Tolerance Analysis in Straight-Build Mechanical Assemblies Using a Probabilistic Approach-2D Assembly

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ABSTRACT

Product quality in mechanical assemblies is determined by the dispersion of manufacturing variance during the structure building. This paper focuses on straight-build assembly and proposes a probabilistic approach based on connective assembly model to analyze the effect of individual component variations on the eccentricity of the straight-build assembly. The probabilistic approach calculates the pdf (probability density function) of key assembly variation of rotor assembly of high speed rotating machines. The probabilistic approach considers two straight-build scenarios: (i) Best Build; and (ii) Direct Build, for 2D (Two-Dimensional) "axi-symmetric" assemblies. Numerical examples are presented to investigate the probabilistic approach for its efficiency and accuracy in comparison to MCS (Monte Carlo simulation).

Key Words: **Monte Carlo Simulation; Assembly Tolerance Analysis; Variance Dispersion in Straight Build Assemblies.**

1. INTRODUCTION

Variations always exist in mechanical components due to imperfections in the manufacturing process. These variations are observed as small deviations in the dimensions of individual components from their nominal design. These variations disperse and accumulate as components are assembled together. The accumulated variations result in incorrect dimensions of the final assembly [1].

Various tools for analyzing tolerance stack-up in an assembly have been reported in literature. These include WOW (Worst-on-Worst) [2-4] and RSS (Root Sum Square) [5-8] methods of assembly tolerance analysis. These methods are not suitable for geometric tolerances analysis [8]. In particular, the WOW analysis gives results that are not optimistic, and RSS analysis gives low probability result for those occurring in the worst-case combination. There is still a need of taking account the randomness of component features for the determination of the probability of assembly failure.

The MCS method is the most common and popular method for statistical tolerance analysis, where, the random dimensions are generated based on known statistical distributions [1]. The assembly KC (Key Characteristic) is found for each part values. In MCS, to obtain correct estimate of potentially small probabilities of failure, a large number of samples is to be generated. Authors who have discussed the MCS method include: Grossman [9], Nadler [10], DeDoncker and Spencer [11], Turner, et. al. [12], Pandit and Starkey [13], Turner [14], and Hussain, et. al. [15]. Research of these authors reveals that MCS is

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computationally intensive, and if the simulations are performed with an inadequate number of samples, MCS method will result in inaccurate estimation. An alternative approach to using MCS to predict the probability of an assembly failure is the FORM (First Order Reliability Method) [16]. This method approximates the probability of failure by linearising the limit state function at the most likely failure point. This method is not well suited for analyzing mechanical assemblies, because the limit state function is likely to have multiple failure points.

In this paper a probability based analysis for variation dispersion in straight-build assemblies is proposed. The analytical models are developed by reducing connective assembly model [17] into a linearised form. These models are used to analyze BBA (Best Build Assembly) and DBA (Direct Build Assembly) environments. The BBA minimizes the eccentricity of the build by taking advantage of the axi-symmetric property. The DBA process ignores the control of the eccentricity of assembly.

To understand the benefits of controlling the eccentricity, the component variance is considered to be random and statistical variations in the eccentricity are predicted. Hussain, et. al. [18] used non-linear connective models in conjunction with the Monte Carlo simulation method to investigate different optimization strategies for straight-build, including BBA, which was referred to as Table axis based combinatorial approach. In this paper, the linearised assembly models are used for applying a probability based approach to determine the pdf for the eccentricity, using BBA and DBA. The pdf is then used to check that eccentricity remains below a set value.

Throughout this paper, the components are considered to be nominally 2D axi-symmetric structures, like those considered by Hussain, et. al. [18]. This assumption is made to aid visualisation of the problem and to simplify the presentation. Section 2 presents an overview of the linearised model used in straight build assembly, whilst Section 3 applies a probabilistic approach to Best Build and DBA. In Section 4 the proposed probabilistic approach is compared with MCS, and Section 5 summarises the conclusions from the study.

2. MODELLING FOR STRAIGHT-BUILD ASSEMBLY

For 2D components, the transformation matrix T representing the geometric relationship between mating features of a 2D component as shown in Fig. 1 is given by:

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}^{T}_{[2\times1]} & 1 \end{bmatrix} \tag{1}$$

Where **R** and **p** are defined by:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \mathbf{p} = \begin{bmatrix} X \\ Y \end{bmatrix} \tag{2}$$

Where X and Y defines the translation of upper feature relative to lower feature along X and Y axis, and θ represents the orientation of the upper feature relative to the lower feature.

The assembly model uses the concept of part mating theory [2]. Fig. 2 shows an example for a two part axi-symmetric assembly. Here, the mating features are defined by coordinate reference frames: $O_0X_0Y_0$, $O_1X_1Y_1$ and $O_2X_2Y_2$, where $O_0X_0Y_0$ is global reference frame.

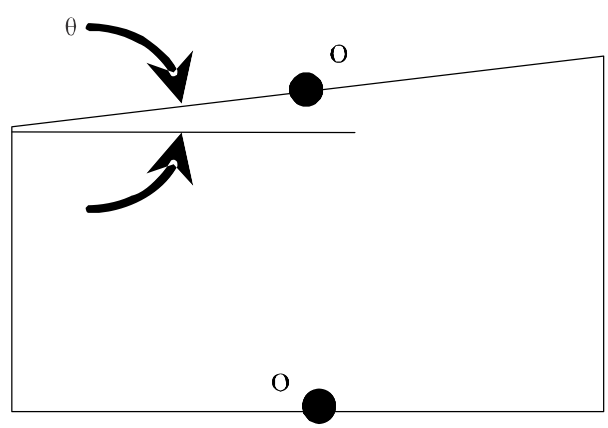


FIG. 1. GEOMETRIC RELATIONSHIP BETWEEN MATING FEATURES OF A 2D COMPONENT.

Fig. 2 shows the case, when no manufacturing variations are present. Therefore, transformation matrix representing nominal dimensions of Part-1 can be given as:

$$\mathbf{T}_{0-1}^{N} = \begin{bmatrix} \mathbf{R}_1^N & \mathbf{p}_1^N \\ \mathbf{0}_{[2\times1]}^T & 1 \end{bmatrix} \tag{4}$$

Similarly, transformation matrix $\mathbf{T}_{1-2}^{N}$ representing nominal dimensions of Component 2 can be expressed as:

$$\mathbf{T}_{1-2}^{N} = \begin{bmatrix} \mathbf{R}_2^N & \mathbf{p}_2^N \\ \mathbf{0}_{[2\times1]}^T & 1 \end{bmatrix} \tag{5}$$

Using part mating theory, two components are assembled by joining coordinate frame $O_0X_1Y_1$ on the upper surface of Part-1 to same coordinate frame on lower surface of Part-2 (Fig. 2(b)), and transformation matrix $\mathbf{T}_{0-2}^{N}$ expressing the component-to-component relationships is given by:

$$\mathbf{T}_{0-2}^{N} = \mathbf{T}_{0-1}^{N}\mathbf{T}_{1-2}^{N} \tag{6}$$

$\mathbf{T}_{0-2}^{N}$ can be expressed in the same form as Equations (1,4,5), such that:

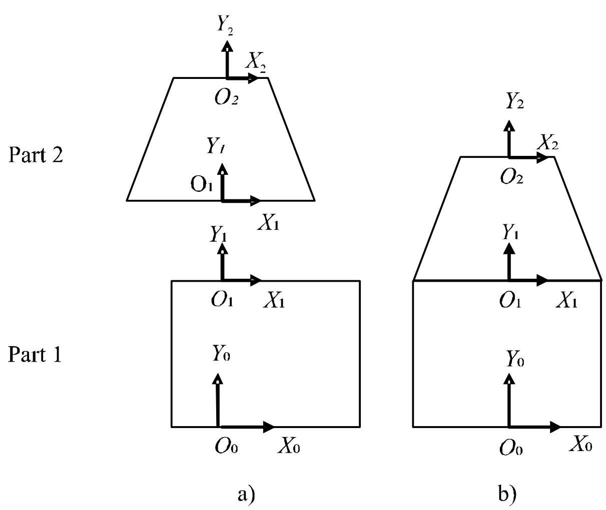


FIG. 2. AN EXAMPLE OF A 2 COMPONENT ASSEMBLY, (i) UNASSEMBLED, (ii) ASSEMBLED

$$\mathbf{T}_{0-2}^{N} = \begin{bmatrix} \mathbf{R}_{12}^N & \mathbf{p}_{12}^N \\ \mathbf{0}_{[2\times1]}^T & 1 \end{bmatrix} \tag{7}$$

where $\mathbf{R}_{12}^N$ is a 2x2 matrix representing nominal orientation of frame 2 with respect to frame *0*, and $\mathbf{p}_{12}^N$ represents 2x1 displacement vector having the nominal location of origin O_2 with respect to origin O_0.

As Components 1 and 2 are axi-symmetric components, the nominal assembly is axi-symmetric such that the nominal orientation of each reference frame is zero (i.e. θ=0° in Equation (2) and $\mathbf{R}_1^N = \mathbf{R}_2^N = \mathbf{I}$) and the nominal horizontal offset between reference frames is zero (i.e. X=0 in Equation (3)). Hence, Equations (4,5,7) may be written as:

$$\mathbf{T}_{0-1}^{N} = \begin{bmatrix} \mathbf{I} & \mathbf{p}_1^N \\ \mathbf{0}_{[2\times1]}^T & 1 \end{bmatrix} \tag{8}$$

$$\mathbf{T}_{1-2}^{N} = \begin{bmatrix} \mathbf{I} & \mathbf{p}_2^N \\ \mathbf{0}_{[2\times1]}^T & 1 \end{bmatrix} \tag{9}$$

$$\mathbf{T}_{0-2}^{N} = \begin{bmatrix} \mathbf{I} & \mathbf{p}_1^N + \mathbf{p}_2^N \\ \mathbf{0}_{[2\times1]}^T & 1 \end{bmatrix} \tag{10}$$

where **I** is the 2x2 identity matrix and

$$\mathbf{p}_1^N = \begin{bmatrix} 0 \\ Y_1 \end{bmatrix} \tag{11}$$

$$\mathbf{p}_2^N = \begin{bmatrix} 0 \\ Y_2 \end{bmatrix} \tag{12}$$

Equation (10) for the two-component axi-symmetric assembly can be generalized readily to n axi-symmetric components. Now, the generalized transformation matrix $\mathbf{T}_{0-n}^{N}$ for the nominal connecting feature on lower end of the first component and the upper end of n[th] component may be written as:

$$\mathbf{T}_{0-n}^{N} = \begin{bmatrix} \mathbf{I} & \sum_{i=1}^{n} \mathbf{p}_i^N \\ \mathbf{0}_{[2\times1]}^{T} & 1 \end{bmatrix} \tag{13}$$

where

$$\mathbf{p}_i^N = \begin{bmatrix} 0 \\ Y_i \end{bmatrix} \tag{14}$$

The above transformations have been developed for nominal components, and the nominal assembly.

If manufacturing variations are considered in the assembly components, the transformation matrix $\mathbf{T}_{0\text{-}1}$ which relates frame 1-0 may be written as:

$$\mathbf{T}_{0-1} = \mathbf{T}_{0-1}^{N}\mathbf{D}_1 \tag{15}$$

where $\mathbf{T}_{0-1}^{N}$ is given by Equation (4) and Equations (16-18) are based on Equations (2-3) and make the practical assumption that the rotation error is little. Transformation matrix $\mathbf{D}_1$ consider the manufacturing variations present in Part-1.

$$\mathbf{D}_1 = \begin{bmatrix} \mathbf{I} + \mathbf{dR}_1 & \mathbf{dp}_1 \\ \mathbf{0}_{[2\times1]}^{T} & 1 \end{bmatrix} \tag{16}$$

$$\mathbf{dR}_1 = \begin{bmatrix} 0 & -d\theta_1 \\ d\theta_1 & 0 \end{bmatrix} \tag{17}$$

$$\mathbf{dp}_1 = \begin{bmatrix} dX_1 \\ dY_1 \end{bmatrix} \tag{18}$$

Likewise, manufacturing variations in Part-2 change frame 2 the orientation and location with respect to frame 1, and transformation matrix $\mathbf{T}_{1\text{-}2}$ relating frame 2-1 may be written as:

$$\mathbf{T}_{1-2} = \mathbf{T}_{1-2}^{N}\mathbf{D}_2 \tag{19}$$

$\mathbf{T}_{1-2}^{N}$ is given by Equation (5) and transformation matrix $\mathbf{D}_2$ is given by:

$$\mathbf{D}_2 = \begin{bmatrix} \mathbf{I} + \mathbf{dR}_2 & \mathbf{dp}_2 \\ \mathbf{0}_{[2\times1]}^{T} & 1 \end{bmatrix} \tag{20}$$

$$\mathbf{dR}_2 = \begin{bmatrix} 0 & -d\theta_2 \\ d\theta_2 & 0 \end{bmatrix} \tag{21}$$

$$\mathbf{dp}_2 = \begin{bmatrix} dX_2 \\ dY_2 \end{bmatrix} \tag{22}$$

If Parts 1 and 2 are assembled, the transformation matrix $\mathbf{T}_{0\text{-}2}$ between feature 0 on Part 1 and feature 2 on Part 2 may be stated as:

$$\mathbf{T}_{0\text{-}2} = \mathbf{T}_{0\text{-}1}\text{-}\mathbf{T}_{1\text{-}2} \tag{23}$$

Using Equations (15-23), it can be shown that $\mathbf{T}_{0\text{-}2}$ can be expressed in the form:

$$\mathbf{T}_{0-2} = \begin{bmatrix} \mathbf{R}_{12} & \mathbf{p}_{12} \\ \mathbf{0}_{[2\times1]}^{T} & 1 \end{bmatrix} \tag{24}$$

where

$$\mathbf{R}_{12} = \mathbf{R}_1^N\mathbf{R}_2^N + \mathbf{R}_1^N\mathbf{dR}_1\mathbf{R}_2^N + \mathbf{R}_1^N\mathbf{R}_2^N\mathbf{dR}_2 + \mathbf{R}_1^N\mathbf{dR}_1\mathbf{R}_2^N\mathbf{dR}_2 \tag{25}$$

$$\mathbf{p}_{12} = \left(\mathbf{R}_1^N + \mathbf{R}_1^N\mathbf{dR}_1\right)\left(\mathbf{p}_2^N + \mathbf{R}_2^N\mathbf{dp}_2\right) + \left(\mathbf{p}_1^N + \mathbf{R}_1^N\mathbf{dp}_1\right) \tag{26}$$

Assuming that the dimensional variations (translational and rotational) are small, such that matrix-vector multiplications $\mathbf{dR}_1\mathbf{dR}_2$ and $\mathbf{dR}_1$dp1 are negligibly small, and noting that for nominally axi-symmetric components $\mathbf{R}_1^N = \mathbf{R}_2^N = \mathbf{I}$, Equations (25-26) can be approximated as:

$$\mathbf{R}_{12} \approx \mathbf{I} + \mathbf{dR}_1 + \mathbf{dR}_2 \tag{27}$$

$$\mathbf{p}_{12} \approx \left(\mathbf{p}_1^N + \mathbf{dp}_1\right) + \left(\mathbf{p}_2^N + \mathbf{dp}_2\right) + \mathbf{dR}_1\mathbf{p}_2^N \tag{28}$$

Substituting Equations (27-28) into Equation (24) gives:

$$\mathbf{T}_{0-2}^{Approx} = \begin{bmatrix} \mathbf{I} + \sum_{i=1}^{2}\mathbf{dR}_i & \sum_{i=1}^{2}\left(\mathbf{p}_i^N + d\mathbf{p}_i\right) + \mathbf{dR}_1\mathbf{p}_2^N \\ \mathbf{0}_{[2\times1]}^T & 1 \end{bmatrix} \tag{29}$$

Equation (29) is an approximate transformation matrix for the two-component axi-symmetric assembly including the presence of dimensional variations, and is based on neglecting second-order products of $\mathbf{dR}_1$,$\mathbf{dR}_2$ and $\mathbf{dp}_1$ which is consistent with the "small" rotation approximation made in the definition of Equations (16,20).

The process used to develop the exact transformation matrix (Equation (24)) and the approximate transformation matrix (Equation (29)) for the two-component example can be generalized readily to *n* axi-symmetric components. The generalized exact transformation matrix $\mathbf{T}_{0\text{-}n}$ between the connecting feature on lower surface of first part and the upper surface of n[th] part can be written as:

$$\mathbf{T}_{0-n} = \prod_{i=1}^{n}\mathbf{T}_{(i-1)-i} \tag{30}$$

where $\mathbf{T}_{(i,1)\text{-}i}$ is the transformation matrix relating frame (i-1) to frame *i* on Component *i*. In practice, it is easier to calculate Equation (30) numerically by multiplying together the transformation matrices describing the mating features on each component, rather than developing analytical expressions, as was done for Equation (24). This approach is used in the numerical examples section to determine "exact" results for the eccentricity of the build.

Using the same approach as that to derive Equation (29), it may be seen readily that generalized approximate transformation matrix for an assembly may be written as:

$$\mathbf{T}_{0-n}^{Approx} = \begin{bmatrix} \mathbf{I} + \sum_{i=1}^{n}\mathbf{dR}_i & \sum_{i=1}^{n}(\mathbf{p}_i^N + d\mathbf{p}_i) + \sum_{i=2}^{n}\left(\sum_{j=2}^{i}\mathbf{dR}_{j-1}\right)\mathbf{p}_i^N \\ \mathbf{0}_{[2\times1]}^T & 1 \end{bmatrix} \tag{31}$$

Here, all products of order greater than one of $\mathbf{dR}_k$(k=1,…,n) and $\mathbf{dp}_l$ (l=1,..,n) have been neglected. Equation (31) may now be written as:

$$\mathbf{T}_{0-n}^{Approx} = \begin{pmatrix} 1 & -\sum_{i=1}^{n}d\theta_i & \sum_{i=1}^{n}dX_i - \sum_{i=1}^{n-1}d\theta_i\left(\sum_{j=i+1}^{n}Y_j\right) \\ \sum_{i=1}^{n}d\theta_i & 1 & \sum_{i=1}^{n}(Y_i + dY_i) \\ 0 & 0 & 1 \end{pmatrix} \tag{32}$$

Where dX_i and dY_i are the translational errors and $d\theta_i$ is the orientation error for the ith part. By comparing the entries in matrix Equation (32) with Equations (1-3,13), it can be deduced that: (i) the assembly orientation error for an *n* part assembly is sum of the component orientation variations $\left(d\theta_n^{assembly} = \sum_{i=1}^{n}d\theta_i\right)$; (ii) the assembly height error for an n-component assembly is the sum of component height variations $\left(dY_n^{assembly} = \sum_{i=1}^{n}dY_i\right)$; and (iii) the assembly horizontal error for an n component assembly depends on the horizontal error for all *n* components $\left(\sum_{i=1}^{n}dX_i\right)$ and the product of the orientation errors and nominal vertical component heights of the previous *n-1* components $\left(\sum_{i=1}^{n-1}d\theta_i\left(\sum_{j=i+1}^{n}Y_j\right)\right)$.

In this work, the horizontal assembly error governs the quality of the build, and for an n component assembly this is given by:

$$dX_n^{assembly} = \sum_{i=1}^{n}dX_i - \sum_{i=1}^{n-1}d\theta_i\left(\sum_{j=i+1}^{n}Y_j\right) \tag{33}$$

As expected, this error does not depend on component height variations dY_i. Equation (33) is used as the basis for predicting the build eccentricity using 3 straight-build scenarios: (i) Direct Build; and (ii) Best Build. Each of these scenarios is described next.

Direct-Build Assembly

Here direct build eccentricity ε_n^{Direct} is obtained by taking the absolute value of Equation (33), i.e.:

$$\varepsilon_n^{Direct} = \left|dX_{assembly}^{n}\right| = \left|\sum_{i=1}^{n} dX_i - \sum_{i=1}^{n-1} d\theta_i \left(\sum_{j=i+1}^{n} Y_j\right)\right| \quad (34)$$

Best Build Assembly

In BBA method each part is rotated by 180° about its symmetrical axis and that configuration of assembly is selected which results in minimum eccentricity. The operation of rotating a component has the effect of changing the sign of its contribution to the eccentricity of the assembly and is equivalent to changing the signs for dX_i and $d\theta_i$ simultaneously in Equation (33), where $i=2, 3, \ldots, n-1$. The build eccentricity for one, two, three and n-component assemblies are considered below. The results presented are based on re-ordering Equation (33), so that the signs of the terms involving dX_i and $d\theta_i$ can be changed easily, and then determining the minimum absolute value.

For a 1-part assembly, minimum eccentricity is:

$$\varepsilon_1^{\min} = \left|dX_1\right| \quad (35)$$

This is the same as the result obtained using Direct Build (Equation (34) with $n=1$). For a 2-part assembly, the minimum eccentricity is:

$$\varepsilon_2^{\min} = \min\left(\left|\left(dX_1 - Y_2 d\theta_1\right) \pm dX_2\right|\right) = \left|\left|dX_1 - Y_2 d\theta_1\right| - \left|dX_2\right|\right| \quad (36)$$

For a 3 part assembly, the minimum eccentricity is:

$$\varepsilon_3^{\min} = \min\left(\left|\left(dX_1 - (Y_2 + Y_3) d\theta_1\right) \pm (dX_2 - Y_3 d\theta_2)\right) \pm dX_3\right|\right)$$
$$= \left|\left|\left|dX_1 - \left(Y_2 + Y_3\right) d\theta_1\right| - \left|dX_2 - Y_3 d\theta_2\right|\right| - \left|dX_3\right|\right| \quad (37)$$

For an n-part assembly, the minimum eccentricity is:

$$\varepsilon_n^{\min} = \min\left(\left|(dX_1 - (\sum_{i=2}^{n} Y_i) d\theta_1) \pm (dX_2 - (\sum_{i=3}^{n} Y_i) d\theta_2) \pm \ldots \pm dX_n\right|\right) \quad (38)$$

3. A PROBABILITY BASED METHOD

The dimensional variations dX_i and $d\theta_i$ are modeled as random variables and pdf's are produced from DBA and BBA. The pdf's so obtained give an insight of improvements of Best Build over Direct Build approach.

The part variance (dX_i and $d\theta_i$) are considered to be independent, zero-mean Gaussian random variables having known standard deviations i.e.

$$E[dX_i] = 0, E[dX_i^2] = \sigma_{Xi}^2$$

$$E[dX_i] = 0\ , \ E[dX_i^2] = \sigma_{Xi}^2$$

$$E[dX_i d\theta_j] = 0\ , \ E[d\theta_i] = 0$$

$$E[d\theta_i^2] = \sigma_{\theta i}^2\ , \ E[d\theta_i] = 0$$

$$E[d\theta_i^2] = \sigma_{\theta i}^2$$

where $\boldsymbol{\sigma}_{Xi}$ and $\boldsymbol{\sigma}_{\theta j}$ are the standard deviations for the horizontal location of the mating feature.

3.1 Direct Build Assembly

As random part variations are statistically independent and Gaussian, it can be shown that the horizontal assembly error (Equation (33)) has standard deviation:

$$\sigma_n = \sqrt{\sum_{i=1}^{n} \sigma_{Xi}^2 + \sum_{i=1}^{n-1} \sigma_{\theta i}^2 \left(\sum_{j=i+1}^{n} Y_j\right)^2} \quad (39)$$

From Equation (34) the probability density function ε_n^{Direct} for n part axi-symmetric assembly with Direct Build may be written as:

$$p\left(\varepsilon_n^{Direct}\right) = \begin{cases} \sqrt{\dfrac{2}{\pi\sigma_n^2}} \exp\left(-\dfrac{(\varepsilon_n^{Direct})^2}{2\sigma_n^2}\right), & \varepsilon_n^{Direct} \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (40)$$

This pdf is a half-Gaussian distribution based on standard deviation .

3.2. Best Build Assembly

The minimum build eccentricity ε_1^{min} for 1 part assembly can be obtained from Equation (35).

$$p\left(\varepsilon_1^{min}\right)=\begin{cases}\sqrt{\dfrac{2}{\pi\sigma_{X1}^2}}\exp\left(-\dfrac{(\varepsilon_1^{min})^2}{2\sigma_{X1}^2}\right), & \varepsilon_1^{min}\geq 0\\ 0 & \text{otherwise}\end{cases} \tag{41}$$

This pdf is a half-Gaussian distribution based on standard deviation σ_{X1}.

For a 2 part assembly, the minimum build eccentricity ε_2^{min} is obtained from Equation (36). Noting that d_{X1}-$Y_2 d\theta_1$ is a zero-mean Gaussian random variable d_{X1} and $d\theta_1$ are statistically independent, $\left|dX_1 - Y_2 d\theta_1\right|$ has a half-Gaussian distribution based on standard deviation $\sigma_{2,1}^* = \sqrt{\sigma_{X1}^2 + \sigma_{\theta 1}^2 Y_2^2}$. Given that $\left|dX_2\right|$ and $\left|dX_1 - Y_2 d\theta_1\right|$ are independent, pdf for the minimum eccentricity ε_2^{min} may be found as:

$$p\left(\varepsilon_2^{min}\right)=\begin{cases}\sqrt{\dfrac{8}{\pi\sigma_2^2}}\exp\left(-\dfrac{(\varepsilon_2^{min})^2}{2\sigma_2^2}\right)\left[2-\Phi\left(\dfrac{\varepsilon_2^{min}\sigma_{X2}}{\sigma_2\sigma_{2,1}^*}\right)-\Phi\left(\dfrac{\varepsilon_2^{min}\sigma_{2,1}^*}{\sigma_2\sigma_{X2}}\right)\right], & \varepsilon_2^{min}\geq 0\\ 0 & \text{otherwise}\end{cases} \tag{42}$$

where σ_2 is found from Equation (39) with $n=2$ and Φ is the cumulative normal distribution [19].

For a 3 -part assembly, the minimum build eccentricity ε_3^{min} may be found from Equation (37). As dX_1-$(Y_2+Y_3)d\theta_1$, dX_2-$Y_2 d\theta_1$ and dX_3 are zero-mean Gaussian random variables and dX_1, dX_2, dX_3, $d\theta_1$ and $d\theta_2$ are statistically independent random vartiables, $\left|dX_1-(Y_2+Y_3)d\theta_1\right|$, $\left|dX_2 - Y_2 d\theta_1\right|$ and $\left|dX_3\right|$ are all half-Gaussian distribution based on standard deviations $\sigma_{3,1}^* = \sqrt{\sigma_{X1}^2+\sigma_{\theta 1}^2(Y_1+Y_2)^2}$, $\sigma_{3,2}^* = \sqrt{\sigma_{X2}^2+\sigma_{\theta 2}^2 Y_3^2}$ and σ_{X3}, respectively. From distributions, pdf for the minimum eccentricity ε_3^{min} may be found [19]. Hence, pdf for ε_3^{min} can be found as:

$$p\left(\varepsilon_3^{min}\right)=\begin{cases}\dfrac{4}{\pi\sigma_3\sigma_{X3}}\left\{\displaystyle\int_0^\infty \exp\left(-\dfrac{\sigma_{X3}^2x^2+\sigma_3^2(x+\varepsilon_3^{min})^2}{2\sigma_3^2\sigma_{x3}^2}\right)\left[2-\Phi\left(\dfrac{x\sigma_{3,1}^*}{\sigma_3\sigma_{3,2}^*}\right)-\Phi\left(\dfrac{x\sigma_{3,2}^*}{\sigma_3\sigma_{3,1}^*}\right)\right]dx+\right.\\ \left.\displaystyle\int_{\varepsilon_3^{min}}^\infty \exp\left(-\dfrac{\sigma_{X3}^2x^2+\sigma_3^2\left(x-\varepsilon_3^{min}\right)^2}{2\sigma_3^2\sigma_{X3}^2}\right)\left[2-\Phi\left(\dfrac{x\sigma_{3,1}^*}{\sigma_3\sigma_{3,2}^*}\right)-\Phi\left(\dfrac{x\sigma_{3,2}^*}{\sigma_3\sigma_{3,1}^*}\right)\right]dx,\right\} & \varepsilon_3^{min}\geq 0\\ 0 & \text{otherwise}\end{cases} \tag{43}$$

In practice it is necessary to evaluate the integrals appearing in Equation (39) numerically.

4. RESULTS

In this section, a three-component assembly example is investigated to evaluate the efficiency and accuracy of the proposed probabilistic methods. Each of the three component is a rectangle having nominal width W and height H equal to 70 and 100mm [18], respectively, such that the coordinates of the top mating feature of the rectangle with respect to its lower mating feature are [0,70mm]. Location and orientation errors of top surface relative to its lower surface for each components are considered to be (dX_i and dY_i) and ($d\theta_i$). The tolerances of errors (dX_i,dY_i and $d\theta_i$) are assumed to be normally distributed random variable with zero mean and standard deviation (σ) equal to one third of the specified tolerance. The value of location tolerances for errors dX_i and dY_i is chosen to be 0.1mm and the tolerance for orientation error for each component is taken to be $2h_i/W$ [18].

Results calculated numerically using proposed probabilistic methods are compared to standard MCS method. The results calculated by MCS are based on the Equation (30). Convergence studies for MCS were conducted to find number of simulations required to obtain accurate results. In the convergence study, 50,000 simulations were identified based on the predictions of average, standard deviation, kurtosis, and skewness being accurate to 1%. Unlike MCS, the proposed probability based method does not require convergence study.

Fig. 3 shows the distributions of eccentricity error arising from three-component assembly with height and width tolerances of 0.1mm in linear and logarithmic scale. In Fig. 3 comparison of the results obtained using the proposed method (based on Equation (36) for DBA and Equation (39) for BBA) and MCS method is presented. Fig. 3 reveals that the probability-based approach produces results in agreement with MCS and the obtained statistical distributions are highly non-Gaussian. At tails of log-pdf for high eccentricities, it can be seen that the MCS is less accurate as compared to the proposed method. This shows the competence of the derived pdf expressions compared to MCS. Also these results validate the proposed linear approximations and the derived pdf expressions.

In order to evaluate the effectiveness of the proposed methods with MCS, comparison of the execution time required to perform calculations is made for the two methods. Table 1 shows the timing of the CPU in fractions of DBA for the proposed method and the MCS for the results shown in Fig. 3.

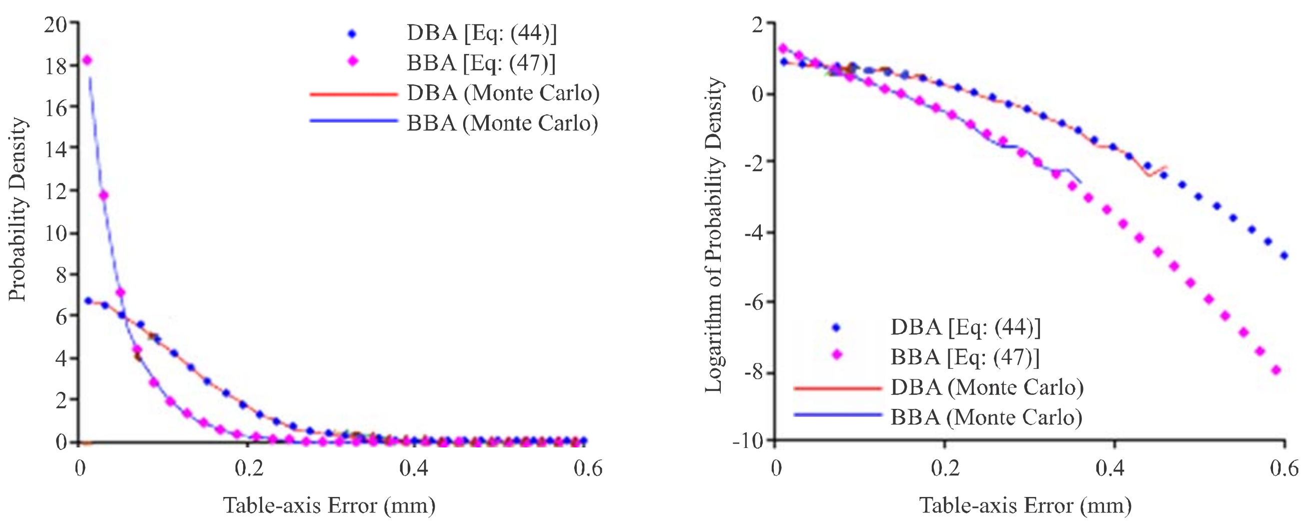
Table 1 shows greater efficiency of proposed analytical method over MCS. This is because, it is easier to calculate the pdf expression (Equation (40)) than the repetitive simulations. It is also observed that the results of DBA are achieved more efficiently than the best-build assembly. This is because additional integrations are required to calculate Equation (43).

The expression for calculating the probability $P(\alpha)$ that the eccentricity ε does not exceed a particular value α is given by:

$$P(\alpha) = \int_0^\alpha p(\varepsilon)d\varepsilon \tag{44}$$

TABLE 1. COMPARISON BETWEEN THE MCS AND THE PROPOSED METHOD FOR THE EXECUTION TIME

Assembly Procedures	Execution Time in Fractions of DBA
Direct Build Assembly (Proposed)	1.0
Best Build Assembly (Proposed)	35.3
Direct Build Assembly (Monte Carlo)	242.6
Best Build Assembly (Monte Carlo)	353.3



(a) LINEAR

(b) LOGARITHMIC SCALE

FIG. 3. DISTRIBUTION OF ECCENTRICITY ERROR FOR THREE-COMPONENT ASSEMBLY

For a three-part assembly, $P(\varepsilon)$ is given by Equations (40,43) respectively, for Direct Build and Best Build Assembly.

Fig. 4 shows results for the probability that the three-component build eccentricity does not exceed a value α when the tolerance is 0.1mm, and compares the results obtained using MCS and the proposed methods, calculated using Equation (44). Fig. 4 shows that the probability results from proposed method are in agreement with MCS.

The comparison of above results for 2D assembly with those calculated for similar 3D assembly case study analyzed by the authors of this paper in [20] reveal that both 2D and 3D assembly analyses give similar trend of assembly variation. Thus 2D representation of assembly is equivalently important as 3D assembly analysis.

5. CONCLUSIONS

A Linearised model based on connective assembly model for straight-build assembly is developed. Eccentricity error expressions for the assembly of rigid circular components are derived. The expressions are used to analyse (a) BBA, and (b) DBA. In BBA, each component is rotated to select the best orientations for minimum eccentricity. On the other hand, Direct Build considers the assembly without optimising orientations. Linearised expressions for the build eccentricity were developed using linearised model. The expressions for the pdf's for the build eccentricity are developed from the derived linearised expressions. An example of 2D axi-symmetric components is analysed to investigate the accuracy and efficiency of probability based approach in comparison to MCS. Numerical results are calculated for an assembly comprising three 2D axi-symmetric components with ideally identical dimensions. The results showed that the derived probabilistic approach yields accurate and efficient results. The proposed approach is expected to provide a valuable tool for tolerance assignment and assembly process design.

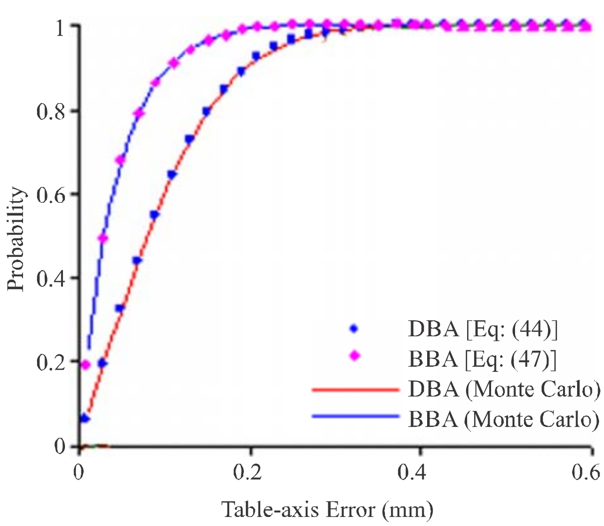


FIG. 4. TABLE-AXIS ERROR VERSUS PROBABLITY FOR PROPOSED AND MONTE CARLO METHODS

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