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BI-QUASI IDEALS AN FUZZY BI-IDEALS OF Γ - SEMIGROUPS

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ABSTRACT. In this paper,as a further generalization of ideals, we introduce the notion of bi-quasi ideal of $\Gamma-$ semigroup as a generalization of bi-ideal of $\Gamma-$semigroup. We study the properties of bi-quasi ideal, characterize the bi-quasi simple $\Gamma-$semigroup and regular $\Gamma-$semigroup using bi-quasi ideals. We introduce the notion of fuzzy bi-quasi ideal of $\Gamma-$semigroup as a generalization of fuzzy bi-ideal and we characterize the regular $\Gamma-$semigroup in terms of fuzzy bi-quasi ideals of $\Gamma-$semigroup.

1. Introduction

Semigroup, as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. The notion of ideals introduced by Dedekind for the theory of algebraic numbers was generalized by E. Noether for associative rings. The one and two sided ideals introduced by her, are still central concepts in ring theory. In 1952, the concept of bi-ideals for semigroup was introduced by Good and Hughes [1]. The notion of bi-ideals in associative rings was introduced by Lazos and Szasz [5]. An interesting particular case of the bi-ideal is the notion of quasi-ideal was first introduced for semigroup and then for rings by Steinfeld [14] in 1956. The notion of quasi ideals is a generalization of left and right ideals whereas the bi-ideals are generalization of quasi-ideals. Iseki [2] studied ideals in semirings. The notion of the bi-ideal in semigroups is a special case of (m,n) ideals introduced by S. Lajos [4]. As a generalization of ring, the notion of a $\Gamma-$ring was introduced by Nobusawa [11] in 1964. In 1995, M. Murali Krishna Rao [10] introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ring, ring, ternary semiring and semiring. In 1981, Sen [13] introduced the notion of a $\Gamma-$semigroup as a generalization of semigroup.The notion of a ternary algebraic system was introduced by Lehmer [6] in 1932, Lister [8] introduced the notion of a ternary ring .

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The fuzzy set theory was developed by Zadeh [16] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. The fuzzification of algebraic structure was introduced by Rosenfeld [12] with the notion of fuzzy subgroups in 1971. K.L.N.Swamy and U.M.Swamy [15] studied fuzzy prime ideals in rings in 1988. In 1982, Liu[7] defined and studied fuzzy subrings as well as fuzzy ideals in rings. Applying the concept of fuzzy sets to the theory of $\Gamma-$ring, Jun and Lee introduced the notion of fuzzy ideals in $\Gamma-$ring and studied the properties of fuzzy ideals of $\Gamma-$ring. Mandal [9] studied fuzzy ideals and fuzzy interior ideals in an ordered semiring. Kuroki [3] studied fuzzy interior ideals in semigroups.

In this paper, as a further generalization of ideals, we introduce the notion of bi-quasi ideal of $\Gamma-$ semigroup which is a generalization of bi-ideal of $\Gamma-$semigroup. We study the properties of bi-quasi ideal, characterize the bi-quasi simple $\Gamma-$ semigroup and regular $\Gamma-$semigroup using bi-quasi ideals.In the section 4, we introduce the notion of fuzzy bi-quasi ideal of $\Gamma-$semigroup and characterize the regular $\Gamma-$semigroup in terms of fuzzy bi-quasi ideals of $\Gamma-$semigroup.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

A semigroup is a non-empty set S together with an associative binary operation. An element 0 of semigroup S with at least two elements is called a zero element of S if x0 = 0x = 0 for all x in S. A semigroup which contains a zero element is called a semigroup with zero. If a semigroup S has no zero element then it is easy to adjoin a zero element 0 to the set by defining 0x = x0 = 0 and 00=0, for all x in S. An element 1 of a semigroup S is called an identity element of S if x1 = 1x = x for all x in S. A semigroup which contains an identity element is called a semigroup with identity or a monoid. If a semigroup S has no identity element then it is easy to adjoin an element 1 to the set by defining 1x = x1 = x and 11 = 1 for all x in S. A subsemigroup T of S is a non-empty subset T of S such that $TT \subseteq T$. A non-empty subset T of S is called a left (right) ideal of S if $ST \subseteq T(TS \subseteq T)$. A non-empty subset T of S is called an ideal of S if it is both a left ideal and a right ideal of$S$. A non-empty subset Q of S is called a quasi ideal of S if $QS \cap SQ \subseteq Q$. A subsemigroup T of S is called a bi-ideal of S if $TST \subseteq T$. A subsemigroup T of S is called an interior ideal of S if $STS \subseteq T$. An element a of semigroup S is called a regular element if there exists an element b of S such that $a = aba$. A semigroup S is called a regular semigroup if every element of S is a regular element.

Let M and Γ be nonempty sets. Then we call M a $\Gamma-$semigroup, if there exists a mapping $M \times \Gamma \times M \to M$ (images of (x, α, y) will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfies $x\alpha(y\beta z) = (x\alpha y)\beta z$. for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

A non-empty subset A of $\Gamma-$semigroup M is called

(i) a $\Gamma-$subsemigroup of M if $A\Gamma A \subseteq A$.
(ii) a quasi ideal of M if $A\Gamma M \cap M\Gamma A \subseteq A$.
(iii) a bi-ideal of M if $A\Gamma A \subseteq A$ and $A\Gamma M\Gamma A \subseteq A$.
(iv) an interior ideal of M if $A\Gamma A \subseteq A$ and $M\Gamma A\Gamma M \subseteq A$.

(v) a left (right) ideal of M if $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$.
(vi) an ideal if $A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.

Let M be a $\Gamma-$semigroup. M is said to be commutative if $a\alpha b = b\alpha a$, for all $a, b \in M$, for all $\alpha \in \Gamma$. An element $a \in M$ is said to be idempotent of M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α idempotent. If every element of M is an idempotent M then $\Gamma-$semigroup M is said to be band. An element $a \in M$ is said to be regular element of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. If every element of M is a regular element of M then M is said to be regular $\Gamma-$semigroup M.

Let M be a non-empty set. A mapping $\mu : M \to [0, 1]$ is called a fuzzy subset of M. If μ is a fuzzy subset of M, for $t \in [0, 1]$ then the set $\mu_t = \{x \in M \mid \mu(x) \geqslant t\}$ is called a level subset of M with respect to a fuzzy subset μ. A fuzzy subset $\mu : M \to [0, 1]$ is a non-empty fuzzy subset if μ is not a constant function. For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leqslant \mu(x)$ for all $x \in M$.

Let A be a non-empty subset of M. The characteristic function of A and it is a fuzzy subset of M is defined by

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Let μ and γ be two fuzzy subsets of $\Gamma-$semigroup M and $x, y, z \in M, \alpha \in \Gamma$. We define

$$\mu \circ \gamma(x) = \begin{cases} \sup\limits_{x=y\alpha z} & \{\, min(\mu(y), \gamma(z))\} : \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu \cap \gamma(x) = \min\{\mu(x), \gamma(x)\}, \text{ for all } x \in M.$$

A fuzzy subset μ of $\Gamma-$semigroup M is called

(i) a fuzzy $\Gamma-$subsemigroup of M if $\mu(x\alpha y) \geqslant min\ \{\mu(x), \mu(y)\}$.
(ii) a fuzzy left (right) ideal of M if $\mu(x\alpha y) \geqslant \mu(y)\ (\mu(x))$.
(iii) a fuzzy ideal of M if $\mu(x\alpha y) \geqslant max\ \{\mu(x), \mu(y)\}$.
(iv) a fuzzy left (right)ideal of M if $\chi_M \circ \mu \subseteq \mu(\mu \circ \chi_M \subseteq \mu)$.
(iv) a fuzzy bi-ideal of M if $\mu \circ \chi_M \circ \mu \subseteq \mu$.
(vi) a fuzzy quasi -ideal of M if $\mu \circ \chi_M \cap \chi_M \circ \mu \subseteq \mu$.

3. Bi-quasi ideals of $\Gamma-$Semigroups

In this section,as a further generalization of ideals, we introduce the notion of left bi-quasi ideal, right bi-quasi ideal and bi-quasi ideal of $\Gamma-$semigroup.. We study the properties of bi-quasi ideals of $\Gamma-$semigroup.

DEFINITION 3.1. Let M be a $\Gamma-$semigroup. A non-empty subset L of M is said to be left bi-quasi ideal (right bi-quasi ideal) of M if L is a subsemigroup of M and

$$M\Gamma L \cap L\Gamma M\Gamma L \subseteq L\ (L\Gamma M \cap L\Gamma M\Gamma L \subseteq L).$$

DEFINITION 3.2. Let M be a $\Gamma-$ semigroup. L is said to be bi-quasi ideal if it is both a left bi-quasi ideal and a right bi-quasi ideal of M.

DEFINITION 3.3. A $\Gamma-$ semigroup M is called a bi-quasi simple $\Gamma-$ semigroup if M has no bi-quasi ideal other than M itself.

EXAMPLE 3.4. Let ***Q*** and ***N*** be the set of all rational numbers and the set of all rational numbers respectively.

$$M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Q \right\}, \quad \Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in N \right\}$$

and ternary operation $A\alpha B$ is defined as usual matrix multiplication of A, α, B, for all $A, B \in M$. and $\alpha \in \Gamma$. then M is a $\Gamma-$semigroup $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$ then R is a bi-quasi ideal of $\Gamma-$semigroup M and R is neither a left ideal nor a right ideal.

EXAMPLE 3.5. Let

$$M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\} and \Gamma = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in N \right\}$$

Ternary operation is defined as usual matrix multiplication then M is a $\Gamma-$semigroup $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$. Then A is a bi-quasi ideal and A is not a bi-ideal of $\Gamma-$semigroup M.

THEOREM 3.6. *Every left ideal of* $\Gamma-$*semigroup* M *is a left bi-quasi ideal of* M.

PROOF. Let L be the left ideal of $\Gamma-$semigroup M. Then

$$M\Gamma L \subseteq L \Rightarrow M\Gamma L \cap L\Gamma M\Gamma L \subseteq M\Gamma L \subseteq L.$$

Hence L is a left bi-quasi ideal of M. □

THEOREM 3.7. *Every left ideal of* $\Gamma-$*semigroup* M *is a right bi-quasi ideal of* M.

PROOF. Let L be the left ideal of $\Gamma-$semigroup M. Then $M\Gamma L \subseteq L$. Then

$$L\Gamma M \cap L\Gamma M\Gamma L \subseteq L\Gamma M\Gamma L \subseteq L\Gamma L \subseteq L.$$

Therefore $L\Gamma M \cap L\Gamma M\Gamma L \subseteq L$. Hence L is a right bi-quasi ideal of M. □

COROLLARY 3.8. *Every left ideal of* $\Gamma-$*semigroup* M *is a bi-quasi ideal of* M.

THEOREM 3.9. *Every quasi ideal is a bi-quasi ideal of* $\Gamma-$*semigroup*

PROOF. Let L be a quasi ideal of $\Gamma-$semigroup M. Then $L\Gamma M \cap M\Gamma L \subseteq L$. We have $L\Gamma M\Gamma L \subseteq L\Gamma M$, since $M\Gamma L \subseteq M$. Therefore $M\Gamma L \cap L\Gamma M\Gamma L \subseteq M\Gamma L \cap L\Gamma M \subseteq L$. Similarly we can prove every quasi ideal is a right bi-quasi ideal. Hence every quasi ideal is a bi-quasi ideal of $\Gamma-$semigroup of M. □

THEOREM 3.10. *Every bi-ideal of* $\Gamma-$*semigroup* M *is a bi-quasi ideal of* M.

PROOF. Let L be a bi-ideal of $\Gamma-$semigroup M. Then $L\Gamma M\Gamma L \subseteq L$. Therefore $M\Gamma L \cap L\Gamma M\Gamma L \subseteq L\Gamma M\Gamma L \subseteq L$. Every bi-ideal is a left bi-quasi ideal. Similarly we can prove bi-ideal is a right bi-quasi ideal. □

THEOREM 3.11. *Arbitrary intersection of bi-quasi ideals of* $\Gamma-$*semgroup* M *is either empty or a left bi-quasi ideal of* M.

PROOF. Let M be a $\Gamma-$semigroup and $L = \bigcap_{i\in I} L_i$, where L_i is a bi-quasi ideal of M for all i. Obviously L is a subsemigroup of M Therefore M$\Gamma \cap L_i \cap \cap L_i\Gamma M\Gamma \cap L_i \subseteq M\Gamma L_i \cap L_i\Gamma M\Gamma L_i \subseteq L_i$, for all $i \in I$ implies $M\Gamma \cap L_i \cap \cap L_i\Gamma M\Gamma \cap L_i \subseteq \cap L_i$. Hence L is a left bi-quasi ideal of $\Gamma-$semigroup M.Similarly we can prove L is a right bi-quasi ideal. □

COROLLARY 3.12. *If L is a left ideal and R is a right ideal of $\Gamma-$semigroup M then $L \cap R$ is a bi-quasi ideal of $\Gamma-$semgroup M.*

THEOREM 3.13. *Let M be a $\Gamma-$semigroup.Then the following statements are equivalent*

(i) *M is a bi-quasi simple $\Gamma-$semigroup M.*
(ii) *$M\Gamma a = M$, for all $a \in M$*
(iii) *$(a) = M$, for all $a \in M$ where (a) is the smallest bi-quasi ideal generated by a*

PROOF. Let M be a $\Gamma-$semigroup

(i) $\Rightarrow$ (ii) Suppose that M is a bi-quasi simple $\Gamma-$semigroup, $a \in M$ and $L = M\Gamma a$. Then L is a left ideal. Therefore by Corollary 3.8, L is a bi-quasi ideal of $\Gamma-$semigroup M. Hence $M\Gamma a = M$, for all $a \in M$.

(ii) $\Rightarrow$ (iii) Suppose that $M\Gamma a = M$, for all $a \in M$ and (a) is the smallest bi-quasi ideal of M containing a. Then $M\Gamma a \subseteq (a) \subseteq M \Rightarrow M \subseteq (a) \subseteq M$. Therefore $M = (a)$.

(iii) $\Rightarrow$ (i) Suppose (a) is the smallest bi-quasi ideal generated by a, $(a) = M$, for all $a \in M$ and A is a bi-quasi ideal of M and $a \in A$. Then $(a) \subseteq A \subseteq M$. Therefore $A = M$. Hence M is a bi-quasi simple $\Gamma-$semigroup.

□

THEOREM 3.14. *Let M be a $\Gamma-$semigroup. Then M is a bi-quasi simple $\Gamma-$semigroup if and only if $M\Gamma a \cap a\Gamma M\Gamma a = M$, for all $a \in M$.*

PROOF. Let M be a $\Gamma-$semigroup. Suppose M is a bi-quasi simple $\Gamma-$semigroup and $a \in M$. By Corollary 3.12, $M\Gamma a \cap a\Gamma M\Gamma a$ is a bi-quasi ideal of $\Gamma-$semigroup M. Therefore $M\Gamma a \cap a\Gamma M\Gamma a = M$, for all $a \in M$, since M is a bi-quasi simple $\Gamma-$semigroup.

Conversely suppose that $M\Gamma a \cap a\Gamma M\Gamma a = M$, for all $a \in M$. Let T be a bi-quasi ideal of $\Gamma-$semigroup M and $a \in T$.

$$M = M\Gamma a \cap a\Gamma M\Gamma a \subseteq M\Gamma T \cap T\Gamma M\Gamma T \subseteq T \subseteq M$$

Therefore $M = T$. Hence M is a bi-quasi simple $\Gamma-$semigroup. □

THEOREM 3.15. *Let M be a $\Gamma-$semigroup. If $M = M\Gamma a$, for all $a \in M$ then every left bi-quasi ideal of $\Gamma-$semigroup is a quasi ideal.*

PROOF. Suppose M is a $\Gamma-$semigroup with $M = M\Gamma a$, for all $a \in M$ and L is a left bi-quasi ideal of $\Gamma-$semigroup. Then $M\Gamma L \cap L\Gamma M\Gamma L \subseteq L$. Let $a \in L$. Thus $M\Gamma a \subseteq M\Gamma L$ and $M \subseteq M\Gamma L \subseteq M$. Finally, $M\Gamma L = M$. Therefore $M\Gamma L \cap L\Gamma M \subseteq L$. □

COROLLARY 3.16. *Let M be a $\Gamma-$semigroup. If $M = M\Gamma a$, for all $a \in M$ then every bi-quasi ideal of $\Gamma-$semigroup is a quasi ideal.*

The following two theorems are wellborn theorems on $\Gamma-$semigroups.

THEOREM 3.17. *Let M be a regular Γ - semigroup. Then every quasi ideal of Γ -semigroup M is an ideal of Γ - semigroup M.*

THEOREM 3.18. *M is a regular $\Gamma-$semigroup if and only if $A\Gamma B = A \cap B$, for any right ideal A and left ideal B of $\Gamma-$semigroup M.*

THEOREM 3.19. *Let M be a regular $\Gamma-$semigroup. Then every left bi-quasi ideal of Γ - semigroup M is an ideal of Γ - semigroup M.*

PROOF. Let M be a regular $\Gamma-$semigroup and L be a left bi-quasi ideal of Γ - semigroup M. Then $M\Gamma L \cap L\Gamma M\Gamma L \subseteq L$. We know that $L\Gamma M$ and $M\Gamma L$ are right ideal and left ideal of $\Gamma-$semigroup M. By Theorem 3.18, we have $L\Gamma M\Gamma M\Gamma L = L\Gamma M \cap M\Gamma L$. Therefore $L\Gamma M \cap M\Gamma L = L\Gamma M\Gamma M\Gamma L \subseteq M\Gamma L$ and $L\Gamma M \cap M\Gamma L = L\Gamma M\Gamma M\Gamma L \subseteq L\Gamma M\Gamma L$ Hence $L\Gamma M \cap M\Gamma L \subseteq M\Gamma L \cap L\Gamma M\Gamma L \subseteq L$. Thus L is a quasi ideal of $\Gamma-$semigroup M. Therefore, by Theorem[3.17]. L is an ideal of $\Gamma-$semigroup M. □

COROLLARY 3.20. *Let M be a regular $\Gamma-$semigroup. Then every bi-quasi ideal of $\Gamma-$semigroup M is an ideal of $\Gamma-$semigroup M.*

THEOREM 3.21. *Let M be a $\Gamma-$semigroup. M is a regular $\Gamma-$semigroup if and only if $B = M\Gamma B \cap B\Gamma M\Gamma B$ for every left bi-quasi ideal B of M.*

PROOF. Suppose B is a left bi-quasi ideal of a regular $\Gamma-$semigroup M. and $x \in B$. We have $M\Gamma B \cap B\Gamma M\Gamma B \subseteq B$, since B is a left bi-quasi ideal of M. As M is a regular, there exist $y \in M, \alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$. Then $x \in M\Gamma B$ and $B\Gamma M\Gamma B$. Therefore $B \subseteq B\Gamma M\Gamma B \cap M\Gamma B$. Hence $B\Gamma M\Gamma B \cap M\Gamma B = B$.

Conversely suppose that $B = M\Gamma B \cap B\Gamma M\Gamma B$, for any left bi-quasi ideal B of M. Let R and L be right ideal and left ideal of M respectively. Then by Corollary 3.12, $R \cap L$ is a bi-quasi ideal. Then

$$\begin{aligned} R \cap L &= M\Gamma(R \cap L) \cap (R \cap L)\Gamma M\Gamma(R \cap L) \\ &\subseteq (R \cap L)\Gamma M\Gamma(R \cap L) \\ &\subseteq R\Gamma M\Gamma L \\ &\subseteq R\Gamma L. \end{aligned}$$

We have $R\Gamma L \subseteq L$ and $R\Gamma L \subseteq R$. Therefore $R\Gamma L \subseteq R \cap L$. Hence $R\Gamma L = R \cap L$. By Theorem [3.18] , M is a regular $\Gamma-$semigroup. □

COROLLARY 3.22. *Let M be a $\Gamma-$semigroup. M is a regular $\Gamma-$semigroup if and only if $B = M\Gamma B \cap B\Gamma M\Gamma B$ for every bi-quasi ideal B of M.*

4. Fuzzy bi-quasi ideals of $\Gamma-$semigroups:

In this section we introduce the notion of fuzzy bi-quasi ideal as a generalization of fuzzy bi-ideal of $\Gamma-$semigroup and study the properties of fuzzy bi-quasi ideals

DEFINITION 4.1. A fuzzy subset μ of $\Gamma-$semigroup M is called a fuzzy left (right) bi-quasi ideal if $\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu \subseteq \mu\big(\mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu \subseteq \mu\big)$.

A fuzzy subset μ of $\Gamma-$semigroup M is called a fuzzy bi-quasi ideal if it is both a fuzzzy left bi-quasi ideal and a fuzzy right bi-quasi ideal of M.

EXAMPLE 4.2. Let Q be the set of all rational numbers,

$$M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\} and \Gamma = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in N \right\}$$

Ternary operation is defined as the usual matrix multiplication then M is a $\Gamma-$semigroup. If $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, 0 \neq b \in Q \right\}$, then A is a bi-quasi ideal but not a bi-ideal of $\Gamma-$semigroup M. Define $\mu : M \to [0, 1]$ such that $\mu(x) == \begin{cases} 1 & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$ Then μ is a fuzzy bi-quasi ideal of M.

THEOREM 4.3. *Every fuzzy left ideal of* $\Gamma-$*semigroup* M *is a fuzzy bi-quasi ideal of* M.

PROOF. Let μ be a fuzzy left ideal of $\Gamma-$ semigroup M and $x \in M$.

$$\begin{aligned} \chi_M \circ \mu(x) &= \sup_{x=a\alpha b} \{\min\{\chi_M(a), \mu(b)\}\} \\ &= \sup_{x=a\alpha b} \{\min\{1, \mu(b)\}\} \\ &= \sup_{x=a\alpha b} \{\mu(b)\} \\ &\leqslant \sup_{x=a\alpha b} \{\mu(a\alpha b)\} \\ &= \sup_{x=a\alpha b} \{\mu(x)\}\} \\ &= \mu(x) \\ &\Rightarrow \ \chi_M \circ \mu(x) \leqslant \mu(x). \end{aligned}$$

$\mu \circ \chi_M \circ \mu(x) = \sup\limits_{x=u\alpha v\beta s} \{\min\{\mu(u), \chi_M \circ \mu(v\beta s)\}\} \leqslant \sup\limits_{x=u\alpha v\beta s} \{\min\{\mu(u), \mu(v\beta s)\}\} = \mu(x)$. Therefore $\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu \subseteq \mu$. Now $\mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu(x) = \min\{\mu \circ \chi_M(x), \mu \circ \chi_M \circ \mu(x)\} \leqslant \mu(x)$. Hence μ is a fuzzy bi-quasi ideal of M. □

THEOREM 4.4. *Every fuzzy right ideal of* Γ *- semigroup* M *is a fuzzy bi-quasi ideal of* M.

PROOF. Let μ be a fuzzy right ideal of $\Gamma-$ semigroup M and $x \in M$.

$$\begin{aligned} \mu \circ \chi_M(x) &= \sup_{x=a\alpha b} \min\{\mu(a), \chi_M(b)\} \\ &= \sup_{x=a\alpha b} \mu(a) \\ &\leqslant \sup_{x=a\alpha b} \mu(a\alpha b) \\ &= \mu(x). \end{aligned}$$

Therefore $\mu \circ \chi_M(x) \leqslant \mu(x)$. Now

$$\begin{aligned}\mu \circ \chi_M \circ \mu(x) &= \sup_{x=u\alpha v\beta s} \min\{\mu \circ \chi_M(u\alpha v), \mu(s)\} \\ &\leqslant \sup_{x=u\alpha v\beta s} \min\{\mu(u\alpha v), \mu(s)\} \\ &= \mu(x)\end{aligned}$$

and

$$\mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu(x) = \min\{\mu \circ \chi_M(x), \mu \circ \chi_M \circ \mu(x)\} \leqslant \mu(x).$$

Therefore μ is a fuzzy right bi-quasi ideal of $\Gamma-$ semigroup M. Similarly we can prove $\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu \subseteq \mu$. Hence μ is a fuzzy bi-quasi ideal of M. □

THEOREM 4.5. *Let M be a $\Gamma-$semigroup and μ be a non-empty fuzzy subset of M. Then μ is a fuzzy bi-quasi ideal of $\Gamma-$ semigroup M if and only if the level subset μ_t of μ is a bi-quasi ideal of $\Gamma-$ semigroup M for every $t \in [0, 1]$,*

PROOF. Let M be a $\Gamma-$ semigroup and μ be a non-empty fuzzy subset of M. Suppose μ is a fuzzy bi-quasi ideal of $\Gamma-$ semigroup , $\mu_t \neq \phi, t \in [0, 1]$ and $a, b \in \mu_t, \alpha \in \Gamma$. Let $x \in M\Gamma\mu_t \cap \mu_t\Gamma M\Gamma\mu_t$. Then $x = b\alpha a = c\beta d\gamma e$, where $b, d \in M, a, c, e \in \mu_t$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$$\begin{aligned}&\chi_M \circ \mu(x) \geqslant t \text{ and } \mu \circ \chi_M \circ \mu(x) \geqslant t \\ \Rightarrow &\mu(x) \geqslant (\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu)(x) \geqslant t\end{aligned}$$

Therefore $x \in \mu_t$. Hence μ_t is a left bi-quasi ideal of M.

Similarly we can prove μ_t is a right bi-quasi ideal of M. Conversely suppose that μ_t is a bi-quasi ideal of $\Gamma-$semiring M, for all $t \in Im(\mu)$. Let $x, y \in M, \mu(x) = t_1, \mu(y) = t_2$ and $t_1 \geqslant t_2$. Then $x, y \in \mu_{t_2}$ and $x\alpha y \in \mu_{t_2} \Rightarrow \mu(x\alpha y) \geqslant t_2 = \min\{\mu(x), \mu(y)\}$. We have $M\Gamma\mu_l \cap \mu_l\Gamma M\Gamma\mu_l \subseteq \mu_t$, for all $l \in Im(\mu)$. Suppose $t = \min\{Im(\mu)\}$. Then $M\Gamma\mu_t \cap \mu_t\Gamma M\Gamma\mu_t \subseteq \mu_t$. Therefore $\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu \subseteq \mu$. Hence μ is a fuzzy left bi-quasi ideal of $\Gamma-$ semigroup M.

Similarly we can prove μ is a fuzzy right bi-quasi ideal of $\Gamma-$semigroup M. □

THEOREM 4.6. *Let I be a non-empty subset of a $\Gamma-$semigroup M and χ_I be the characteristic function of I. Then I is a bi-quasi ideal of $\Gamma-$semigroup M if and only if χ_I is a fuzzy bi-quasi ideal of $\Gamma-$semigroup M.*

PROOF. Let I be a non-empty subset of a $\Gamma-$semigroup M and χ_I be the characteristic function of I. Suppose I is a bi-quasi ideal of $\Gamma-$semigroup M. Obviously χ_I is a fuzzy $\Gamma-$subsemigroup of M. We have $M\Gamma I \cap I\Gamma M\Gamma I \subseteq I$. Then

$$\chi_M \circ \chi_I \cap \chi_I \circ \chi_M \circ \chi_I = \chi_{M\Gamma I} \cap \chi_{I\Gamma M\Gamma I} = \chi_{M\Gamma I \cap I\Gamma M\Gamma I} \subseteq \chi_I.$$

Therefore χ_I is a fuzzy left bi-quasi ideal of $\Gamma-$semigroup M. Similarly we can prove χ_I is a fuzzy right bi-quasi ideal of $\Gamma-$semigroup M. Conversely suppose that χ_I is a fuzzy bi-quasi ideal of M. Then I is a $\Gamma-$subsemiring of M. We have

$$\begin{aligned}
&\chi_M \circ \chi_I \cap \chi_I \circ \chi_M \circ \chi_I \subseteq \chi_I\\
\Rightarrow &\chi_{M\Gamma I} \cap \chi_{I\Gamma M\Gamma I} \subseteq \chi_I\\
\Rightarrow &\chi_{M\Gamma I \cap I\Gamma M\Gamma I} \subseteq \chi_I
\end{aligned}$$

Therefore $M\Gamma I \cap I\Gamma M\Gamma I \subseteq I$. Hence I is a left bi-quasi ideal of $\Gamma-$semigroup M.

Similarly we can prove I is a right bi-quasi ideal of $\Gamma-$semigroup M. Therefore I is a fuzzy bi-quasi ideal of $\Gamma-$semigroup M. □

THEOREM 4.7. *If μ and λ are fuzzy bi-quasi ideals of $\Gamma-$semigroup M then $\mu \cap \lambda$ is a fuzzy bi-quasi ideal of $\Gamma-$ semigroup M.*

PROOF. Let μ and λ be fuzzy left bi- quasi ideals of $\Gamma-$ semigroup M and $x \in M$. Then

$$\begin{aligned}
\chi_M \circ \mu \cap \lambda(x) &= \sup_{x=a\alpha b}\{\min\{\chi_M(a), \mu \cap \lambda(b)\}\}\\
&= \sup_{x=a\alpha b}\{\min\{\chi_M(a), \min\{\mu(b), \lambda(b)\}\}\}\\
&= \sup_{x=a\alpha b}\{\min\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\}\}\\
&= \min\{\sup_{x=a\alpha b}\{\min\{\chi_M(a), \mu(b)\}\ , \sup_{x=a\alpha b}\{\min\{\chi_M(a), \lambda(b)\}\}\}\\
&= \min\{\chi_M \circ \mu(x).\chi_M \circ \lambda(x)\}\\
&= \chi_M \circ \mu \cap \chi_M \circ \lambda(x)
\end{aligned}$$

Therefore

$$\chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \cap \chi_M \circ \lambda.$$

and

$$\begin{aligned}
&\mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda(x)\\
&= \sup_{x=a\alpha b\beta c}\{\min\{\mu \cap \lambda(a), \chi_M \circ \mu \cap \lambda(b\beta c)\}\}\\
&= \sup_{x=a\alpha b\beta c}\{\min\{\mu \cap \lambda(a), \chi_M \circ \mu \cap \chi_M \circ \lambda(b\beta c)\}\}\\
&= \sup_{x=a\alpha b\beta c}\{\min\{\min\{\mu(a), \lambda(a)\}\}, \min\{\chi_M \circ \mu(b\beta c), \chi_M \circ \lambda(b\beta c)\}\}\\
&= \sup_{x=a\alpha b\beta c}\{\min\{\min\{\mu(a), \chi_M \circ \mu(b\beta c)\}\}, \min\{\lambda(a), \chi_M \circ \lambda(b\beta c)\}\}\\
&= \min\{\sup_{x=a\alpha b\beta c}\{\min\{\mu(a), \chi_M \circ \mu(b\beta c)\}\}, \sup_{x=a\alpha b\beta c}\min\{\lambda(a), \chi_M \circ \lambda(b\beta c)\}\}\}\\
&= \min\{\mu \circ \chi_M \circ \mu(x), \lambda \circ \chi_M \circ \lambda(x)\}\\
&= \mu \circ \chi_M \circ \mu \cap \lambda \circ \chi_M \circ \lambda(x).
\end{aligned}$$

Therefore $\mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \mu \circ \chi_M \circ \mu \cap \lambda \circ \chi_M \circ \lambda$. Hence

$$\begin{gathered}
\chi_M \circ \mu \cap \lambda \cap \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda =\\
(\chi_M \circ \mu) \cap (\mu \circ \chi_M \circ \mu) \cap (\chi_M \circ \lambda) \cap (\lambda \circ \chi_M \circ \lambda) \subseteq \mu \cap \lambda.
\end{gathered}$$

Hence $\mu \cap \lambda$ is a fuzzy left bi-quasi ideal of M.

Similarly we can prove $\mu \cap \lambda$ is a fuzzy right bi-quasi ideal of $\Gamma-$semigroup M. Therefore $\mu \cap \lambda$ is a fuzzy bi-quasi ideal of $\Gamma-$semigroup M. □

THEOREM 4.8. *Let μ and λ be fuzzy right ideal and fuzzy left ideal of $\Gamma-$ semigroup M respectively. Then $\mu \cap \lambda$ is a fuzzy bi-quasi ideal of $\Gamma-$ semigroup M.*

PROOF. Let μ and λ be fuzzy right ideal and fuzzy left ideal of $\Gamma-$ semigroup M respectively. Then By Theorem 4.7, we have

$$\chi_M \circ (\mu \cap \lambda) = \chi_M \circ \mu \cap \chi_M \circ \lambda$$

and

$$\begin{aligned}
\mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda &= \mu \circ \chi_M \circ \mu \cap \lambda \circ \chi_M \circ \lambda. \\
\Rightarrow \chi_M \circ (\mu \cap \lambda) \cap \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda & \\
= (\chi_M \circ \mu) \cap (\mu \circ \chi_M \circ \mu) \cap (\chi_M \circ \lambda) \cap (\lambda \circ \chi_M \circ \lambda) & \\
&\subseteq \mu \cap \lambda.
\end{aligned}$$

Hence $\mu \cap \lambda$ is a fuzzy left bi-quasi ideal of $\Gamma-$ semigroup M. Similarly we can prove $\mu \cap \lambda$ is a fuzzy right bi-quasi ideal of $\Gamma-$semigroup M. Hence the theorem. □

The following two theorems are well-known theorems in $\Gamma-$ semigroups. So we omit the proofs.

THEOREM 4.9. *If μ is a fuzzy quasi-ideal of a regular $\Gamma-$semigroup M then μ is a fuzzy ideal of M.*

THEOREM 4.10. *A $\Gamma-$ semigroup M is a regular if and only if $\lambda \circ \mu = \lambda \cap \mu$, for any fuzzy right ideal λ and fuzzy left ideal μ of M.*

THEOREM 4.11. *Let M be a $\Gamma-$semigroup. Then M is a regular if and only if $\mu = \chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu$, for any fuzzy bi-quasi ideal μ of $\Gamma-$semigroup M.*

PROOF. Let μ be a fuzzy bi-quasi ideal of regular $\Gamma-$semigroup M.
Then $\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu \subseteq \mu$.

$$\begin{aligned}
\chi_M \circ \mu(x) &= \sup_{x = x\alpha y\beta x} \{\min\{\chi_M(x\alpha y), \mu(x)\}\} = \mu(x) \\
\mu \circ \chi_M \circ \mu(x) &= \sup_{x = x\alpha y\beta x} \{\min\{\mu(x), \chi_M \circ \mu(y\beta x)\}\} \\
&= \sup_{x = x\alpha y\beta x} \{\min\{\mu(x), \sup_{y\beta x = r\delta s} \min\{\chi_M(r), \mu(s)\}\}\} \\
&= \sup_{x = x\alpha y\beta x} \{\min\{\mu(x), \sup_{y\beta x = r\delta s} \min\{1, \mu(s)\}\}\} \\
&\geqslant \sup_{x = x\alpha y\beta x} \{\min\{\mu(x), \mu(x)\}\} \\
&= \mu(x).
\end{aligned}$$

Therefore $\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu = \mu$. Conversely suppose that $\mu = \chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu$, for any fuzzy bi-quasi ideal μ of $\Gamma-$semigroup M. Let B be a bi-quasi ideal of $\Gamma-$semigroup

M. Then by Theorem 4.6, χ_B is a fuzzy bi-quasi ideal of $\Gamma-$semigroup M. Therefore

$$\chi_B = \chi_M \circ \chi_B \cap \chi_B \circ \chi_M \circ \chi_B = \chi_{M\Gamma B} \cap \chi_{B\Gamma M\Gamma B}$$

and $B = M\Gamma B \cap B\Gamma M\Gamma B$. By Theorem 3.21, M is a regular $\Gamma-$ semigroup. □

THEOREM 4.12. *Let M be a regular $\Gamma-$ semigroup. Then μ is a fuzzy bi-quasi ideal of M if and only if μ is a fuzzy quasi ideal of M.*

PROOF. Let μ be a fuzzy bi-quasi ideal of $\Gamma-$ semigroup M and $x \in M$. Then

$$\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu \subseteq \mu$$

Suppose $\chi_M \circ \mu(x) > \mu(x)$. Since M is a regular, there exist $y \in M, \alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$. Then

$$\begin{aligned} \mu \circ \chi_M \circ \mu(x) &= \sup_{x=x\alpha y\beta x} \min\{\mu(x\alpha y), \chi_M \circ \mu(x)\}) \\ &> \sup_{x=x\alpha y\beta x} \min\{\mu(x\alpha y), \mu(x)\} \\ &= \mu(x) \end{aligned}$$

Which is a contradiction. Therefore $\mu \circ \chi_M \cap \chi_M \circ \mu \subseteq \mu$. By Theorem 4.9, converse is true. □

THEOREM 4.13. *Let M be a $\Gamma-$ semigroup. Then M is a regular if and only if $\mu \cap \gamma \subseteq \gamma \circ \mu \cap \mu \circ \gamma \circ \mu$, for every fuzzy bi-quasi ideal μ and every fuzzy idealγ of $\Gamma-$ semigroup M.*

PROOF. Let M be a regular $\Gamma-$ semigroup,μ be a fuzzy bi-quasi ideal,γ be a fuzzy ideal and $x \in M$. Then there exist $\alpha, \beta \in \Gamma$ and $y \in M$ such that $x = x\alpha y\beta x$.

$$\begin{aligned} \mu \circ \gamma \circ \mu(x) &= \sup_{x=y\delta z} \{min\{\mu \circ \gamma(y), \mu(z)\}\} \\ &= \sup_{x=x\alpha y\beta x} \{min\{\mu \circ \gamma(x\alpha y), \mu(x)\}\} \\ &\geqslant \min\{ \sup_{x\alpha y=x\alpha y\beta x\alpha y} \{min\{\mu(x), \gamma(y\beta x\alpha y)\}, \mu(x)\}\} \\ &\geqslant \min\{\min\{\mu(x), \gamma(x)\}, \mu(x)\} \\ &= \min\{\mu(x), \gamma(x)\} \\ &= \mu \cap \gamma(x). \end{aligned}$$

Therefore $\mu \cap \gamma \subseteq \mu \circ \gamma \circ \mu$.

$$\begin{aligned} \gamma \circ \mu(x) &= \sup_{x=a\alpha b} \{min\{\gamma(a), \mu(b)\}\} \\ &= \sup_{x=x\alpha y\beta x} \{\min\{\gamma(x\alpha y), \mu(x)\}\} \\ &\geqslant \min\{\gamma(x), \mu(x)\} = \mu \cap \gamma(x). \end{aligned}$$

Hence $\mu \cap \gamma \subseteq \gamma \circ \mu \cap \mu \circ \gamma \circ \mu$.

Conversely suppose that the condition holds. Let μ be a fuzzy bi-quasi ideal of $\Gamma-$semigroup M.

$$\mu \cap \chi_M \subseteq \chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu$$

Therefore $\mu \subseteq \chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu$. By Theorem 4.14, M is a regular $\Gamma-$ semigroup. □

5. Conclusion

As a further generalization of ideals, we introduced the notion of bi-quasi ideal of $\Gamma-$ semigroup as a generalization of bi-ideal of $\Gamma-$semigroup and studied the properties of bi-quasi ideals, characterized the bi-quasi simple $\Gamma-$semigroup and regular $\Gamma-$semigroup using bi-quasi ideals.We also introduced the notion of fuzzy bi-quasi ideal of $\Gamma-$semigroup as a generalization of fuzzy bi-ideal and characterized the regular $\Gamma-$semigroup in terms of fuzzy bi-quasi ideals of $\Gamma-$semigroup.We plan to study maximal and minimal bi-quasi ideals of $\Gamma-$ semigroup.

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