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WEAK SEMI COMPATIBILITY AND FIXED POINT THEOREMS FOR EXPANSIVE MAPPINGS IN G-METRIC SPACES

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Abstract

In this paper we introduce the concept of g-compatible mapping, $f-$ compatible mapping, absorbing mappings and compatible mappings of type (E) in the setting of G-metric space and establish some examples to show their independency. Further we prove some fixed point theorems for weak semi compatibility which also includes property E.A. Our results are actually generalization of results of Manro [9].

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1 Introduction

In 1963, Gahler[5] introduced the concept of 2- metric spaces and claimed that 2-metric is a generalization of the usual notion of metric, but some authors proved that there is no relation between these two functions. It is clear that in 2- metric space, $d(x, y, z)$ is to be taken as the area the triangle with vertices x, y and z in R^2. In 1992, Dhage[4] introduced the concept of a D- metric space. The central concept of D- metric space is different from 2- metric spaces. Geometrically, a D-metric $D(x, y, z)$ represents the perimeter of the triangle with vertices x, y and z in R^2. However, Mustafa et al. [12] have demonstrated that most of

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the claims concerning the fundamental topological structure of D- metric space are incorrect. Alternatively, they have introduced [13] more appropriate notion of generalized metric space which called G-metric space. They generalized the concept of metric, in which the real number is assigned to every triplet of an arbitrary set.

2 Preliminary

Definition 2.1 ([13]) *Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying:*

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$, $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x)$ (Symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric space or, more specifically, a G-metric on X and the pair (X, G) is G -metric space.

Example 2.1 ([13]) *Let (X, d) be a usual metric space, and define G_s and G_m on $X \times X \times X$ to R^+ by*

$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z)$

$G_m(x, y, z) = max\{d(x, y), d(y, z), d(x, z)\}$.

Proposition 2.1 ([13]) *Let (X, G) is a $G-$ metric space. Then for any x, y, z and $a \in X$ it follows that:*

(i) $G(x, y, z) = 0 \Rightarrow x = y = z$.

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$.

(iii) $G(x, y, y) \leq 2G(y, x, x)$.

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$.

(v) $G(x, y, z) \leq \frac{2}{3}\{G(x, y, a) + G(x, a, z) + G(a, y, z)\}$.

(vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Definition 2.2 ([13]) *Let (X, G) be a G-metric space, and $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $lim_n G(x, x_n, x_m) = 0$, one say that the sequence $\{x_n\}$ is $G-$ convergent to x.*

Definition 2.3 ([13]) *Let (X, G) be a G-metric space. Then for a sequence $\{x_n\}$ in X and a point $x \in X$, the following are equivalent:*

(i) $\{x_n\}$ is convergent to x.

(ii) $G(x_n, x_n, x) \to 0$, as $n \to \infty$.

(iii) $G(x_n, x, x) \to 0$, as $n \to \infty$.

(iv) $G(x_n, x_m, x) \to 0$, as $n, m \to \infty$.

Definition 2.4 ([13]) *Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-cauchy if given $\epsilon > 0$, there exist natural number N such that $G(x_n, x_m, x_l) < \epsilon$ for all $l, m, n \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $l, m, n \to \infty$.*

Definition 2.5 ([13]) *In a G-metric space (X, G), the following are equivalent:*
(i) the sequence $\{x_n\}$ is G-cauchy sequence.
(ii) for every $\epsilon > 0$, there exist natural number N such that $G(x_n, x_m, x_l) < \epsilon$ for all $l, m, n \geq N$.

Definition 2.6 ([13]) *Let (X, G) and (X', G') be two G-metric spaces, and let $f : (X, G) \to (X', G')$ be a function, then $f-$ is said to be $G-$ continuous at a point $a \in X$ iff, given $\epsilon > 0$, there exist $\delta > 0$ such that $x, y \in X$, and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous in X iff it is G-continuous at all $a \in X$.*

Definition 2.7 (13) *Let (X, G) and (X', G') be two G-metric spaces. Then a function $f : (X, G) \to (X', G')$ is G-continuous at a point $x \in X$ iff it is G-sequentially continuous at x; that is, whenever $\{x_n\}$ is G-convergent to x, we have $\{f(x_n)\}$ is G-continuous to $f(x)$.*

Definition 2.8 ([13]) *A G-metric (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.*

Proposition 2.2 ([13]) *Every G-metric space (X, G) induces a metric space (X, d_G) defined by*
$d_G(x, y) = G(x, y, y) + G(y, x, x)$ for all $x, y \in X$.
Note that if (X, G) is symmetric, then
$d_G(x, y) = 2G(x, y, y)$ for all $x, y \in X$.
However, if (X, G) is not symmetric then it holds by the G -metric properties, that is
$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y)$ for all $x, y \in X$.

Definition 2.9 ([13]) *A G-metric space (X, G) is said to be G-complete if every G-cauchy sequence in (X, G) is G-convergent in (X, G).*

Proposition 2.3 ([13]) *A G-metric space (X, G) is G-complete iff (X, d_G) is a complete metric space.*

Proposition 2.4 ([13]) *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Two self maps f and g of metric space (X, d) are said to be f-compatible of type (E) ([26]) if $lim_n ffx_n = lim_n fgx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$, for some $t \in X$. Similarly, two self maps f and g of metric space (X, d) are said to be g compatible of type (E)[26]

if $lim_n ggx_n = lim_n gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$, for some $t \in X$.

On the same account, we introduce the concept of f-compatible of type (E) and g-compatible of type (E) in G-metric space setting as follows:

Definition 2.10 *Let (X, G) be a G-metric space. Then self mappings f and g are said to be f - compatible of type (E) if*

$$lim_n G(fgx_n, gt, gt) = lim_n G(ffx_n, gt, gt) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$, for some $t \in X$. Similarly, self mappings f and g are said to be g-compatible of type (E) if

$$lim_n G(gfx_n, ft, ft) = lim_n G(ggx_n, ft, ft) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$, for some $t \in X$.

Remark 2.1 *f-compatible of type (E) and g-compatible of type (E) are independent notion as following example supports:*

Example 2.2 *Let $X = [1, \infty)$ and (X, G) be a G-metric space defined by $G(x, y, z) =\mid x - y \mid + \mid y - z \mid + \mid z - x \mid$ for all $x, y, z \in X$ and f and g be self mappings of G-metric space (X, G) defined by*

$fx = x + 3$ if $x \in [1, 4)$ and $fx = 7$ if $x \geq 4$.

$gx = 4$ if x is not an integer and $gx = 7$ if x is an integer.

Then for sequence $x_n = 1 + \epsilon_n$ where $\epsilon_n = \frac{1}{n}$,, $n > 1$ and also as $n \to \infty$, $\epsilon_n \to 0$,

$lim_n fx_n = lim_n f(1 + \epsilon_n) = 4$ and $lim_n gx_n = lim_n g(1 + \epsilon_n) = 4$.

Therefore $lim_n fx_n = lim_n gx_n = 4$(say t).

Now $lim_n fgx_n = lim_n fg(1 + \epsilon_n) = lim_n f(4) = 7$.

Also $lim_n ffx_n = lim_n ff(1 + \epsilon_n) = 7$. Therefore

$lim_n G(fgx_n, gt, gt) = G(7, 7, 7) = 0$.

Also, $lim_n G(ffx_n, gt, gt) = G(7, 7, 7) = 0$.

These conclude that f and g are f-compatible of type (E), that is

$lim_n G(fgx_n, gt, gt) = lim_n G(ffx_n, gt, gt) = 0$. Again $lim_n gfx_n = lim_n gf(1+ \epsilon_n) = 4$ also $lim_n ggx_n = lim_n gg(1 + \epsilon_n) = 7$. Therefore $lim_n G(ggx_n, ft, ft) = G(7, 7, 7) = 0$.

But $lim_n G(gfx_n, ft, ft) = G(4, 7, 7) \neq 0$. These conclude that f and g are not g-compatible of type (E).

Two self maps f and g of metric space (x, d) are called compatible [7] if

$lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$ for some in $t \in X$. Further, f and g are called g-compatible ([22]cited from [27]) if $lim_n d(gfx_n, ffx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$ for some in $t \in X$. Similarly, maps f and

g are called f -compatible ([22], [27]) if $lim_n d(fgx_n, ggx_n) = 0$,, whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$ for some in $t \in X$.

On the same account, the following definition of compatible mappings in G-metric setting is given:

Definition 2.11 ([10], [11], [28]) *Let (X, G) be a G-metric space. Then self mappings f and g are said to be compatible if $lim_n G(fgx_n, gfx_n, gfx_n) = 0$ and $lim_n G(gfx_n, fgx_n, fgx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$ for some in $t \in X$.*

Now we define f-compatibility and g-compatibility in the setting of G-metric space as follows:

Definition 2.12 *Let (X, G) be a G-metric space. Then self mappings f and g are said to be f-compatible if $lim_n G(fgx_n, ggx_n, ggx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$ for some in $t \in X$. Similarly, self mappings f and g are said to be g-compatible if $lim_n G(gfx_n, ffx_n, ffx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$ for some in $t \in X$.*

Remark 2.2 *f-compatibility and g-compatibility are independent notion as following example supports:*

Example 2.3 *Let $X = R$ and (X, G) be a G-metric space defined by $G(x, y, z) = \mid x - y \mid + \mid y - z \mid + \mid z - x \mid$ for all $x, y, z \in X$ and f and g be self mappings of G-metric space (X, G) defined by*

$fx = x + 1$ if $x \in R$ and

$gx = 2$ if x is not an integer and $gx = 3$ if x is an integer.

Then for sequence $x_n = 1 + \epsilon_n$ where $\epsilon_n = \frac{1}{n}$,, $n > 1$ and also as $n \to \infty$, $\epsilon_n \to 0$,

$lim_n fx_n = lim_n f(1 + \epsilon_n) = 2$ and $lim_n gx_n = lim_n g(1 + \epsilon_n) = 2$.

Therefore $lim_n fx_n = lim_n gx_n = 2$(say t).

Now $lim_n fgx_n = lim_n fg(1 + \epsilon_n) = lim_n f(2) = 3$.

Also $lim_n ggx_n = lim_n gg(1 + \epsilon_n) = 3$. Therefore
$lim_n G(fgx_n, ggx_n, ggx_n) = G(3, 3, 3) = 0$.

On the other hand, $lim_n gfx_n = lim_n gf(1 + \epsilon_n) = 2$. Also $lim_n ffx_n = lim_n ff(1 + \epsilon_n) = 3$. Therefore $lim_n G(gfx_n, ffx_n, ffx_n) = G(2, 3, 3) \neq 0$.

This concludes that f and g are f- compatible but not g-compatible.

Compatible mappings are independent from f- compatibility and g- compatibility.

Now we define f-absorbing and g-absorbing in the setting of G-metric space on the same lines as [6] in metric spaces as follows:

Definition 2.13 *Let (X, G) be a G-metric space. Then self mapping f and g will be called g -absorbing if there exists a real number $R > 0$ such that*

$G(gfx, gx, gx) \leq RG(fx, gx, gx)$ for all $x \in X$. Similarly, self mapping f and g will be called f -absorbing if there exists a real number $R > 0$ such that $G(fgx, fx, fx) \leq RG(fx, gx, gx)$ for all $x \in X$.

f-absorbing and g-absorbing are independent notions.

Definition 2.14 ([10], [21]) *Let (X, G) be a G-metric space. Then self mapping f and g are said to be R-weak commuting of type (A_f) if there exists a real number $R > 0$ such that $G(fgx, ggx, ggx) \leq RG(fx, gx, gx)$ for all $x \in X$.imilarly, f and g are said to be R-weak commuting of type (A_g) if there exists a real number $R > 0$ such that $G(gfx, ffx, ffx) \leq RG(fx, gx, gx)$ for all $x \in X$*

Definition 2.15 ([1]) *Let f and g are two self mappings of $G-$metric space (X, G). Then maps f and g satisfy the E.A. property if there exists a sequence $\{x_n\}$ in X such that $lim_n fx_n = lim_n gx_n = t$ for some $t \in X$.*

Definition 2.16 ([20]) *Two self mappings f and g on a $G-$metric space (X, G) are called conditionally commuting if they commute on nonempty subset of the set of coincidence points whenever the set of their coincidence points is nonempty.*

Pant, Bisht and Arora [19] introduced a notion of weak reciprocal continuity as follows:

Definition 2.17 *Two self mappings f and g of $G-$metric space (X, G) will be called weakly reciprocally continuous if $lim_n fgx_n = ft$ or $lim_n gfx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$ for some $t \in X$.*

Further Saluja et al. [23] introduced a notion weak semi compatibility as follows:

Definition 2.18 *Two self mappings f and g of a $G-$metric space (X, G) will be called weak semi compatible mappings if $lim_n fgx_n = gt$ or $lim_n gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$ for some $t \in X$.*

3 Main Results

Theorem 3.1 *Let f and g be two weak semi compatible, R-weak commuting type of A_f self mappings, of complete symmetric G-metric space (X, G) satisfying the following conditions:*

(a) $f(X) \subseteq g(X)$,

(b) $G(gx, gy, gz) \geq hG(fx, fy, fz)$ for every $x, y, z \in X$ and $h > 1$,

(c) f and g are either f- compatible of type (E) or g- compatible of type (E).

If f and g are conditionally commuting, then f and g have a common fixed point in X.

Proof- Let x_0 be any point in X. Since $f(X) \subseteq g(X)$, there exist a point x_1 in X such that $fx_0 = gx_1$. Similarly we can have a sequence $\{x_n\}$ in X such that $fx_n = gx_{n+1}$. Let $y_n = fx_n = gx_{n+1}$. Now we shall show that $\{y_n\}$ is a Cauchy sequence in X.

For proving this take (b) with $x = x_n, y = x_{n+1}, z = x_{n+1}$.

$G(gx_n, gx_{n+1}, gx_{n+1}) \geq hG(fx_n, fx_{n+1}, fx_{n+1})$

$G(y_n, y_{n+1}, y_{n+1}) \leq \frac{1}{h}G(y_{n-1}, y_n, y_n)$. ....(1)

Similarly we have,

$G(y_{n-1}, y_n, y_n) \leq \frac{1}{h}G(y_{n-2}, y_{n-1}, y_{n-1})$.

This yields with (1),

$G(y_n, y_{n+1}, y_{n+1}) \leq \frac{1}{h^2}G(y_{n-2}, y_{n-1}, y_{n-1})$.

By continuing this process we have

$G(y_n, y_{n+1}, y_{n+1}) \leq \frac{1}{h^n}G(y_0, y_1, y_1)$.

Let $\frac{1}{h} = k$ then we have

$G(y_n, y_{n+1}, y_{n+1}) \leq k^n G(y_0, y_1, y_1)$....(2)

Now for all $n, m \in N$(set of natural number) and $n < m$ we have from $G(5)$

$G(y_n, y_m, y_m) \leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_m, y_m)$

$\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + ... + G(y_m, y_m, y_m)$.

This yields with (2)

$G(y_n, y_m, y_m) \leq k^n G(y_o, y_1, y_1) + ... + k^{m-1}G(y_0, y_1, y_1)$.

Let $m = n + p$, we have

$G(y_n, y_{n+p}, y_{n+p}) \leq k^n G(y_o, y_1, y_1) + ... + k^{n+p-1}G(y_0, y_1, y_1)$

$\leq k^n G(y_0, y_1, y_1)(1 + k + ...) = k^n G(y_0, y_1, y_1)\frac{1}{(1-k)}$.

Now limiting $n \to \infty$, yields $G(y_n, y_m, y_m) \to 0$. Thus $\{y_n\}$ is a Cauchy sequence. Since (X, G) is complete. Then there exist a point $t \in X$ such that $lim_n y_n = t$ or $lim_n fx_n = lim_n gx_{n+1} = t$.

Case 1- f and g are f -compatible of type (E).

Since f and g are weak semi compatible mappings, this yields either $lim_n fgx_n = gt$ or $lim_n gfx_n = ft$.

First we take $lim_n gfx_n = ft$.

Since f and g are f compatible of type (E), this yields

$lim_n G(fgx_n, gt, gt) = 0$ and $lim_n G(ffx_n, gt, gt) = 0$.

Now by (b),

$G(gfx_n, gt, gt) \geq hG(ffx_n, ft, ft)$.

Now limiting $n \to \infty$ yields $G(ft, gt, gt) \geq hG(gt, ft, ft)$. Since (X, G) is symmetric, this yields $G(ft, gt, gt) \geq hG(ft, gt, gt)$. Since $h > 1$ yields $ft = gt$. Since f and g are conditionally commuting this yields $fgt = gft$ or $fgt = gft = fft = ggt$. Now by (b), $G(ggt, gt, gt) \geq hG(fgt, ft, ft)$ implies $G(fgt, gt, gt) \geq hG(fgt, gt, gt)$. Since $h > 1$ yields $fgt = gt$ or $fgt = ggt = gt$. Hence gt is common fixed point of f and g.

Now we take $lim_n fgx_n = gt$.

Since f and g are f compatible of type (E), this yields

$lim_n G(fgx_n, gt, gt) = 0$ and $lim_n G(ffx_n, gt, gt) = 0$.
Since f and g are R-weak commuting type of A_f, yields
$G(fgx_n, ggx_n, ggx_n) \leq RG(fx_n, gx_n, gx_n)$. Limiting $n \to \infty$ yields
$G(lim_n fgx_n, lim_n ggx_n, lim_n ggx_n) \leq RG(t, t, t)$.
This implies $lim_n ggx_n = gt$. Now by (b)
$G(ggx_n, gt, gt) \geq hG(fgx_n, ft, ft)$.
Again limiting $n \to \infty$ yields $G(gt, gt, gt) \geq hG(gt, ft, ft)$. Since $h > 1$ yields $ft = gt$. Now by same above working it can be easily get that gt is common fixed point of f and g.

Case 2- f and g are g compatible of type (E).
Since f and g are weak semi compatible mappings, this yields either $lim_n fgx_n = gt$ or $lim_n gfx_n = ft$.
First we take $lim_n gfx_n = ft$.
Since f and g are g compatible of type (E), this yields
$lim_n G(gfx_n, ft, ft) = 0$ and $lim_n G(ggx_n, ft, ft) = 0$.
Since f and g are R-weak commuting type of A_f, yields
$G(fgx_n, ggx_n, ggx_n) \leq RG(fx_n, gx_n, gx_n)$. Limiting $n \to \infty$ yields
$G(lim_n fgx_n, lim_n ggx_n, lim_n ggx_n) \leq RG(t, t, t)$.
This implies $lim_n fgx_n = ft$. Now by (b)
$G(ggx_n, gx_n, gx_n) \geq hG(fgx_n, fx_n, fx_n)$.
Again limiting $n \to \infty$ yields $G(ft, t, t) \geq hG(ft, t, t)$. Since $h > 1$ yields $ft = t$. Since $f(x) \subseteq g(X)$, then there exist a point $u \in X$ such that $ft = gu$. Now by (b), by taking $x = u$ and $y = x_n$, we easily get $fu = t = gu$. This shows that gu is common fixed point of f and g.
Similarly, we easily prove that gt is common fixed point of f and g if we take $lim_n fgx_n = gt$.

Example 3.1 *Let (X, G) be a G-metric space where $X = [1, 7]$ and $G(x, y, z) = | x - y | + | y - z | + | z - x |$. For all $x, y, x \in X$ define $f, g : X \to X$ by*
$fx = \frac{x+5}{2}$ *if* $x > 5$,
$fx = 1$ *if* $1 \leq x \leq 5$
and
$gx = x$ *for all* x.
When we take sequence $x_n = 5 + \epsilon_n$ where $\epsilon_n = \frac{1}{n},$, $n > 1$ and also as $n \to \infty$, $\epsilon_n \to 0$,
$lim_n fx_n = lim_n f(5 + \epsilon_n) = 5$ *and* $lim_n gx_n = lim_n g(5 + \epsilon_n) = 5$.
Therefore $lim_n fx_n = lim_n gx_n = 5$(say t).
Now $lim_n fgx_n = lim_n fg(5 + \epsilon_n) = 5 = g(5)$ also $lim_n gfx_n = lim_n gf(5 + \epsilon_n) = 5 \neq f(5)$. Therefore f and g are weak semi-compatible. Also, f and g are f compatible of type (E) and R- weakly commuting of type (A_f).
Also, f and g satisfies condition (b) for $h = \frac{6}{5}$. Thus, f and g satisfies all conditions of above theorem and $x = 1$ is a common fixed point of f and g.

Theorem 3.2 *Let f and g be two weak semi compatible self mappings of G -metric space (X, G) satisfying the following conditions:*

(a) $f(X) \subseteq g(X)$,
(b) $G(gx, gy, gz) \geq hG(fx, fy, fz)$ for all $x, y, z \in X$ and $h > 1$
(c) f and g are either compatible or f -compatible or g -compatible.
If f and g satisfy E.A. property, then f and g have a common fixed point in X.

Proof. Since f and g satisfy E.A. property, then there exist a sequence $\{x_n\}$ in X such that $lim_n fx_n = lim_n gx_n = t$ ... (3) for some $t \in X$.

f and g are compatible. Since f and g are weak semi compatible mappings, this yields either $lim_f gx_n = gt$ or $lim_n gfx_n = ft$.

First we take $lim_n gfx_n = ft$.

Since f and g are compatible, this yields

$lim_n G(fgx_n, gfx_n, gfx_n) = 0$ or $lim_n G(gfx_n, fgx_n, fgx_n) = 0$.

By (b), we easily get $lim_n ffx_n = ft$. Now by (b),

$G(gfx_n, gx_n, gx_n) \geq hG(ffx_n, fx_n, fx_n)$.

Now limiting $n \to \infty$ yields $G(ft, t, t) \geq hG(ft, t, t)$. Since $h > 1$ yields $ft = t$. By (a), there exist a point $u \in X$ such that $ft = gu$. Now by (b), $G(gu, gx_n, gx_n) \geq hG(fu, fx_n, fx_n)$. This gives, $fu = t = gu$. Hence gu is common fixed point of f and g.

Similarly, if f is g- absorbing or g is f- absorbing, the f and g have common fixed point in X.

Now, we give example to support validity of above theorem:

Example 3.2 *Let (X, G) be a G-metric space where $X = [1, 7]$ and $G(x, y, z) = \mid x - y \mid + \mid y - z \mid + \mid z - x \mid$. For all $x, y, x \in X$ define $f, g : X \to X$ by*

$fx = \frac{x+5}{2}$ *if* $x > 5$,

$fx = 1$ *if* $1 \leq x \leq 5$

and

$gx = x$ *for all* x.

Same as done in Example 3.1, it is easy to observe that pair (f, g) satisfy all the condition of above theorem except condition (c) and $x = 1$ is common fixed point of f and g.

Theorem 3.3 *Let f and g be two weak semi compatible, R-weak commuting type of A_f self mappings of G-metric space satisfying the following conditions:*
(a) $f(X) \subseteq g(X)$,
(b) $G(gx, gy, gz) \geq hG(fx, fy, fz)$ for all $x, y, z \in X$ and $h > 1$
(c) f is g -absorbing or g is f -absorbing.
If f and g satisfy E.A. property, then f and g have a common fixed point in X.

Proof. Since f and g satisfy E.A. property, then there exist a sequence $\{x_n\}$ in X such that $lim_n fx_n = lim_n gx_n = t$ ... (6) for some $t \in X$.

Let f is g - absorbing. Since f and g are weak semi compatible mappings, this yields either $lim_f gx_n = gt$ or $lim_n gfx_n = ft$.

First we take $lim_n gfx_n = ft$.

Since f is g absorbing, this gives, $G(gfx_n, gx_n, gx_n) \leq RG(fx_n, gx_n, gx_n)$. This gives, $ft = t$. By (a), we get, $ft = gu$.

Now by (b),

$G(gu, gx_n, gx_n) \geq hG(fu, fx_n, fx_n)$.

Now limiting $n \to \infty$ yields $G(t, t, t) \geq hG(fu, t, t)$. Since $h > 1$ yields $fu = t = gu$. R Weak commuting of type (A_f) gives, $fgu = ggu$.

Now, f is g absorbing yielding $gfu = gu$ and hence fu is common fixed point of f and g.

Similarly, if g is f absorbing, then f and g have common fixed point in X.

Now, we give example to support validity of above theorem:

Example 3.3 *Let (X, G) be a G-metric space where $X = [1, 7]$ and $G(x, y, z) = \mid x - y \mid + \mid y - z \mid + \mid z - x \mid$. For all $x, y, x \in X$ define $f, g : X \to X$ by*

$fx = \frac{x+5}{2}$ *if* $x > 5$,

$fx = 1$ *if* $1 \leq x \leq 5$

and

$gx = x$ *for all* x.

It is easy to observe that pair (f, g) satisfy all the condition of above theorem except condition (c) as done in Example 3.1 and $x = 1$ is common fixed point of f and g.

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References

[1] Aamri, M. and Moutawakil, D. El., *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl., **270**(1)(2002), 181-188.

[2] Abbas, M., Khan, S.H. and Nazir, T., *Common fixed points of R-weakly commuting maps in generalized metric spaces*, Fixed point theory and applications, 2011, **2011**:41.

[3] Al-Thagafi, M.A. and Shahjad, N., *Generalized I-nonexpansive self maps and invariant approximations*, Acta Mathematica Sinica, **24**(5) (2008), 867-876.

[4] Dhage, B.C., *Generalized metric space and mapping with fixed point*, Bull. Calcutta Math. Soc., **84**(4)(1992), 329-336.

[5] Gähler, S., *2-metrische Räume und ihre topologische structure*, Math. Nachr., **26**(1963), 115-148.

[6] Gopal, D., Ranadive, A.S. and Pant, R.P., *Common fixed points of absorbing maps*, Bulletin of Marathwada Mathematical Society, **9** (1)(2008), 43-48

[7] Jungck, G., Compatible mappings and common fixed points, Int. J. Math. Math. Sci., **9**(4) (1986), 771-779.

[8] Jungck, G. Murthy, P.P. and Cho,Y.J., *Compatible mappings of type(A) and common fixed points*, Math. Japon., **38**(2)(1993), 381-390.

[9] Manro, S. and Kumam, P., *Common fixed point theorems for expansion mappings in various abstract spaces using the concept of weak reciprocal continuity,* Fixed point theory and applications, 2012, **2012**:221.

[10] Manro, S., Kumar, S. and Bhatia, S.S., *Expansion mapping theorems in G-metric spaces*, Int. J. Contemp. Math. Sciences, **5**(51)(2010), 2529-2535.

[11] Manro, S., Kumar, S. and Bhatia, S.S., *Weakly compatible maps of type (A) in G-metric spaces*, Demostratio mathematica, **45**(4) (2012), 901-909.

[12] Mustafa, Z. and Sims, B., *Some remarks concerning D-metric spaces*, In: Garcia-Falset, J. et al., (Eds.), *Proceeding of the International conferences on fixed point theory and applications*, July 13-19, Yokohama publishers, Valencia , Spain (pp. 189-198), Yokohama Publishers, Yokohama, 2004.

[13] Mustafa, Z, and Sims, B., *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7**(2)(2006), 289-297.

[14] Mustafa, Z., Awawdeh, F. and Shatanawi,W., *Fixed point theorem for expansive mappings in G-metric space*, Int. J. Contemp. Math. Sciences, **5**(50) (2010), 2463-2472.

[15] Mustafa, Z., Obiedat, H. and Awawdeh, F., *Some fixed point theorem for mapping on complete G-metric space*, Fixed Point Theory and Applications, Volume **2008**, Article ID 189870, 12 pages.

[16] Mustafa, Z., Shatanawi, W. and Bataineh, M., *Fixed point theorem on uncomplete G-metric spaces*, Journal of Mathematics and Statistics, **4**(4)(2008), 196-201.

[17] Pant, R.P., *Common fixed points of non commuting mappings*, J. Math. Anal. Appl., **188**(2)(1994), 436-440.

[18] Pant, R.P. and Bisht, R.K., *Common fixed point theorems under a new continuity condition*, Ann. Univ. Ferrara, **58**(1)(2012), 127-141.

[19] Pant, R.P., Bisht, R.K. and Arora, D., *Weak reciprocal continuity and fixed point theorems*, Ann. Univ. Ferrara, **57**(1)(2011), 181-190.

[20] Pant,V. and Pant, R.P., *Common fixed points of conditionally commuting maps*, Fixed Point Theory, **11**(1)(2010), 113-118.

[21] Pathak, H.K., Cho, Y.J. and Kang, S.M., *Remarks of R-weakly commuting mappings and common fixed point theorems*, Bull. Korean Math. Soc., **34**(2)(1997), 247-257.

[22] Pathak, H.K. and Khan, M.S., *A comparison of various types of compatible maps and common fixed points*, Indian J. Pure Appl. Math., **28**(4)(1997), 477-485.

[23] Saluja, A.S., Jain, M.K., and Jhade, P.K., *Weak semi compatibility and fixed point theorems*, Bull. Internat. Math. Virtual Inst., **2**(2)(2012), 205-217.

[24] Sessa, S., *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math., **32**(46)(1982), 149-153.

[25] Singh, B. and Jain, S., *Semi compatibility, compatibility and fixed point theorems in fuzzy metric spaces*, J. Chungcheong Math. Soc., **18**(2005), 1-22.

[26] Singh, M.R. and Singh, Y.M., *On various types of compatible maps and common fixed point theorems for non- continuous maps*, Hacettepe Journal of Mathematics and Statistics, **40**(4)(2011), 503-513.

[27] Singh, S.L. and Tomar, A., *Weaker forms of commuting maps and existence of fixed points*, J. Korea Soc. Math. Educ., Ser. B: Pure Appl. Math., **10**(3)(2003), 145-161.

[28] Vats, R.K., Kumar, S. and Sihag, V., *Common fixed point theorem for expansive mappings in G-metric Spaces*, Journal of Mathematics and Computer Science, **6**(2013), 60-71.

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