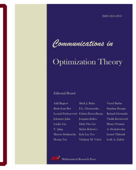
Commun. Optim. Theory 2015 (2015), Article ID 1 Copyright © 2015 Mathematical Research Press.

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org



ON EXISTENCE OF OPERATOR SOLUTIONS OF GENERALIZED VECTOR QUASI- VARIATIONAL INEQUALITIES

TIRTH RAM

Department of Mathematics, University of Jammu, Jammu 180 006, J & K, India

Abstract. In this work, we intend to present existence results for solutions of generalized operator vector quasi-variational inequalities involving multi-valued mapping in topological vector spaces both under compact and non-compact assumptions by employing 1-person game theorems. The results of this paper generalize and unify the corresponding results of several authors and can be considered as a significant extension of the previously known results.

Keywords: Multivalued mapping; Compactness; Vector variational inequality.

2010 AMS Subject Classification: 49J40, 47H04.

1. Introduction

Variational inequalities, which include many important problems in nonlinear analysis and optimization such as the Nash equilibrium problem, complementarity problems, vector optimization problems, fixed point problems, saddle point problems and game theory, recently have been studied as an effective and powerful tool for studying many real world problems which arise in economics, finance, image reconstruction, ecology, transportation, and network; see [1-7] and the references therein.

In this work, we present existence results for solutions of generalized operator vector quasi-variational inequalities involving multi-valued mapping in topological vector spaces both under compact and noncompact assumptions by employing 1-person game theorems. The results of this paper generalize and unify many recent results.

Received November 1, 2014

2. Preliminaries

Since its birth in the mid 1960's the area of variational inequality theory has experienced a phenomenal growth. It is now considered a field in its own right. In 1980, Giannessi [7] introduced the vector variational inequality (In short, VVI) in a finite dimesional Euclidean space, many authors have intensively studied (VVI) and its various extensions in abstact spaces; see, for example, [2,9,10] and the references therein

In 2002, Domokos and Kolumban [6] introduced and studied a class of operator variational inequalities (In short OVVI). These operator variational inequalities include not only scalar and vector variational inequalities as special cases, see for example [2] but also have sufficent evidence for their importance to study, see [6]. They designed (OVVI) to provide a suitable unified approach to several kinds of variational inequalities and vector variational inequality problems in banach spaces, and sucessively described these problems in a wider context of (OVVI). Inspired by their work, in recent papers [12,13], Kum and Kim developed the scheme of (OVVI) from single valued into general multi- valued settings.

Motivated and inspired by the work of Domokos and Kolumban [6] Kum and Kim [12,13]. In this paper, we prove existence results for solutions of generalized operator vector quasi-variational inequalities involving multivalued mapping in topological vector spaces both under compact and noncompact assumptions by employing 1- person game theorems.

We begin with taking a brief look at the standard definition of continuities of multi-valued mappings.Let X and Y be nonempty topological spaces and $T : X \to 2^Y$ be a multi-valued mappings. A multi-valued map $T : X \to 2^Y$ is said to *upper semi continuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$; and a multi-valued map is said to be *lower semi continuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V = \Phi$, there exists an open neighborhood U of x in X such that$T(y) \cap V \neq \Phi$ for each $y \in U$ and T is said to be continuous if it is both *lower semi continuous* and *upper semi continuous* It is also known that $T : X \to 2^Y$ is lower semi continuous if and only if for each closed set V in Y, the set $\{x \in X : T(x) \subset V\}$ is closed in X. we define partial ordering $\preceq_{C_x}$ on Y by $y \preceq_{C_x} z$ if and only if $z - y \in C(x)$. We shall write $y \prec_{C_x} z$ if and only if $z - y \in intC(x)$ in the case $intC(x) \neq \Phi$.

Let E be a Hausdorff topological vector space, X a nonempty convex subset of E, F is another Hausdorff topological vector space. A nonempty subset P of E is called *convex cone* if $\lambda P \subseteq P$, for all $\lambda > 0$ and $P + P = P$.

From now on, unless otherwise specified, we work under the following settings:

Let $L(E,F)$ be the space of all continuous linear operators from E to F, and X' a nonempty convex subset of $L(E,F)$, X a nonempty convex subset of E. Let $T : X' \to 2^E$ be a multi-valued map. Let $C : X' \to 2^F$ be a multi-valued map such that for each $f \in X'$, $C(f)$ is convex cone in F with $0 \notin C(f)$.

Then *Generalized Vector Variational Inequalities with Operator Solution* (In short, GOVVI) is defined as follows:

$$\text{Find } f_0 \in X^{'} \text{ such that} \forall f \in X^{'}, \exists x \in T(f_0)$$

$$\text{with } \langle f - f_0, x\rangle + H(f_0, g) \notin -int_F C(f_0).$$

When T is single valued and $H \equiv 0$, (GOVVI) reduces to (OVVI) due to Domokos and Kolumban [6]. As pointed out in [6], the notion (GOVVI) is motivated by the fact that the solutions are sought in the space of continuous linear operators. We also introduce the quasi version of (GOVVI) called the *Generalized Vector Quasi-variational Inequalities with Operator Solution* (In short, GOQVVI). Let $A : X^{'} \to 2^{X^{'}}$ be a multi-valued mapping. We define (GOQVVI) as follows:

$$\text{Find } f_0 \in X^{'} \text{ such that } f_0 \in cl_X A(f_0) \text{ and}$$

$$\forall f \in X^{'},\ \exists x \in T(f_0) \text{ with } \langle f - f_0, x\rangle + H(f_0, g) \notin -int_F C(f_0).$$

Now we need the following definitions.

The Graph of a multi-valued map $T : X^{'} \subset L(E,F) \to 2^F$ denoted by $G(T)$ is

$$G(T) = \{(f,x) \in X^{'} \times F : f \in X^{'},\ x \in T(f)\}.$$

The inverse T^{-1} of T is the multi-valued map from $R(T)$, the range of T, to $X^{'}$ defined by

$$f \in T^{-1}(x) \text{ iff } x \in T(f).$$

T is called *upper semi continuous* on $X^{'}$ if for each $f \in X^{'}$ and any open set V in F containing $T(f) \subseteq V$ for all $g \in U$.

The following one person game theorems will be used to establish the main results of this paper.

Theorem 2.1. *Let $\Gamma = (X, A, P)$ be a 1- person game such that*

- (i) *X is nonempty compact convex subset of a Hausdorff topological vector space,*
- (ii) *$A, cl_E(A) : X \to 2^X$ be a multi-valued mappings such that for each $f \in X, A(f)$ is nonempty convex set in X, for each $g \in X$, $A^{-1}(g)$ is open set in $X$and $cl_E A$ is upper semi continuous,*
- (iii) *$P : X \to 2^X$ be a multi-valued mappings such that for each $f \in X$, $f \notin coP(f)$ and for each $g \in X, P^{-1}(g)$ is open set X.*

Then there exist $f^* \in X$ such that $f^* \in cl_X A(f^*)$ and $A(f^*) \cap P(f^*) = \Phi$.

Theorem 2.2 *Let $\Gamma = (X, A, P)$ be a 1- person game such that*

- (i) *X is nonempty compact convex subset of a Hausdorff topological vector space,and $X^{'}$ be a nonempty compact subset of X,*
- (ii) *$A : X \to 2^{X^{'}}$ and $cl_E A : X \to 2^X$ be a multi-valued mappings such that for each $f \in X, A(f)$ is nonempty convex set, for each $g \in X^{'}$, $A^{-1}(g)$ is open set in $X$and $cl_E A$ is upper semi continuous,*

(iii) $P : X \to 2^{X'}$ *be a multi-valued mappings such that for each* $f \in X$, $f \notin coP(f)$ *and for each* $g \in X'$, $P^{-1}(g)$ *is open set* X.

Then there exist $f^* \in X$ *such that* $f^* \in cl_X A(f^*)$ *and* $A(f^*) \cap P(f^*) = \Phi$.

Remark 2.3 Theorem 2.1 is a special case of [4, Theorem 2] and Theorem 2.2 is a special case of [5, of Theorem 2].

3. Existence results in compact setting

In this section we establish some existence results under the compact assumptions. We need the following.

Lemma 3.1 [3] *Let* X *and* Y *be be topological vector space and let* $L(X,Y)$ *be equipped with the uniform convergence topology* δ. *then the bilinear form* $\langle .,. \rangle : L(X,Y) \times X \to Y$ *is continuous on* $(L(X,Y),\delta) \times X$.

Now we are ready to establish the main results of this paper on the existence of a solution of (GOQVVI).

Theorem 3.1. *Let* X *is nonempty compact convex subset of a Hausdorrf topological vector space* E*,and* F *be an ordered be an Hausdorff topological vector space.Let* $H : X \times X \to F$ *be vector valued bifunction and* $T : X \to 2^F$ *and* $A : X \to 2^{L(E,F)}$ *be a multi-valued mappings. Assume that*

(i) *for each* $f \in X, H(f,f) = 0$,

(ii) H *is continuous in the first argument and* $C(f)-$ *convex in the second argument,*

(iii) *the mapping* $W : X \to 2^{X'}$ *defined by* $W(f) = F \setminus (-int_F C(f))$ *for each* $f \in K$, *has a closed graph in* $X \times F$,

(iv) *for each* $f \in X, C(f)$ *is closed , convex and pointed cone in* F *such that* $int_F C(f)$ *is nonempty,*

(v) *for each* $f \in X, A(f)$ *is nonempty convex and for each* $g \in X, A^{-1}$ *is open in* X*.Also* $cl_X A : X \to 2^X$ *is upper semi continuous.*

Then there exist $f^* \in X$ *such that for all* $g \in A(f^*), \exists x \in T(f^*)$ *such that*

$$f^* \in cl_X A(f^*) \text{ and} \langle g - f^*, x \rangle + H(f^*, g) \notin -int_F C(f^*).$$

Proof. Define a multi-valued mapping $P : X \to 2^X$ as

$$P(f) = \{g \in X : \langle T(f), g - f \rangle + H(f,g) \subseteq -int_F C(f)\} \text{ for all} f \in X.$$

We show that $f \notin coP(f)$, for each $f \in X$. Suppose that $f \in coP(f)$ for some $f \in X$. Then there exists $f_0 \in X$ such that $f_0 \in coP(f_0)$. This implies that f_0 can be expressed as

$$f_0 = \sum_{i \in I} \lambda_i g_i, \text{ with } \lambda_i \geq 0, \sum_{i \in I} \lambda_i = 1,$$

where $\{g_i : i \in N\}$ be a finite subset of X, $I \subset N$ be arbitrary nonempty subset and $\mathbb{N}$ denotes the set of natural numbers. This follows that

$$\langle T(f_0), g_i - f_0\rangle + H(f_0, g_i) \subseteq -int_F C(f_0)\} \text{ for all } i = 1, 2, \cdots n.$$

Therefore for each $x \in T(f_0)$, we have

$$\sum_{i\in I} \lambda_i[\langle g_i - f_0, x\rangle + H(f_0, g_i)] \in -int_F C(f_0). \tag{1}$$

$$0 = \langle f_0 - f_0, x\rangle + H(f_0 . f_0) \preceq \sum_{i\in I} \lambda_i[\langle g_i - f_0, x\rangle + H(f_0, g_i)].$$

Hence, we have

$$\sum_{i\in I} \lambda_i[\langle g_i - f_0, x\rangle + H(f_0, g_i)] \in C(f_0). \tag{2}$$

From equation (3.1) and (3.2) we have

$$\sum_{i\in I} \lambda_i[\langle g_i - f_0, x\rangle + H(f_0, g_i)] \in \{-in_F C(f_0)\} \cap C(f_0) = \Phi,$$

which is a contradiction. Now we show that for each $g \in X$ the set

$$\begin{aligned} [P^{-1}(g)]^c &= \{f \in X : g \in P(f)\} \\ &= \{f \in X : \langle T(f), g - f\rangle + H(f, g) \subseteq -int_F C(f)\} \end{aligned}$$

is open in X, which is equivalent to showing that the set

$$\begin{aligned} [P^{-1}(g)]^c &= X \setminus P^{-1}(g) \\ &= \{f \in X : \langle T(f), g - f\rangle + H(f, g) \nsubseteq -int_F C(f)\} \\ &= \{f \in X : \exists x \in T(f) \text{ such that } \langle g - f, x\rangle + H(f, g) \notin -int_F C(f)\} \end{aligned}$$

is closed in X. For this purpose, let $\{f_\lambda\}_{\Lambda\in\Lambda}$ be a net $[P^{-1}(g)]^c$ converging to $h \in X$. Then for each λ there is a $x_\lambda \in T(f_\lambda)$ such that

$$\langle g - f_\lambda, x_\lambda\rangle + H(f_\lambda, g) \in W(f_\lambda).$$

Since $T(f)$ is compact, without loss of generality we may assume that x_λ converges to some $x \in T(f)$. By (ii) H is continuous in the first argument and by Lemma 3.1 we have for each $g \in X$ and for all $x \in T(f)$, $f \mapsto \langle g - f, x\rangle$ is continuous. Since W is closed graph in $X \times F$ by assumption (iii) we have

$$\langle g - h, x\rangle + H(h, g) \in W(h),$$

that is, $\langle g - h, x\rangle + H(h, g) \notin -int_F C(h)$. Hence $h \in [P^{-1}(g)]^c$. From assumption (v), it follows that all the hypothesis of Theorem 2.1 are satisfied. Hence there exists $f^* \in X$ such that

$$f^* \in cl_X A(f^*) \text{ and } A(f^*) \bigcap P(f^*) = \Phi.$$

which implies that there exists $f^* \in X$ such that for all $g \in A(f^*)$ there is $x \in T(f^*)$ such that

$$f^* \in cl_X A(f^*) \text{ and} \langle g - f^*, x\rangle + H(f^*, g) \notin -int_F C(f^*).$$

The proof is complete.

Corollary 3.2. *Let X is nonempty compact convex subset of a Hausdorrf topological vector space E,and F be an ordered be an Hausdorrf topological vector space. $T : X \to 2^{L(E,F)}$ be a multivalued mapping with compact values and Let $G : X \to F$ be continuous convex vector valued valued function. Let $C : X \to 2^F$ and $A : X \to 2^X$ be a multi-valued mappings. Assume that conditions (i)-(v) of Theorem 3.1 holds. Then there exists $f^* \in X$ such that for all $g \in A(f^*)$ there is $t^* \in T(f^*)$ such that*

$$f^* \in cl_X A(f^*) \text{ and} \langle t^*, g - f^*\rangle + G(g) - G(f^*, g) \notin -int_F C(f^*).$$

Proof. If we set $H(f, g) = G(g) - G(f)$, then we see that all the assumptions of Theorem 3.1 holds and hence the conclusion follows from Theorem 3.1.

4. Existence results in noncompact setting

For the noncompact case we need the concept of the escaping sequence introduced in Border [1].

Definition 4.1. Let E be a topological vector space and X a subset of E such that $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact sets in the sense that $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$. A sequence $\{f_n\}_{n=1}^{\infty}$ in K is said to be escaping sequence from X (relative to $\{X_n\}_{n=1}^{\infty}$) if for each n there is an M such that $k \geq M, f_k \notin X_n$.

Theorem 4.1. *Let X is nonempty subset of a Hausdorrf topological vector space E,and $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty, compact and convex subsets of X. Let $F, H, , T, C, W$ and A be the same as in Theorem 3.1 and satisfies all the conditions.In additions, suppose that for each sequence $\{f_n\}_{n=1}^{\infty}$ in X with $f_n \in X_n, n \in \mathbb{N}$ which is escaping from X relative to $\{X_n\}_{n=1}^{\infty}$, there exists $m \in \mathbb{N}$ and $g_m \in X_m \bigcup A(f_m)$ such that for each $f_m \in cl_X A(f_m)$, there is $t_m \in T(f_m)$ such that*

$$\langle t_m, g_m - f_m\rangle + H(f_m, g_m) \in -int_F C(f_m). \tag{4.1}$$

Then there exists $f^* \in X$ such that for all $g \in A(f^*)$ there is $t^* \in T(f^*)$ such that

$$f^* \in cl_X A(f^*) \text{ and } \langle t^*, g - f^*\rangle + H(f^*, g) \notin -int_F C(f^*).$$

Proof. Since for each $n \in \mathbb{N}, X_n$ is compact and convex set in X, Theorem 3.1 implies that for all $n \in \mathbb{N}$, there is $t_n \in T(f_m)$ such that

$$f_n \in cl_X A(f_n) \text{ and } \langle t_n, h - f_n \rangle + H(f_n, h) \notin -int_F C(f_n). \tag{4.2}$$

Suppose that the sequence be $\{f_n\}_{n=1}^{\infty}$ escaping from X relative to $\{f_n\}_{n=1}^{\infty}$. By Assumption 4.1, there exists $m \in \mathbb{N}$ and $h_m \in X_m \bigcup A(f_n)$ such that

$$\langle t_m, h_m - f_m \rangle + H(f_m, h_m) \notin -int_F C(f_m),$$

which contradicts (4.2).Hence $\{f_n\}_{n=1}^{\infty}$ is not an escaping sequence from X relative to $\{X_n\}_{n=1}^{\infty}$. Since T is multi-valued mapping with compact values, thus using the arguments similar to those used in proving [8, Theorem 3.2] and [9, Theorem 2], there exists $r \in \mathbb{N}$ and $f^* \in X_r$ such that $f_n \to f^*$ and there is $t \in T(f^*)$ such that

$$\langle t, g - f \rangle + H(f^*, g) \in W(f^*).$$

Since $cl_X A : X \to 2^X$ is upper semicontinuous with compact values, hence there exists $f^* \in A(f^*)$ there is $t^* \in T(f^*)$ such that

$$f^* \in cl_X A(f^*) \text{ and } \langle t^*, g - f^* \rangle + H(f^*, g) \notin -int_F C(f^*).$$

The proof is complete.

Theorem 4.2. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E,and X' be a nonempty compact subset of X. Let F be an ordered Hausdorff topological vector space. Let $H : X \times X \to F$ be a vector- valued bifunction, $T : X \to 2^{L(E,F)}$ a multi-valued mapping with the compact values and $C : X \to 2^F$ a multi-valued mapping such that for each $f \in X, C(f)$ is closed , convex and pointed cone in F with $int_F C(f) \neq \Phi$. Let $A, cl_X A : X \to 2^F$ be a multi-valued mappings such that for each $f \in X, A(f)$ is nonempty, for each $g \in X, A^{-1}(g)$ is open in X and $cl_X A$ is upper semicontinuous. Suppose that conditions (i)-(iii) of Theorem 3.1.Then there exists $f^* \in X$ such that for all $g \in A(f^*)$ there is $t^* \in T(f^*)$ such that*

$$f^* \in cl_X A(f^*) \textit{ and } \langle t^*, g - f^* \rangle + H(f^*, g) \notin -int_F C(f^*).$$

Proof. Define a multi-valued mapping $P : X \to 2^X$ as

$$P(f) = \{g \in X : \ \langle T(f), g - f \rangle + H(f, g) \subseteq -int_F C(f)\} \text{ for all} f \in X.$$

Then using the argument similar to those used in proving Theorem 3.1, we have $f \in coP(f)$ for each $f \in X$ and P^{-1} is open for each $g \in X'$. Thus all the conditions of Theorem 2.2 are satisfied. Hence there exists f^X such that

$$f^* \in cl_X A(f^*) \text{ and } A(f^*) \cap P(f^*) = \Phi.$$

Which implies that there exists $f^* \in X$such that for all $g \in A(f^*)$ there is $t^* \in T(f^*)$ such that

$$f^* \in cl_X A(f^*) \text{ and } \langle t^*, g - f^* \rangle + H(f^*, g) \notin -int_F C(f^*).$$

The proof is complete.

Conflict of Interests

The author declares that there is no conflict of interests.

References

[1] K.C. Border, Fix Point Theorems with Applications to Economics and game Theory, Cambridge University Press, Cambridge, 1985.

[2] G.Y. Chen, Existence of solutions for a vector variational inequality: an extension of Hartman- Stampachhia theorem, J. Optim. Theory Appl. 74 (1992), 445-456.

[3] X.P. Ding, The generalized vector quasi variational-like inequalities, Comput. Math. Appl. 37 (1999), 57-67.

[4] X.P. Ding, W.K Kim, K.K.Tan, Equilibria of generalized games with L- majorized preferences, Int. J. Math. Math. Sci. 17 (1994), 783-790.

[5] X.P. Ding, W.K. Kim, K.K. Tan, Equilibria of noncompact generalized games with L- majorized preferences, J. Math. Anal. Appl. 164 (1992), 508-517.

[6] A. Domokos, J. Kolumban, Variational inequalities with operator solutions, J. Global Optim. 23 (2002), 99-110.

[7] F. Giannessi,Theorems of alternatives, quadratic programms and complimentarity problems. In Cottle, R.W., Giannessi, F and Lions, J.L.(eds.), Variational and Complimentarity Problems, John Wiley and Sons, Chichester, England,pp.151-186.

[8] A. Khaliq, Generalized nonlinear vector quasi variational-like inequalities, Nonlinear Anal. Forum 8 (2003), 115-126.

[9] A. Khaliq, S. Krishan, Vector quasi equilibrium problems, Bull. Aust. Math. Soc. 68 (2003), 295-302.

[10] P.Q Khanh, L.M. Luu, On the Existence of Solutions to vector quasi-variational inequalities and quasi compolimentarity with applications to traffic network equilibria, J. Optim. Theory Appl. 123 (2004), 533-548.

[11] I.V. Konnov, J.C. Yao, On the generalized vector variational inequality problem, J. Math. Anal. Appl. 206 (1997), 42-58.

[12] S. Kum, W.K. Kim, Generalized variational and quasi-variational inequalities with operator solutions in a toplogical vector space, J. Optim. Theory Appl. 32 (2005), 581-595.

[13] S. Kum, W.K. Kim, Applications of generalized vector variational and quasi-variational inequalities with operator solutions, J. Global Optim. 133 (2007), 65-75.