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# FIXED POINT THEOREMS FOR SET VALUED CARISTI TYPE CONTRACTIONS ON GAUGE SPACES

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**Abstract.** In this paper, we prove some fixed point theorems for mappings satisfying the Caristi type $G$-contraction conditions on gauge spaces.

**Keywords:** Gauge spaces; Graph; Fixed point; Contraction condition.

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## 1. Introduction

Let $(X,d)$ be a metric space and $T : X \to X$ be a mapping such that there exists a lower semi-continuous function $\phi : X \to [0,\infty)$ satisfying $d(x,Tx) \leq \phi(x) - \phi(Tx)$, then $T$ is called Caristi mapping. Caristi proved in [1] that every Caristi mapping on a complete metric space has a

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fixed point. Then, Kirk [2] proved that the metric space $(X,d)$ is complete if and only if every Caristi mapping for $(X,d)$ has a fixed point. Jachymaski [3] introduced the notion of Banach $G$-contraction and proved some fixed point theorems for mappings satisfying this notion on complete metric space with a graph. Afterwards, many authors extended Banach $G$-contraction in different ways, see for examples, [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. In literature we have several interesting results on gauge spaces, see for example [14, 15, 16, 17, 18, 19, 20]. In this paper we prove some fixed point theorems for Caristi type multivalued mappings on a complete gauge space. For completeness we collect the following definitions.

**Definition 1.1.** [20] Let $X$ be a nonempty set. A function $d : X \times X \to [0,\infty)$ is called pseudo metric in $X$ if for each $x,y,z \in X$

- **:** (i) $d(x,x) = 0$ for each $x \in X$;
- **:** (ii) $d(x,y) = d(y,x)$;
- **:** (iii) $d(x,z) \leq d(x,y) + d(y,x)$.

**Definition 1.2.** [20] Let $X$ be a nonempty set endowed with the pseudo metric $d$. The $d$-ball of radius $\varepsilon > 0$ centered at $x \in X$ is the set

$$B(x,d,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}.$$

**Definition 1.3.** [20] A family $\mathfrak{F} = \{d_a | a \in \mathfrak{A}\}$ of pseudo metrics is said to be separating if for each pair $(x,y)$ with $x \neq y$, there exists $d_a \in \mathfrak{F}$ with $d_a(x,y) \neq 0$.

**Definition 1.4.** [20] Let $X$ be a nonempty set and $\mathfrak{F} = \{d_a | a \in \mathfrak{A}\}$ be a family of pseudo metrics on $X$. The topology $\mathfrak{T}(\mathfrak{F})$ having subbases the family

$$\mathfrak{B}(\mathfrak{F}) = \{B(x,d_a,\varepsilon) : x \in X, d_a \in \mathfrak{F} \text{ and } \varepsilon > 0\}$$

of balls is called topology induced by the family $\mathfrak{F}$ of pseudo metrics. The pair $(X,\mathfrak{T}(\mathfrak{F}))$ is called a gauge space. Note that $(X,\mathfrak{T}(\mathfrak{F}))$ is Hausdorff if we take $\mathfrak{F}$ is separating.

**Definition 1.5.** [20] Let $(X,\mathfrak{T}(\mathfrak{F}))$ be a gauge space with respect to the family $\mathfrak{F} = \{d_a | a \in \mathfrak{A}\}$ of pseudo metrics on $X$. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$. Then,

**:** (i) the sequence $\{x_n\}$ converges to $x$ if for each $a \in \mathfrak{A}$ and $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $d_a(x_n, x) < \varepsilon$ for each $n \geq N_0$. We denote it as $x_n \to^{\mathfrak{F}} x$;

**:** (ii) the sequence $\{x_n\}$ is a Cauchy sequence if for each $a \in \mathfrak{A}$ and $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $d_a(x_n, x_m) < \varepsilon$ for each $n, m \geq N_0$;

**:** (iii) $(X, \mathfrak{T}(\mathfrak{F}))$ is complete if each Cauchy sequence in $(X, \mathfrak{T}(\mathfrak{F}))$ is convergent in $X$;

**:** (iv) a subset of $X$ is said to be closed if it contains the limit of each convergent sequence of its elements.

A directed graph $(V, E)$ consists of a set of vertices $V$, and a set of directed edges $E$. The elements of $E$ are denoted by ordered pairs of vertices. A directed graph, can have loops and permits one or more edges joining the same vertices. More than one edges going in the same direction between same vertices called parallel edges, which are not allowed in our results. For basic terminologies of graph theory we refer [21].

**Definition 1.6.** [3] A mapping $T : X \to CL(X)$ is said to be $G$-continuous if for each sequence $\{x_n\}$ in $X$ such that $(x_n, x_{n+1}) \in E$ and $x_n \to x$ as $n \to \infty$, we have $Tx_n \to Tx$ as $n \to \infty$.

**Definition 1.7.** A mapping $g : X \to [0, \infty)$ is said to be $G$-lower semi continuous, if for each sequence $\{x_n\}$ in $X$ such that $(x_n, x_{n+1}) \in E$ and $x_n \to x$ as $n \to \infty$, we have $g(x) \leq \liminf_{n\to\infty} g(x_n)$.

## 2. **Main results**

Through out this section we assume that $G = (V, E)$ is a directed graph such that $V = X$ and $\{(x, x) : x \in V\} \subset E$ and $G$ has no parallel edges.

**Theorem 2.1.** *Let $X$ be endowed with a graph $G$ and complete gauge structure $\{d_a : a \in \mathfrak{A}\}$ which is separating. Let $T : X \to CL(X)$ be an edge preserving mapping and let for each $a \in \mathfrak{A}$, we have $\phi_a : X \to [0, \infty)$ be a lower semi continuous function such that for each $x \in X$ and $y \in Tx$ with $(x, y) \in E$, we have*

$$d_a(y, Ty) \leq \phi_a(x) - \phi_a(y) \text{ for each } a \in \mathfrak{A}. \tag{1}$$

*Assume that the following conditions hold:*

- **:** *(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;*
- **:** *(ii) there exists a sequence $\{q_a : q_a > 1\}_{a\in\mathfrak{A}}$ such that for each $x \in X$, we have $y \in Tx$ satisfying*

$$d_a(x,y) \leq q_a d_a(x,Tx) \text{ for each } a \in \mathfrak{A}.$$

- **:** *(iii) for each $a \in \mathfrak{A}$, a function $g_a : X \to [0,\infty)$ define by $g_a(x) = d_a(x,Tx)$ is G-lower semi continuous.*

*Then $T$ has a fixed point.*

*Proof.* By hypothesis (i), we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E$. From (1), we have

$$d_a(x_1,Tx_1) \leq \phi_a(x_0) - \phi_a(x_1) \text{ for each } a \in \mathfrak{A}. \tag{2}$$

By using (ii) and (2), we have $q_a > 1$ for each $a \in \mathfrak{A}$ and $x_2 \in Tx_1$ such that

$$d_a(x_1,x_2) \leq q_a d_a(x_1,Tx_1) \leq q_a\phi_a(x_0) - q_a\phi_a(x_1) \text{ for each } a \in \mathfrak{A}. \tag{3}$$

Since $T$ is edge preserving, we have $(x_1,x_2) \in E$. Continuing in the same way we get a sequence $\{x_m\}$ in $X$ such that $(x_m,x_{m+1}) \in E$ and

$$d_a(x_m,x_{m+1}) \leq q_a d_n(x_m,Tx_m) \leq q_a\phi_a(x_{m-1}) - q_a\phi_a(x_m) \text{ for each } m \in \mathbb{N} \text{ and } a \in \mathfrak{A}.$$

This implies that for each $a \in \mathfrak{A}$ the sequence $\{\phi_a(x_m)\}$ is a nonincreasing sequence, there exists $r_a \geq 0$ such that $\phi_a(x_m) \to r_a$ as $m \to \infty$. Now consider $m, p \in \mathbb{N}$, we have

$$\begin{aligned}
d_a(x_m,x_{m+p}) &\leq d_a(x_m,x_{m+1}) + d_a(x_{m+1},x_{m+2}) + d_a(x_{m+2},x_{m+3}) \\
&\quad + \cdots + d_a(x_{m+p-1},x_{m+p}) \\
&\leq q_a(\phi_a(x_{m-1}) - \phi_a(x_m)) + q_a(\phi_a(x_m) - \phi_a(x_{m+1})) + \\
&\quad q_a(\phi_a(x_{m+1}) - \phi_a(x_{m+2})) + \cdots + q_a(\phi_a(x_{m+p-2}) - \phi_a(x_{m+p-1})) \\
&= q_a(\phi_a(x_{m-1}) - \phi_a(x_{m+p-1})) \text{ for each } a \in \mathfrak{A}. \qquad (4)
\end{aligned}$$

This implies that $\{x_m\}$ is a Cauchy sequence in $X$, since $\phi_a \to r_a$ for each $a \in \mathfrak{A}$. By completeness of $X$, we have $x^* \in X$ such that $x_m \to x^*$ as $m \to \infty$. Since each $g_a(x)$ is $G$-lower semi continuous then we have $d_a(x^*,Tx^*) \leq \liminf_m d_a(x_m,Tx_m) = 0$ for each $a \in \mathfrak{A}$. This implies $x^* \in Tx^*$. □

**Theorem 2.2.** *Let $X$ be endowed with a graph $G$ and complete gauge structure $\{d_a : a \in \mathfrak{A}\}$ which is separating. Let $T : X \to CL(X)$ be an edge preserving mapping and let for each $a \in \mathfrak{A}$, we have $\phi_a : X \to [0,\infty)$ be a lower semi continuous function such that for each $x \in X$ and $y \in Tx$ with $(x,y) \in E$, we have*

$$d_a(x,y) \leq \phi_a(x) - \phi_a(y) \text{ for each } a \in \mathfrak{A}. \tag{5}$$

*Assume that the following conditions hold:*

**:** *(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E$;*

**:** *(iii) $T$ is $G$-continuous.*

*Then $T$ has a fixed point.*

*Proof.* By hypothesis (i), we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E$. From (5), we have

$$d_a(x_0,x_1) \leq \phi_a(x_0) - \phi_a(x_1) \text{ for each } a \in \mathfrak{A}. \tag{6}$$

Since $T$ is edge preserving, we have $(x_1,x_2) \in E$. Thus, continuing in same way we get a sequence $\{x_m\}$ in $X$ such that $(x_m,x_{m+1}) \in E$ and

$$d_a(x_m,x_{m+1}) \leq \phi_a(x_m) - \phi_a(x_{m+1}) \text{ for each } m \in \mathbb{N} \text{ and } a \in \mathfrak{A}.$$

This implies that for each $a \in \mathfrak{A}$ the sequence $\{\phi_a(x_m)\}$ is a nonincreasing sequence, there exists $r_a \geq 0$ such that $\phi_a(x_m) \to r_a$ as $m \to \infty$. Now consider $m, p \in \mathbb{N}$, we have

$$\begin{aligned} d_a(x_m,x_{m+p}) &\leq d_a(x_m,x_{m+1}) + d_a(x_{m+1},x_{m+2}) + d_a(x_{m+2},x_{m+3}) \\ &\quad + \cdots + d_a(x_{m+p-1},x_{m+p}) \\ &\leq \phi_a(x_m) - \phi_a(x_{m+1}) + \phi_a(x_{m+1}) - \phi_a(x_{m+2}) + \\ &\quad \phi_a(x_{m+2}) - \phi_a(x_{m+3}) + \cdots + \phi_a(x_{m+p-1}) - \phi_a(x_{m+p}) \\ &= \phi_a(x_m) - \phi_a(x_{m+p}) \text{ for each } a \in \mathfrak{A}. \end{aligned}$$

This implies that $\{x_m\}$ is a Cauchy sequence in $X$, since $\phi_a \to r_a$ for each $a \in \mathfrak{A}$. By completeness of $X$, we have $x^* \in X$ such that $x_m \to x^*$ as $m \to \infty$. As $T$ is $G$-continuous then we have $x^* \in Tx^*$. □

**Theorem 2.3.** *Let $X$ be endowed with a graph $G$ and complete gauge structure $\{d_a : a \in \mathfrak{A}\}$ which is separating. Let $T : X \to CL(X)$ be an edge preserving mapping and let for each $a \in \mathfrak{A}$, we have $\psi_a : X \to [0,\infty)$ be a upper semi continuous function such that for each $x, y \in X$ with $(x,y) \in E$, we have*

$$d_a(y,Ty) \leq \psi_a(x) - \psi_a(y) \text{ for each } a \in \mathfrak{A}. \tag{7}$$

*Assume that the following conditions hold:*

- **:** *(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E$;*
- **:** *(ii) there exists a sequence $\{q_a : q_a > 1\}_{a\in\mathfrak{A}}$ such that for each $x \in X$, we have $y \in Tx$ satisfying*

$$d_a(x,y) \leq q_a d_a(x,Tx) \text{ for each } a \in \mathfrak{A}.$$

- **:** *(iii) If $\{x_n\}$ is a sequence in $X$ such that $(x_n,x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $(x_n,x) \in E$ for each $n \in \mathbb{N}$.*

*Then $T$ has a fixed point.*

*Proof.* By hypothesis (i), we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E$. From (7), we have

$$d_a(x_1,Tx_1) \leq \psi_a(x_0) - \psi_a(x_1) \text{ for each } a \in \mathfrak{A}. \tag{8}$$

By using (ii) and (8), we have $q_a > 1$ for each $a \in \mathfrak{A}$ and $x_2 \in Tx_1$ such that

$$d_a(x_1,x_2) \leq q_a d_a(x_1,Tx_1) \leq q_a\psi_a(x_0) - q_a\psi_a(x_1) \text{ for each } a \in \mathfrak{A}. \tag{9}$$

Since $T$ is edge preserving, we have $(x_1,x_2) \in E$. Continuing in the same way we get a sequence $\{x_m\}$ in $X$ such that $(x_m,x_{m+1}) \in E$ and

$$d_a(x_m,x_{m+1}) \leq q_a d_a(x_m,Tx_m) \leq q_a\psi_a(x_{m-1}) - q_a\psi_a(x_m) \text{ for each } m \in \mathbb{N} and a \in \mathfrak{A}.$$

This implies that for each $a \in \mathfrak{A}$ the sequence $\{\psi_a(x_m)\}$ is a nonincreasing sequence, there exists $r_a \geq 0$ such that $\psi_a(x_m) \to r_a$ as $m \to \infty$. Now consider $m, p \in \mathbb{N}$, we have

$$\begin{aligned} d_a(x_m, x_{m+p}) &\leq d_a(x_m, x_{m+1}) + d_a(x_{m+1}, x_{m+2}) + d_a(x_{m+2}, x_{m+3}) \\ &\quad + \cdots + d_a(x_{m+p-1}, x_{m+p}) \\ &\leq q_a(\psi_a(x_{m-1}) - \psi_a(x_m)) + q_a(\psi_a(x_m) - \psi_a(x_{m+1})) + \\ &\quad q_a(\psi_a(x_{m+1}) - \psi_a(x_{m+2})) + \cdots + q_a(\psi_a(x_{m+p-2}) - \psi_a(x_{m+p-1})) \\ &= q_a(\psi_a(x_{m-1}) - \psi_a(x_{m+p-1})) \text{ for each } a \in \mathfrak{A}. \end{aligned} \tag{10}$$

This implies that $\{x_m\}$ is a Cauchy sequence in $X$, since $\phi_a \to r_a$ for each $a \in \mathfrak{A}$. By completeness of $X$, we have $x^* \in X$ such that $x_m \to x^*$ as $m \to \infty$. By hypothesis (iii), we have $(x_m, x^*) \in E$. From (7), we have

$$d_a(x^*, Tx^*) \leq \psi_a(x_m) - \psi_a(x^*) \text{ for each } a \in \mathfrak{A}.$$

Letting $m \to \infty$ in above inequality, we have $d_a(x^*, Tx^*) = 0$ for each $a \in \mathfrak{A}$. Thus, we have $x^* \in Tx^*$. □

## Conflict of Interests

The authors declare that there is no conflict of interests.

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