{D}\backslash\{0\} : t^-(u) = \frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) = 1\right\}$.

iv) there exists a continuous bijection between sets $U = \{u \in \mathcal{D}\backslash\{0\} : \|u\| = 1\}$ and M_λ^-. In particular, t^- is a continuous function on $\mathcal{D}\backslash\{0\}$.

Furthermore, we must have $\alpha_\lambda = \alpha_\lambda^+ < 0 < \alpha_\lambda^-$.

Proof. 3. *Let λ and u satisfying the hypotheses. Consider $s : (0,\infty) \to \mathbb{R}$, where $s(t) = t^{p-q}\|u\|^p - t^{p^*-q}\int_{\mathbb{R}^N}|u|^{p^*}dx$. Recall that $s \in C^\infty$, $s'(t) > 0$ if $t \in (0,t_{max})$, $s'(t_{max}) = 0$ and $s'(t) < 0$ if $t \in (t_{max},\infty)$. Note that $tu \in M_\lambda$ if, and only if, $s(t) = \lambda\int_{\mathbb{R}^N} f|u|^q dx$. Consider also the function $m : (0,\infty) \to \mathbb{R}$, defined by $m(t) = I_\lambda(tu)$. Note that $m \in C^\infty$ and $m'(t) = 0$ if, and only if, $tu \in M_\lambda$.*

Suppose initially that $\lambda\int_{\mathbb{R}^N} f|u|^q dx \leq 0$. Since we have that $s(t_{max}) > 0$, $s(t) \to -\infty$ as $t \to \infty$ and s is a strictly decreasing function on (t_{max},∞), there exists unique value $t^- > t_{max}$, such that $s(t^-) = \lambda\int_{\mathbb{R}^N} f|u|^q dx$.

Since $< \psi_\lambda'(t^-u),t^-u >= (t^-)^{q+1}s'(t^-) < 0$, by definition we have that $t^-u \in M_\lambda^-$. Now, since $t = t^-$ is a global maximum of function m and by fact of $m(t^-) > 0$, we have that $I_\lambda(t^-u) = \sup_{t\geq 0} I_\lambda(tu) > 0$, then we conclude i).

Now we suppose that $\lambda\int_{\mathbb{R}^N} f|u|^q dx > 0$. Combining the definition of S with $k(q,p,N)$, we have

$$\begin{aligned} s(t_{max}) &\geq \left[S^{\frac{p^*}{p}}\left(\frac{p-q}{p^*-q}\right)\right]^{\frac{p-q}{p^*-p}}\|u\|^q\left[1-\left(\frac{p-q}{p^*-q}\right)\right] \\ &\geq k(q,p,N)S^{\frac{Np-Nq}{p^2}}\|u\|^q. \end{aligned}$$

From (2), (11) and by definition of C_2 follow that

$$\begin{aligned} s(t_{max}) &\geq C_f C_2\|u\|^q \geq C_f\Lambda\|u\|^q \\ &> \lambda C_f\|u\|^q \geq \lambda\int_{\mathbb{R}^N} f|u|^q dx. \end{aligned}$$

Thus, by the properties of function s, there are unique $t^+ < t_{max} < t^-$, where $s(t^\pm) = \lambda\int_{\mathbb{R}^N} f|u|^q dx$. Since $< \psi_\lambda'(t^\pm u),t^\pm u >= (t^\pm)^{q+1}s'(t^\pm)$, we get $t^\pm u \in M_\lambda^\pm$. From the fact $m'(t) < 0$ if $t \in (0,t^+)\cup(t^-,\infty)$ and $m'(t) > 0$ if $t \in (t^+,t^-)$ and also by $m(t^+) < 0 < m(t^-)$, it follows that

$$I_\lambda(t^+u) = \inf_{0\leq t\leq t_{max}} I_\lambda(tu) < 0,$$

and

$$I_\lambda(t^-u) = \sup_{t\geq t_{max}} I_\lambda(tu) > 0,$$

which concludes ii).

Let $u \in M_\lambda^-$ be arbitrary. Considering $w = \frac{u}{\|u\|}$, we get that there exists an unique positive value $t^-(w)$ such that $t^-(w)w \in M_\lambda^-$, this is, $t^-\left(\frac{u}{\|u\|}\right)\frac{u}{\|u\|} \in M_\lambda^-$. Thus $\frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) = t^-(u) = 1$, because $u \in M_\lambda^-$. Therefore

$$M_\lambda^- \subset \left\{u \in \mathcal{D}\backslash\{0\} : t^-(u) = \frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) = 1\right\}.$$

Conversely, if $u \in \mathcal{D}\backslash\{0\}$ is a function that satisfies $\frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) = t^-(u) = 1$, then by the uniqueness of $t^-(u)$, we have that $u \in M_\lambda^-$, which concludes iii).

Fix $u \in U$ arbitrary. Define $G_u : (0,\infty)\times U \to \mathbb{R}$ by

$$G_u(t,w) =< I_\lambda'(tw), tw > .$$

Since $G_u(t^-(u),u) =< I_\lambda'(t^-(u)u),t^-(u)u >= 0$ and

$$\frac{\partial G_u}{\partial x_1}(t^-(u),u) = [t^-(u)]^{-1} < \psi_\lambda'(t^-(u)u),t^-(u)u >< 0,$$

then by the implicit function Theorem, there is a neighborhood W_u of u in U and an unique continuous function $T_u : W_u \to (0,\infty)$ such that $G_u(T_u(w),w) = 0$ for all $w \in W_u$, in particular $T_u(u) = t^-(u)$. Since $u \in U$ is arbitrary, we have that the function $T : U \to (0,\infty)$, given by $T(u) = t^-(u)$ is continuous and one-to-one. Being $T^- : U \to M_\lambda^-$, where $T^-(u) = t^-(u)u$, we get that T^- is continuous and one-to-one. Now if $u \in M_\lambda^-$ then by iii) we have that $T^-(w) = u$, where $w = \frac{u}{\|u\|}$. Since t^- is continuous on U, it follows that t^- is continuous on $\mathcal{D}\backslash\{0\}$.

Remain to show that $\alpha_\lambda^+ < 0 < \alpha_\lambda^-$. Now since $f_+ \not\equiv 0$ is continuous, we can choose a function u in $\mathcal{D}\backslash\{0\}$ such that $\lambda\int_{\mathbb{R}^N} f|u|^q dx > 0$. Then by item ii) we obtain that $\alpha_\lambda^+ \leq I_\lambda(t^+(u)u) < 0$.

Now we will show $\alpha_\lambda^- > 0$. By (4) and definition of S, for all $u \in M_\lambda^-$, we have that

$$\left[S^{\frac{p^*}{p}}\left(\frac{p-q}{p^*-q}\right)\right]^{\frac{1}{p^*-p}} < \|u\|. \tag{12}$$

By (2) *and* (3), *for all* $u \in M_\lambda^-$, *we obtain*

$$I_\lambda(u) \geq \left(\frac{N-p}{N}\right)\|u\|^p - \left(\frac{p^*-q}{qp^*}\right)\lambda C_f\|u\|^q.$$

From (11), (12) *and by definition* C_3 *we have, for all* $u \in M_\lambda^-$, *that*

$$\begin{aligned} I_\lambda(u) &\geq \|u\|^q\left[\frac{N-p}{N}\|u\|^{p-q} - \left(\frac{p^*-q}{qp^*}\right)C_f\Lambda\right] \\ &> C_f\|u\|^q\frac{p^*-q}{qp^*}\left[\frac{N-p}{N}C_f^{-1}\left(S^{\frac{p^*}{p}}\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}}-\Lambda\right] \\ &> C_f\left[S^{\frac{p^*}{p}}\left(\frac{p-q}{p^*-q}\right)\right]^{\frac{q}{p^*-p}}\left(\frac{p^*-q}{qp^*}\right)C_3 \\ &= \quad c > 0. \end{aligned}$$

Hence $\alpha_\lambda^- \geq c > 0$ *and by Lemma* 2.2, *which concludes that* $\alpha_\lambda = \alpha_\lambda^+ < 0 < \alpha_\lambda^-$. □

Now we will ensure that there are $(PS)_{\alpha_\lambda^+}-$sequence and $(PS)_{\alpha_\lambda^-}-$sequence on M_λ and M_λ^- respectively, for the functional I_λ.

Lemma 2.4. *If* $\lambda \in (0,\Lambda)$, *then*

i) there exists a sequence $(u_n) \subset M_\lambda$ *such that*

$$I_\lambda(u_n) = \alpha_\lambda + o(1) = \alpha_\lambda^+ + o(1),$$
$$I'_\lambda(u_n) = o(1) \text{ in } \mathcal{D}^*.$$

ii) there exists a sequence $(u_n) \subset M_\lambda^-$ *such that*

$$I_\lambda(u_n) = \alpha_\lambda^- + o(1),$$
$$I'_\lambda(u_n) = o(1) \text{ in } \mathcal{D}^*.$$

Proof. 4. *Firstly we will prove item i). By the Ekeland variational principle we get a sequence* $(u_n) \subset M_\lambda$, *such that*

$$I_\lambda(u_n) < \alpha_\lambda + \frac{1}{n}, \tag{13}$$

$$I_\lambda(u_n) < I_\lambda(w) + \frac{1}{n}\|u_n - w\|, \tag{14}$$

for all $n \in \mathbb{N}$ *e* $w \in M_\lambda$. *We obtain that* (u_n) *is bounded in* $\mathcal{D}$, *since* $\alpha_\lambda < 0$, *there exists* n_0, *where* $2 \leq -n_0\alpha_\lambda$, *such that by* (3)

$$I_\lambda(u_n) = \frac{p^*-p}{pp^*}\|u_n\|^p - \frac{p^*-q}{qp^*}\lambda\int_{\mathbb{R}^N} f|u_n|^q dx < \frac{\alpha_\lambda}{2},$$

for $n \geq n_0$. *Then without loss of generality, by* (2) *and the fact of* $\alpha_\lambda < 0$, *we obtain for all* n

$$\left(-\frac{\alpha_\lambda}{2\Lambda C_f S^q}\frac{qp^*}{p^*-q}\right)^{\frac{1}{q}} \leq \|u_n\| \leq \left(\frac{p^*-q}{qp^*}\Lambda C_f S^q\right)^{\frac{1}{p-q}}. \tag{15}$$

Now we will show that $\|I'_\lambda(u_n)\|_{\mathcal{D}^*} \to 0$ *as* $n \to \infty$, *for this, we need the following result.*

Claim 2.1. *Let* $\lambda \in (0,\Lambda)$ *be arbitrary. For any function* $u \in M_\lambda(M_\lambda^-)$ *there exists* $0 < \varepsilon = \varepsilon(u) < \frac{\|u\|}{2}$ *and* $\eta : B(0,\varepsilon) \subset \mathcal{D} \to (\frac{1}{2},2)$ *differentiable, such that* $\eta(0) = 1$, $\eta(w)(u - w) \in M_\lambda(M_\lambda^-)$ *for all* $w \in B(0,\varepsilon)$ *and for all* $z \in \mathcal{D}$, *it holds*

$$\begin{aligned} &< \eta'(0), z >= \\ &\qquad \frac{-p\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla z\,dx + q\lambda\int_{\mathbb{R}^N} f|u|^{q-2}uz\,dx}{(p-q)\|u\|^p - (p^*-q)\int_{\mathbb{R}^N}|u|^{p^*}dx} \\ &\qquad + \frac{p^*\int_{\mathbb{R}^N}|u|^{p^*-2}uz\,dx}{(p-q)\|u\|^p - (p^*-q)\int_{\mathbb{R}^N}|u|^{p^*}dx}. \end{aligned} \tag{16}$$

Assume the Claim for while. Let any $u \in \mathcal{D}\backslash\{0\}$. *Applying the Claim* 2.1 *for* $u_n \in M_\lambda$, *we obtain a function* $\eta_n : B(0,\varepsilon_n) \subset \mathcal{D} \to (\frac{1}{2},2)$ *differentiable, where* $0 < \varepsilon_n < \frac{\|u_n\|}{2}$, *with* $\eta_n(0) = 1$ *and* $\eta_n(w)(u_n - w) \in M_\lambda$ *for all* w *in* $B(0,\varepsilon_n)$. *Fix any* $0 < \rho < \varepsilon_n$, *let* $v_\rho = \eta_n(w_\rho)(u_n - w_\rho)$, *where* $w_\rho = \frac{\rho u}{\|u\|}$. *Since* $v_\rho \in M_\lambda$, *by* (14) *it follows that*

$$I_\lambda(v_\rho) - I_\lambda(u_n) \geq -\frac{1}{n}\|v_\rho - u_n\|,$$

and thus by the mean value Theorem, we get that

$$\left\langle I'_\lambda(u_n), v_\rho - u_n\right\rangle + o(\|v_\rho - u_n\|) \geq -\frac{1}{n}\|v_\rho - u_n\|.$$

By definition of v_ρ *we obtain*

$$\begin{aligned} -\frac{1}{n}\|v_\rho - u_n\| \leq\ & o(\|v_\rho - u_n\|) - \rho\left\langle I'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle \\ & + (\eta_n(w_\rho) - 1)\left\langle I'_\lambda(u_n), u_n - w_\rho\right\rangle \\ =\ & o(\|v_\rho - u_n\|) - \rho\left\langle I'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle \\ & + (\eta_n(w_\rho)-1)\left\langle I'_\lambda(u_n) - I'_\lambda(v_\rho), u_n - w_\rho\right\rangle. \end{aligned}$$

Hence for all $0 < \rho < \varepsilon_n$, *we have that*

$$\begin{aligned} \left\langle I'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle \leq\ & \frac{\eta_n(w_\rho)-1}{\rho}\left\langle I'_\lambda(u_n) - I'_\lambda(v_\rho), u_n - w_\rho\right\rangle \\ & + \frac{\|v_\rho - u_n\|}{n\rho} + o(\|v_\rho - u_n\|). \end{aligned}$$

Since $\lim_{\rho \to 0^+}\frac{|\eta_n(w_\rho) - 1|}{\rho} \leq \|\eta'_n(0)\|$, *and noticing that the sequence* (u_n) *is bounded, also the functional* I'_λ *is continuous and* $\lim_{\rho \to 0^+} v_\rho = u_n$, *we infer that there exists* $C > 0$ *independent of* ρ *and* n, *satisfying*

$$\left\langle I'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle \leq \frac{C}{n}\|\eta'_n(0)\|. \tag{17}$$

Now, we will show that $\|\eta'_n(0)\|$ *is bounded for all* $n \in \mathbb{N}$. *By* (15) *and* (16) *we have that there exists* $C > 0$, *independent of* $n \in \mathbb{N}$, *such that*

$$|< \eta'_n(0), w >| \leq C\frac{\|w\|}{\left|(p-q)\|u_n\|^p - (p^*-q)\int_{\mathbb{R}^N}|u_n|^{p^*}dx\right|}.$$

Hence, it is enough to prove that there is $\delta > 0$ such that, for n sufficiently large,

$$\left|(p-q)\|u_n\|^p - (p^*-q)\int_{\mathbb{R}^N} |u_n|^{p^*}dx\right| > \delta. \tag{18}$$

Suppose by absurd that there is a subsequence, which denote simply by (u_n), such that

$$(p-q)\|u_n\|^p - (p^*-q)\int_{\mathbb{R}^N} |u_n|^{p^*}dx = o(1). \tag{19}$$

Since $(u_n) \subset M_\lambda$ and by the relations (3), (15) and (19) we have that

$$\lambda\int_{\mathbb{R}^N} f|u_n|^q dx = \left(\frac{p^*-p}{p^*-q}\right)\|u_n\|^p + o(1). \tag{20}$$

Thus by definition of F_λ (see proof of Lemma 2.2) and by (19) and (20), we obtain

$$F_\lambda(u_n) = o(1). \tag{21}$$

Since $(u_n) \subset M_\lambda$, arguing as in (10), by combining relations (2), (11), (15) with the definition of C_4, we get

$$F_\lambda(u_n) \geq -\frac{\alpha_\lambda}{2}\left(\frac{qp^*}{p^*-q}\right)\Lambda^{\frac{-p}{p-q}}\left(C_4 - \Lambda^{\frac{p}{p-q}}\right) + o(1).$$

Now $C_4 > \Lambda^{\frac{p}{p-q}}$ and since $\alpha_\lambda < 0$, we have that there exists n_0 and $C > 0$, such that

$$F_\lambda(u_n) > C,$$

for all $n \geq n_0$, which contradicts (21). Furthermore there exists $\delta > 0$, such that (18) is satisfied. Thus we have that $\|\eta_n'(0)\| \leq C$, for all $n \in \mathbb{N}$ and then by (17) we obtain $I_\lambda'(u_n) = o(1)$ in $\mathcal{D}^*$.

Proof of Claim 2.1. *Consider $u \in M_\lambda$. Define the function $F : \mathbb{R}^+ \times \mathcal{D} \to \mathbb{R}$, by*

$$\begin{aligned} F(\eta,w) &= \langle I_\lambda'(\eta(u-w)), \eta(u-w)\rangle \\ &= \eta^p\|u-w\|^p - \eta^q\lambda\int_{\mathbb{R}^N} f|u-w|^q dx \\ &\quad -\eta^{p^*}\int_{\mathbb{R}^N}|u-w|^{p^*}dx. \end{aligned}$$

Now since $F(1,0) = 0$ and $\frac{\partial}{\partial x_1}F(1,0) =< \psi_\lambda'(u),u >\neq 0$, because $u \notin M_\lambda^0$, it follows by the implicit function Theorem, that there exists $\varepsilon > 0$ where $\varepsilon = \varepsilon(u) < \frac{\|u\|}{2}$ and $\eta : B(0,\varepsilon) \subset \mathcal{D} \to (\frac{1}{2},2)$ functional differentiable, such that $\eta(0) = 1$, $F(\eta(w),w) = 0$ for all $w \in B(0,\varepsilon)$, this is, $\eta(w)(u-w) \in M_\lambda$, for all $w \in B(0,\varepsilon)$. We also get that $< \eta'(0),w >= -\dfrac{\frac{\partial F(1,0)}{\partial x_2}\cdot w}{\frac{\partial F(1,0)}{\partial x_1}}$, for all $w \in \mathcal{D}$, this is, the equality (16) holds.

Consider now the case $u \in M_\lambda^-$. Similarly we obtain $0 < \varepsilon = \varepsilon(u) < \frac{\|u\|}{2}$ and $\eta : B(0,\varepsilon) \subset \mathcal{D} \to (\frac{1}{2},2)$ differentiable, such that $\eta(0) = 1$, $\eta(w)(u-w) \in M_\lambda$, for all $w \in B(0,\varepsilon)$ and verifying (16). Suppose by absurd, that there is no $0 < \tilde{\varepsilon} < \varepsilon$ such that, $\eta(w)(u-w) \in M_\lambda^-$, for all $w \in B(0,\tilde{\varepsilon})$. Then there exists $(w_n) \subset \mathcal{D}$ where $\varepsilon > \|w_n\|$, $\|w_n\| \to 0$, $n \to \infty$, and $v_n \doteq \eta(w_n)(u-w_n) \notin M_\lambda^-$. Therefore $(v_n) \subset M_\lambda^+$, because $v_n \notin M_\lambda^0$ (Lemma 2.2). By relations (2) and (5) we obtain

$$\begin{aligned} 0 &< < \psi_\lambda'(v_n),v_n > \\ &\leq \|v_n\|^q C_f(p^*-q)\left[\Lambda - \left(\frac{p^*-p}{p^*-q}\right)\left(\frac{\|v_n\|}{C_f^{\frac{1}{p-q}}}\right)^{p-q}\right]. \end{aligned}$$

By the above inequality, definitions of ε, C_5, from (11) and (12) we have

$$\begin{aligned} 0 &< 3^q\|u\|^q C_f(p^*-q)\left[\Lambda - \left(\frac{p^*-p}{p^*-q}\right)\frac{\|u\|^{p-q}}{4^{p-q}C_f}\right] \\ &\leq 3^q\|u\|^q C_f(p^*-q)\left[\Lambda - C_5\right] \leq 0, \end{aligned}$$

for all $n \in \mathbb{R}$, which is a contradiction. Thus there exists $0 < \tilde{\varepsilon} < \varepsilon$ such that $\eta(w)(u-w) \in M_\lambda^-$, for all $w \in B(0,\tilde{\varepsilon})$, which concludes the proof of Claim 2.1. The proof of ii) of Lemma 2.4 is similar to case i). □

As consequence of hypothesis (H) we have the following result.

Lemma 2.5. *If (u_n) is a sequence in $\mathcal{D}$, such that $u_n \rightharpoonup u$ weakly in $\mathcal{D}$, then there exits a subsequence (u_n) such that*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f|u_n|^q\, dx = \int_{\mathbb{R}^N} f|u|^q\, dx.$$

Proof. 5. *Since $u_n \rightharpoonup u$ weakly in $\mathcal{D}$, passing to subsequence if necessary, we have that $u_n \to u$ strongly in $L^q_{loc}(\mathbb{R}^N)$. Now, since (u_n) is bounded in $\mathcal{D}$, there exists $C > 0$ such that $\|u_n\|_{L^{p^*}} \leq C$ for all $n \in \mathbb{N}$. Consider for any $r, R > 0$*

$$A_r = \sup_{|x|\leq r}\frac{|f(x)|}{|x|^b}\left(\int_{|x|\leq r} 2|x|^{\frac{bp^*}{p^*-q}}\, dx\right)^{\frac{p^*-q}{q}}$$

and

$$A^R = \sup_{|x|\geq R}\frac{|f(x)|}{|x|^a}\left(\int_{|x|\geq R} 2|x|^{\frac{ap^*}{p^*-q}}\, dx\right)^{\frac{p^*-q}{q}}.$$

Let $\varepsilon > 0$ arbitrary. It follows by (H) that $A_r \to 0$ as $r \to 0$ and $A^R \to 0$ as $R \to \infty$. Consider r_0, R_0 such that $A_{r_0}, A^{R_0} < \frac{\varepsilon}{3C^q}$. Now

$$\left|\int_{\mathbb{R}^N} f[|u_n|^q - |u|^q]dx\right| \le C^q(A_{r_0} + A^{R_0}) + \sup_{r_0\le|x|\le R_0}|f(x)|\left|\int_{r_0\le|x|\le R_0}[|u_n|^q - |u|^q]dx\right|.$$

Since $u_n \to u$ *in* $L^q_{loc}(\mathbb{R}^N)$ *and* f *is locally bounded in* $\mathbb{R}^N\backslash\{0\}$*, there exists* $n_0 \in \mathbb{N}$ *such that*

$$\sup_{r_0\le|x|\le R_0}|f(x)|\left|\int_{r_0\le|x|\le R_0}[|u_n|^q - |u|^q]\,dx\right| < \frac{\varepsilon}{3},$$

for all $n \ge n_0$*. Then, for* $n \ge n_0$ *we have that,*

$$\left|\int_{\mathbb{R}^N} f[|u_n|^q - |u|^q]\,dx\right| < \varepsilon.$$

Since $\varepsilon > 0$ *is arbitrary, we complete the proof.* □

The following remark of Brézis e Lieb (1983), (see (Kavian, 1993, Lemma 4.8)), follows by applying Fatou's Lemma.

Remark 2.2. *If* $\Omega \subset \mathbb{R}^N$ *is a open set,* (g_n) *is a bounded sequence in* $L^r(\Omega)$*,* $1 < r < \infty$ *such* $g_n \to g$ *a.e in* Ω*, then* $g \in L^r(\Omega)$ *and* $g_n \rightharpoonup g$ *weakly in* $L^r(\Omega)$*.*

A proof a next result can be found (Jianfu, 1995, Lemma 2.2).

Remark 2.3. *Let* (u_n) *be a* $(PS)_c-$*sequence for* I_λ*. Then, there exists a subsequence, denoted again for* (u_n)*, such that*

$$\nabla u_n \to \nabla u \ \text{ a.e in } \mathbb{R}^N,$$

$$|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \rightharpoonup |\nabla u|^{p-2}\frac{\partial u}{\partial x_i} \quad \text{weakly in} \quad (L^p(\Omega))^*,$$

for each $1 \le i \le N$*, for all open set* $\Omega \subset \mathbb{R}^N$*.*

Now, we will give a detailed description of the $(PS)-$sequences for I_λ.

Lemma 2.6. *Let* $\lambda \in (0,\Lambda)$ *be arbitrary. If* (u_n) *is a* $(PS)_c-$*sequence for* I_λ *which* $c \ne 0$ *and*

$$c < \alpha_\lambda + \frac{1}{N}S^{\frac{N}{p}}.$$

Then (u_n) *has a convergent subsequence.*

Proof. 6. *Recall that by Remark* 2.1*, we get that* (u_n) *is bounded and there exists* $u \in \mathcal{D}$ *such that,* $u_n \rightharpoonup u$ *weakly in* $\mathcal{D}$*, where* $u_n \ge 0$ *and* $u \ge 0$*. Suppose by absurd that* $u = 0$*. Then, by Lemma* 2.5 *we have*

$$\int_{\mathbb{R}^N} f u_n^q\,dx = o(1).$$

Thus, by $\langle I'_\lambda(u_n),u_n\rangle = o(1)$ *and the last relation, we have that*

$$\|u_n\|^p = \int_{\mathbb{R}^N} u_n^{p^*}dx + o(1). \tag{22}$$

Then, by the above relations

$$c = I_\lambda(u_n) + o(1) = \frac{1}{N}\|u_n\|^p + o(1).$$

If $c < 0$*, by the above equation, we have a contradiction. If* $c > 0$*, there exists* $n_0 \in \mathbb{N}$ *where* $\|u_n\|^p \ge \frac{c}{2}$ *if* $n \ge n_0$*. Then, by* (22)*, the definition of* S*, the fact of* $\alpha_\lambda < 0$ *and the above equality, we obtain for* $n \ge n_0$*, that*

$$c < \alpha_\lambda + \frac{1}{N}S^{\frac{N}{p}} \le \frac{1}{N}\|u_n\|^p = c + o(1),$$

which is a contradiction. Therefore $u \ne 0$*.*

In order to prove that $u \in M_\lambda$*, it is enough to prove that* $I'_\lambda(u) = 0$*. Firstly, we will show for all* $\varphi \in \mathcal{D}$ *that*

$$\int_{\mathbb{R}^N} f u_n^{q-1}\varphi\,dx \to \int_{\mathbb{R}^N} f u^{q-1}\varphi\,dx, \tag{23}$$

$$\int_{\mathbb{R}^N} u_n^{p^*-1}\varphi\,dx \to \int_{\mathbb{R}^N} u^{p^*-1}\varphi\,dx. \tag{24}$$

Consider A_r, A^R *as in Lemma* 2.5*, then by Hölder inequality we have*

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} f[u_n^{q-1}\varphi - u^{q-1}\varphi]\,dx\right|\\ &\le (A_r + A^R)\|u_n\|^{q-1}_{L^{p^*}}\|\varphi\|_{L^{p^*}}\\ &\quad + \sup_{r\le|x|\le R}|f(x)|\left|\int_{r\le|x|\le R}[|u_n|^q - |u|^q]\,dx\right|,\end{aligned}$$

which shows relation (23)*, since* $(\|u_n\|_{L^{p^*}})$ *is bounded, the limit* $A_r \to 0$ *as* $r \to 0$*,* $A^R \to 0$ *as* $R \to \infty$*,* f *is locally bounded in* $\mathbb{R}^N\backslash\{0\}$ *and* $u_n \to u$ *strongly in* $L^q_{loc}(\mathbb{R}^N)$*. The relation* (24) *follows from Remark* 2.2*, observing that* $\varphi \in \mathcal{D} \hookrightarrow L^{\frac{p^*}{p^*-1}}(\mathbb{R}^N)$*.*

Thus from Remark 2.3 *we get*

$$\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx \to \int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx, \tag{25}$$

for all $\varphi \in C_c^\infty$*. Since* $\mathcal{D}$ *is the closure of* C_c^∞*, we obtain that* (25) *is satisfied for all* $\varphi \in \mathcal{D}$*. It follows by* (23)*,* (24) *and* (25) *that*

$$\langle I'_\lambda(u),\varphi\rangle = 0, \text{ for all } \varphi \in \mathcal{D},$$

this is, $I'_\lambda(u) = 0$*. Then,* $u \in M_\lambda$*, in particular,* $I_\lambda(u) \ge \alpha_\lambda$*. We will show that, up to a subsequence,* $u_n \to u$ *strongly in* $\mathcal{D}$*. Consider* $z_n = u_n - u$*, we affirm that*

$$\|z_n\|^p - \int_{\mathbb{R}^N}|z_n|^{p^*}dx = o(1). \tag{26}$$

Since $z_n \rightharpoonup 0$ weakly in $\mathcal{D}$, from Remark 2.3 and (Brézis e Lieb, 1983, Theorem 1) we have that

$$\|u_n\|^p = \|z_n\|^p + \|u\|^p + o(1), \tag{27}$$

and

$$\int_{\mathbb{R}^N} u_n^{p^*} dx = \int_{\mathbb{R}^N} u^{p^*} dx + \int_{\mathbb{R}^N} |z_n|^{p^*} dx + o(1). \tag{28}$$

We get, by Lemma 2.5 that

$$\int_{\mathbb{R}^N} f \left(u_n^q - u^q \right) dx = o(1). \tag{29}$$

Now, since $\langle I'_\lambda(u_n), u_n \rangle \to 0$ and $\langle I'_\lambda(u), u \rangle = 0$ we have that

$$\begin{aligned} o(1) &= \|u_n\|^p - \lambda \int_{\mathbb{R}^N} f u_n^q dx - \int_{\mathbb{R}^N} u_n^{p^*} dx \\ &= \|u\|^p + \|z_n\|^p - \lambda \int_{\mathbb{R}^N} f u_n^q dx - \int_{\mathbb{R}^N} u^{p^*} dx \\ &\quad - \int_{\mathbb{R}^N} |z_n|^{p^*} dx + o(1) \\ &= \langle I'_\lambda(u), u \rangle + \|z_n\|^p - \lambda \int_{\mathbb{R}^N} f \left(u_n^q - u^q \right) dx \\ &\quad - \int_{\mathbb{R}^N} |z_n|^{p^*} dx + o(1) \\ &= \|z_n\|^p - \int_{\mathbb{R}^N} |z_n|^{p^*} dx + o(1), \end{aligned}$$

and so (26) is proved.

Since I_λ is continuous and $c < \alpha_\lambda + \frac{1}{N} S^{\frac{N}{p}}$, there exists $n_0 \in \mathbb{N}$ and $\varepsilon > 0$, such that if $n \geq n_0$ we have

$$\alpha_\lambda + \frac{1}{N} S^{\frac{N}{p}} - \varepsilon > I_\lambda(u_n),$$

now, by (26), (27), (28), (29) and the fact of $I_\lambda(u) \geq \alpha_\lambda$ we obtain that

$$\begin{aligned} I_\lambda(u_n) &= I_\lambda(u) + \frac{1}{p} \|z_n\|^p - \frac{1}{p^*} \int_{\mathbb{R}^N} |z_n|^{p^*} dx + o(1) \\ &\geq \alpha_\lambda + \frac{1}{N} \|z_n\|^p + o(1). \end{aligned}$$

Then, by the above relations, we get for n large that

$$\|z_n\|^p < S^{\frac{N}{p}}. \tag{30}$$

We affirm that $\|z_n\| \to 0$ as $n \to \infty$. Let us assume by contradiction that there exists $\delta > 0$ such that, up to a subsequence, $\|z_n\| \geq \delta$. Then by the definition of S and the relations (26) and (30) we obtain that

$$S^{\frac{N}{p}} \leq \liminf \left\{ \frac{\|z_n\|^N}{\left(\int_{\mathbb{R}^N} |z_n|^{p^*} dx \right)^{\frac{N}{p^*}}} \right\} = \liminf \{ \|z_n\|^p \} < S^{\frac{N}{p}},$$

which is a absurd. Then, $\|z_n\| \to 0$ as $n \to \infty$, namely, $u_n \to u$ strongly in $\mathcal{D}$. □

3 Existence of a Solution

We have all the tools necessary to obtain a solution of (1), in fact, in the next result we will get a solution u_λ of (1), where $u_\lambda \in M_\lambda^+$.

Proposition 3.1. *Fix any $\lambda \in (0, \Lambda)$. Then there exists a solution of (1), $u_\lambda \in M_\lambda^+$, satisfying $I_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+ < 0$, where $\|u_\lambda\| \to 0$ as $\lambda \to 0$.*

Proof. 7. *By Lemma 2.4 i) follows that there exists $(u_n) \subset M_\lambda$ satisfying*

$$I_\lambda(u_n) = \alpha_\lambda + o(1) = \alpha_\lambda^+ + o(1),$$

$$I'_\lambda(u_n) = o(1) \text{ in } \mathcal{D}^*,$$

this is, (u_n) is a $(PS)_{\alpha_\lambda}$−sequence for I_λ. By fact of $\alpha_\lambda < 0$, it follows from Lemma 2.6 and Remark 2.1 that there exists $u_\lambda \in \mathcal{D}$, where $u_\lambda \geq 0$ and, up to a subsequence if necessary, $u_n \to u_\lambda$ strongly in $\mathcal{D}$, in particular, we obtain that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+ < 0$. Consequently, $u_\lambda \gneq 0$ and since $\alpha_\lambda^+ < 0 < \alpha_\lambda^-$, we obtain that $u_\lambda \in M_\lambda^+$.

By (2) and (5), we infer that there exists $C > 0$, independent of λ, such that

$$\|u_\lambda\| \leq C(\lambda C_f)^{\frac{1}{p-q}},$$

and thus we conclude the proof. □

In order to obtain the second solution for problem (1) we need some conditions on of the regularity of such solutions. For this respect we have the following result.

Proposition 3.2. *Let u a solution of problem (1). Then, for any open set Ω such that $\overline{\Omega} \subset\subset \mathbb{R}^N \backslash \{0\}$ there exists $0 < \gamma < 1$ such that $u \in C^{1,\gamma}(\Omega)$.*

Proof. 8. *Consider Ω and Ω' satisfying the hypothesis, where $\overline{\Omega} \subset \Omega'$. By a similar arguments used in the work (Garcia Azorero e Peral Alonso, 1994, Theorem A.1), we can conclude that the function $u \in L^\infty(\Omega')$. It follows from (Ladyzhenskaëiĕa, 1968, Theorem 1.1, p. 251) that there exists a constant $\gamma = \gamma(\Omega', \|f\|_{L^\infty(\Omega')}, \|u\|_{L^\infty(\Omega')}) \in (0,1)$ such that $u \in C^{0,\gamma}(\Omega')$. Then by (Tolksdorf, 1984, Theorem 1) we obtain that $u \in C^{1,\gamma}(\Omega)$, which conclude the result.* □

4 Second Solution

In order to obtain the second solution to (1), first we will show the inequality

$$\alpha_\lambda^- < \alpha_\lambda + \frac{1}{N} S^{\frac{N}{p}}.$$

For this, consider $u_\lambda \in M_\lambda^+$ the solution of (1) obtained in Proposition 3.1. By definition of M_λ^+ and the relation (5) we get that $\int_{\mathbb{R}^N} f_+ u_\lambda^q \, dx > 0$. Then we consider

$$\Sigma = \Sigma(\lambda) = \left\{ y \in \mathbb{R}^N \backslash \{0\} : f_+(y) > 0 \text{ and } u_\lambda(y) > 0 \right\}.$$

Since f_+ and u_λ are continuous functions, consider for each $y \in \Sigma$ a positive constant $R = R(y) < \|y\|$, where $f_+(x) > 0$ and $u_\lambda(x) \geq \frac{u_\lambda(y)}{2}$ for all $x \in B_{2R}(y)$. Let $\phi_y \in C_c^\infty(\mathbb{R}^N)$ a cut-off function with support $B_{2R}(y)$ such that ϕ_y is identically 1 on $B_R(y)$.

Given $\varepsilon > 0$, define

$$w_\varepsilon(x) = k \left[\frac{\varepsilon^{\frac{1}{p-1}}}{\varepsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}} \right]^{\frac{N-p}{p}},$$

where $k = \left(N \left(\frac{N-p}{p-1} \right)^{p-1} \right)^{\frac{N-p}{p^2}}$. From Talenti (1976), we obtain for all $y \in \mathbb{R}^N$ that $w_{y,\varepsilon}(x) = w_{y,\varepsilon}(x-y)$ solves the special Sobolev critical equation

$$-\Delta_p u = u^{p^*-1}, \quad \text{in } \mathbb{R}^N,$$

and also $\|w_{y,\varepsilon}\|_{L^{p^*}}^{p^*} = \|w_{y,\varepsilon}\|^{p^*} = S^{\frac{N}{p}}$.

Given $\varepsilon > 0$ and $y \in \Sigma$, define

$$\psi_{y,\varepsilon}(x) \doteq \phi_y(x) w_{y,\varepsilon}(x), \quad x \in \mathbb{R}^N,$$

and

$$v_{y,\varepsilon}(x) \doteq \frac{\psi_{y,\varepsilon}(x)}{\|\psi_{y,\varepsilon}\|_{L^{p^*}}}, \quad x \in \mathbb{R}^N.$$

Then, we have the following result.

Lemma 4.1. *Let $\lambda \in (0,\Lambda)$ be arbitrary. Then, for a.e $y \in \Sigma$, there exists $\varepsilon_0 = \varepsilon_0(y,\lambda) > 0$, such that*

$$\sup_{l \geq 0} I_\lambda(u_\lambda + l v_{y,\varepsilon}) < \alpha_\lambda + \frac{1}{N} S^{\frac{N}{p}}, \tag{31}$$

for all $0 < \varepsilon < \varepsilon_0$.

Proof. 9. *Consider, for simplicity, $v_\varepsilon = v_{y,\varepsilon}$. For all $\varepsilon > 0$ we get that $(v_\varepsilon) \subset \mathcal{D}$ is bounded and since I_λ is continuous, with $I_\lambda(u_\lambda) = \alpha_\lambda < \alpha_\lambda + \frac{1}{N} S^{\frac{N}{p}}$, we infer that there exists $l_1 = l_1(\lambda) > 0$ such that*

$$I_\lambda(u_\lambda + l v_\varepsilon) < \alpha_\lambda + \frac{S^{\frac{N}{p}}}{2N},$$

for all $\varepsilon > 0$ and $l \in [0,l_1]$. We also get a $l_2 > l_1$, which is independent of $\varepsilon > 0$, where if $l \geq l_2$, then

$$I_\lambda(u_\lambda + l v_\varepsilon) < \alpha_\lambda + \frac{1}{N} S^{\frac{N}{p}} - 1,$$

because $\int_{\mathbb{R}^N} v_\varepsilon^{p^*} dx = 1$ for all $\varepsilon > 0$. Thus

$$\sup_{l \in [0,l_1] \cup [l_2,\infty)} I_\lambda(u_\lambda + l v_\varepsilon) < \alpha_\lambda + \frac{1}{N} S^{\frac{N}{p}}, \tag{32}$$

for all $\varepsilon > 0$. To complete the proof, by relation (32), it is enough to find $\varepsilon_0 > 0$ where if $\varepsilon \in (0,\varepsilon_0)$ we get

$$I_\lambda(u_\lambda + l v_\varepsilon) < \alpha_\lambda + \frac{1}{N} S^{\frac{N}{p}}, \quad \text{for all} \quad l \in [l_1,l_2].$$

First, we will show the case where $p = 2$. The proof is in Miotto (2010), but for the sake of completeness, we will give a sketch of the proof. Since $f > 0$ in the set where $v_\varepsilon = v_{y,\varepsilon} > 0$, it follows from the estimates obtained by Brezis e Nirenberg (1983, 1989) that

$$\begin{aligned} I_\lambda(u_\lambda + l v_{y,\varepsilon}) \quad &\leq \quad I_\lambda(u_\lambda) + \frac{1}{N} S^{\frac{N}{2}} - l^{2^*-1} \int_{\mathbb{R}^N} u_\lambda v_{y,\varepsilon}^{2^*-1} dx \\ &\quad + O(\varepsilon^{N-2}) + O(\varepsilon^N) + o(\varepsilon^{\frac{N-2}{2}}), \end{aligned}$$

for all $l \in [l_1,l_2]$ and $\varepsilon \in (0,1)$. By using a similar argument to that used in (Tarantello, 1992, Lemma 3.1), we can conclude that for every $l \geq 0$ and a.e $y \in \Sigma$, there exists $\varepsilon_0 = \varepsilon(y,\lambda) > 0$ such that for all $l \in [l_1,l_2]$ and $\varepsilon \in (0,\varepsilon_0)$ that

$$I_\lambda(u_\lambda + l v_{y,\varepsilon}) < \alpha_\lambda + \frac{1}{N} S^{\frac{N}{2}}. \tag{33}$$

Then if $p = 2$, the result follows from (32) and (33).

Now, we will consider the case $p > 2$. We will adapt some arguments used in Jianfu (1995). Note that

$$(a+b)^r - a^r - b^r - r a^{r-1} b \geq 0, \quad a,b \geq 0, \ r > 1. \tag{34}$$

By (Jianfu, 1995, (4.15)) we obtain, for each $\eta, \eta' \in \mathbb{R}^N$ that

$$(|\eta|^{p-2}\eta + |\eta'|^{p-2}\eta')(\eta + \eta') \geq |\eta + \eta'|^p. \tag{35}$$

Since $I'_\lambda(u_\lambda) = 0$ and $u_\lambda \geq 0$, we obtain that

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^{p-2} \nabla u_\lambda \nabla v_\varepsilon \, dx = \int_{\mathbb{R}^N} f u_\lambda^{q-1} v_\varepsilon dx + \int_{\mathbb{R}^N} u_\lambda^{p^*-1} v_\varepsilon dx.$$

Then, by the above equality, the relations (34), (35) and the fact of $\|v_\varepsilon\|_{L^{p^*}}^{p^*} = 1$, we obtain that

$$\begin{aligned} & I_\lambda(u_\lambda + l v_\varepsilon) \\ \leq \ & \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u_\lambda|^{p-2} \nabla u_\lambda + |l \nabla v_\varepsilon|^{p-2} l \nabla v_\varepsilon)(\nabla u_\lambda + l \nabla v_\varepsilon) dx \\ & \quad - \frac{\lambda}{q} \int_{\mathbb{R}^N} f(u_\lambda + l v_\varepsilon)^q dx \\ & \quad - \frac{1}{p^*} \int_{\mathbb{R}^N} u_\lambda^{p^*} + (l v_\varepsilon)^{p^*} + p^* l u_\lambda^{p^*-1} v_\varepsilon dx \\ = \ & I_\lambda(u_\lambda) - \frac{\lambda}{q} \int_{\mathbb{R}^N} f[(u_\lambda + l v_{y,\varepsilon})^q - u_\lambda^q - \frac{q}{p} u_\lambda^{q-1} l v_\varepsilon] dx \\ & \quad + \frac{l^p}{p} \|v_\varepsilon\|^p - \frac{l^{p^*}}{p^*} + \frac{l^{p-1}}{p} \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \nabla u_\lambda dx \\ & \quad - \frac{p-1}{p} l \int_{\mathbb{R}^N} u_\lambda^{p^*-1} v_\varepsilon dx. \end{aligned} \tag{36}$$

Consider for any $\varepsilon > 0$ the function $m_\varepsilon : [0,\infty) \to \mathbb{R}$ where,

$$m_\varepsilon(t) = \frac{t^p}{p}\|v_\varepsilon\|^p - \frac{t^{p^*}}{p^*}.$$

It follows from (Jianfu, 1995, (4.26)) or (Noussair et al., 1993, Lemma 2.2) that

$$\sup_{t\geq 0} m_\varepsilon(t) \leq \frac{1}{N}S^{\frac{N}{p}} + O(\varepsilon^{\frac{N-p}{p-1}}). \tag{37}$$

For any $y \in \Sigma$ we have that $f > 0$ in $B_{2R}(y)$ and $v_\varepsilon \geq 0$ with $v_\varepsilon \equiv 0$ in $\mathbb{R}^N \backslash \overline{B}_{2R}(y)$ and since $1 < q < p$, it follows from (34) that

$$\int_{\mathbb{R}^N} f[(u_\lambda + lv_\varepsilon)^q - u_\lambda^q - \frac{q}{p}u_\lambda^{q-1} lv_\varepsilon]\, dx \geq 0.$$

Then, by $I_\lambda(u_\lambda) = \alpha_\lambda$, the last relation, (36) and (37), we obtain that

$$\begin{aligned} I_\lambda(u_\lambda + lv_\varepsilon) \ &\leq\ \alpha_\lambda + \frac{1}{N}S^{\frac{N}{p}} + K_1\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \nabla u_\lambda dx \\ &\quad -K_2 \int_{\mathbb{R}^N} u_\lambda^{p^*-1} v_\varepsilon dx + O(\varepsilon^{\frac{N-p}{p-1}}), \end{aligned} \tag{38}$$

for any $l \in [l_1,l_2]$ and $\varepsilon > 0$, where K_1, K_2 are positive constants which are independents of ε. Direct calculations (see e.g. Huang (1998); Jianfu (1995)) show that

$$\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^r dx = \begin{cases} O(\varepsilon^{\frac{N(p-r)}{p}}) & \text{if } r > \frac{N(p-1)}{N-1} \\ O(\varepsilon^{\frac{N(p-r)}{p}}\ln(\varepsilon)) & \text{if } r = \frac{N(p-1)}{N-1} \\ O(\varepsilon^{\frac{(N-p)r}{p(p-1)}}) & \text{if } r < \frac{N(p-1)}{N-1} \end{cases} \tag{39}$$

$$\int_{\mathbb{R}^N}|v_\varepsilon|^r dx = \begin{cases} O(\varepsilon^{\frac{N(p-r)+pr}{p}}) & \text{if } r > \frac{p^*}{p'} = \frac{N(p-1)}{N-p} \\ O(\varepsilon^{\frac{N}{p}}\ln(\varepsilon)) & \text{if } r = \frac{p^*}{p'} \\ O(\varepsilon^{\frac{(N-p)r}{p(p-1)}}) & \text{if } r < \frac{p^*}{p'} \end{cases} \tag{40}$$

Therefore,

$$\begin{aligned} &\left|\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \nabla u_\lambda dx\right| \\ \leq\ & \max_{B_{2R}(y)} |\nabla u_\lambda(x)| \int_{B_{2R}(y)}|\nabla v_\varepsilon|^{p-1} dx \\ =\ & O(\varepsilon^{\frac{N-p}{p}}), \end{aligned} \tag{41}$$

$$\begin{aligned} \int_{\mathbb{R}^N} u_\lambda^{p^*-1} v_\varepsilon dx \ &\geq\ \min_{B_{2R}(y)} u_\lambda^{p^*-1}(x) \int_{\mathbb{R}^N} v_\varepsilon dx \\ &=\ C(y,\lambda)\varepsilon^{\frac{N-p}{p(p-1)}}. \end{aligned} \tag{42}$$

Thus, by (38),(41) and (42), for all $l \in [l_1,l_2]$ and $\varepsilon > 0$ we have

$$\begin{aligned} I_\lambda(u_\lambda + lv_\varepsilon) \ &\leq\ \alpha_\lambda + \frac{1}{N}S^{\frac{N}{p}} + O(\varepsilon^{\frac{N-p}{p-1}}) + O(\varepsilon^{\frac{N-p}{p}}) \\ &\quad -K_2 C(y,\lambda)\varepsilon^{\frac{N-p}{p(p-1)}}. \end{aligned} \tag{43}$$

Since $p > 2$, there exists $\varepsilon_0 = \varepsilon(y,\lambda) > 0$ such that

$$O(\varepsilon^{\frac{N-p}{p-1}}) + O(\varepsilon^{\frac{N-p}{p}}) - K_2 C(y,\lambda)\varepsilon^{\frac{N-p}{p(p-1)}} < 0,$$

for all $\varepsilon \in (0,\varepsilon_0)$. Then, by (43) we have for all $l \in [l_1,l_2]$ and $\varepsilon \in (0,\varepsilon_0)$ that

$$I_\lambda(u_\lambda + lv_{y,\varepsilon}) < \alpha_\lambda + \frac{1}{N}S^{\frac{N}{p}}. \tag{44}$$

From (32) and (44) we obtain

$$\sup_{l\geq 0} I_\lambda(u_\lambda + lv_{y,\varepsilon}) < \alpha_\lambda + \frac{1}{N}S^{\frac{N}{p}},$$

for all $\varepsilon \in (0,\varepsilon_0)$, which concludes the proof. □

Now we will get a solution of (1) with positive energy, namely, U_λ solution of (1) where $U_\lambda \in M_\lambda^-$.

Proposition 4.1. *Let $\lambda \in (0,\Lambda)$ be arbitrary. Then there exists $U_\lambda \in \mathcal{D}$ solution of (1), where $I_\lambda(U_\lambda) > 0$.*

Proof. 10. *Let $\lambda \in (0,\Lambda)$, $y \in \Sigma$ and $0 < \varepsilon < \varepsilon_0(y,\lambda)$ be arbitrary such that (31) holds. First, we will show*

$$\alpha_\lambda^- < \alpha_\lambda + \frac{1}{N}S^{\frac{N}{p}}. \tag{45}$$

By Lemma 4.1, it is enough to find $l_0 > 0$, such that the function $u_\lambda + l_0 v_{y,\varepsilon} \in M_\lambda^-$. Consider $A_1 = \{u \in \mathcal{D}\backslash\{0\} : t^-(u) < 1\} \cup \{0\}$ and $A_2 = \{u \in \mathcal{D}\backslash\{0\} : t^-(u) > 1\}$. Follows from Lemma 2.3 iii), that $\mathcal{D}\backslash M_\lambda^- = A_1 \cup A_2$. Since $u_\lambda \in M_\lambda^+$, by Lemma 2.3 that $1 = t^+(u_\lambda) < t^-(u_\lambda)$, this is, $u_\lambda \in A_2$. We will find $\tilde{l} > 0$, such that $u_\lambda + \tilde{l}v_{y,\varepsilon} \in A_1$. We affirm that there exists $c > 0$, such that

$$0 < t^-\left(\frac{u_\lambda + lv_{y,\varepsilon}}{\|u_\lambda + lv_{y,\varepsilon}\|}\right) \leq c,$$

for all $l \geq 0$. Suppose by absurd that there isn't a positive c. Then there exists $(l_n) \subset [0,\infty)$, where $l_n \to \infty$ and if consider $z_n = \frac{u_\lambda + l_n v_{y,\varepsilon}}{\|u_\lambda + l_n v_{y,\varepsilon}\|}$, then $t^-(z_n) \to \infty$. By the bounded convergence Theorem we get that

$$\int_{\mathbb{R}^N} |z_n|^{p^*} dx \to \frac{\int_{\mathbb{R}^N} |v_{y,\varepsilon}|^{p^*} dx}{\|v_{y,\varepsilon}\|^{p^*}} = 1.$$

Since $(t^-(v_n)v_n)_n \subset M_\lambda^-$, by definition of α_λ^-, it follows that $I_\lambda(t^-(z_n)z_n)$ is bounded from below by $\alpha_\lambda^- > 0$, but

$$\begin{aligned} I_\lambda(t^-(z_n)z_n) \;=\; & \frac{1}{p}\left[t^-(z_n)\right]^p - \frac{\lambda\left[t^-(z_n)\right]^q}{q}\int_{\mathbb{R}^N} f|z_n|^q dx \\ & -\frac{\left[t^-(z_n)\right]^{p^*}}{p^*}\int_{\mathbb{R}^N}|z_n|^{p^*}dx, \end{aligned}$$

that follows to $-\infty$ *as* $n \to \infty$. *Thus there exists* $c > 0$, *where* $0 < t^-\left(\frac{u_\lambda + lv_{y,\varepsilon}}{\|u_\lambda + lv_{y,\varepsilon}\|}\right) \leq c$, *for all* $l \geq 0$. *Let* $\tilde{l} > 0$, *such that* $\|u_\lambda + \tilde{l}v_{y,\varepsilon}\| > c$, *then we get that*

$$t^-\left(\frac{u_\lambda + \tilde{l}v_{y,\varepsilon}}{\|u_\lambda + \tilde{l}v_{y,\varepsilon}\|}\right) \leq c < \|u_\lambda + \tilde{l}v_{y,\varepsilon}\|,$$

this is, $t^-(u_\lambda + \tilde{l}v_{y,\varepsilon}) = \frac{1}{\|u_\lambda + \tilde{l}v_{y,\varepsilon}\|}t^-\left(\frac{u_\lambda + \tilde{l}v_{y,\varepsilon}}{\|u_\lambda + \tilde{l}v_{y,\varepsilon}\|}\right) < 1$. *Hence* $u_\lambda + \tilde{l}v_{y,\varepsilon} \in A_1$. *Consider* $F : [0,1] \to (0,\infty)$ *defined by* $F(s) = t^-(u_\lambda + s\tilde{l}v_{y,\varepsilon})$ *for all* $s \in [0,1]$. *Notice that* F *is continuous,* $F(0) = t^-(u_\lambda) > 1$ *and also* $F(1) = t^-(u_\lambda + \tilde{l}v_{y,\varepsilon}) < 1$. *Then there exists* $s_0 \in (0,1)$ *such that* $F(s_0) = 1$, *this is,* $t^-(u_\lambda + s_0\tilde{l}v_{y,\varepsilon}) = 1$. *Defining* $l_0 = s_0\tilde{l}$, *we have by Lemma* 2.3 *that* $u_\lambda + l_0 v_{y,\varepsilon} \in M_\lambda^-$. *Thus by Lemma* 4.1 *we have that* $\alpha_\lambda^- < \alpha_\lambda + \frac{1}{N}S^{\frac{N}{p}}$, *where we conclude relation* (45).

By Lemma 2.4 *ii) follows that there exists* (u_n) *in* M_λ^-, *a* $(PS)_{\alpha_\lambda^-}$*−sequence for* I_λ. *By* (45), *it follows from Lemma* 2.6 *and Remark* 2.1 *that there exists* $U_\lambda \in \mathcal{D}$, *where* $U_\lambda \geq 0$ *and, up to subsequence,* $u_n \to U_\lambda$ *strongly in* $\mathcal{D}$, *in particular, we get that* $I'_\lambda(U_\lambda) = 0$ *and* $I_\lambda(U_\lambda) = \alpha_\lambda^- > 0$, *consequently,* $U_\lambda \gneq 0$. *Since* $(u_n) \subset M_\lambda^-$, *by* (5) *and by Lemma* 2.2 *follow that* $\langle \psi'_\lambda(U_\lambda), U_\lambda \rangle < 0$, *this is,* $U_\lambda \in M_\lambda^-$. □

Now, we will show the proof of Theorem 1.1.

Proof of Theorem 1.1. *Let* $\lambda \in (0,\Lambda)$ *be arbitrary. In Proposition* 3.1 *we get a solution* $u_\lambda \in \mathcal{D}$ *of* (1), *where* $I_\lambda(u_\lambda) < 0$. *By Proposition* 4.1 *we obtain* $U_\lambda \in \mathcal{D}$ *solution of* (1), *where* $I_\lambda(U_\lambda) > 0$, *it follows the result.* □

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