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THE PROOF–THEORETICAL ANALYSIS OF CONTRACTION–LESS RELEVANT LOGICS

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Abstract. This is a short overview on gentzenization of some distributive contraction–less relevant logics. We analyze problems which may occur in formulating sequent calculi for the logics RW_+ and RW and we give their solutions, the well–known as well as some recent ones.

1. Introduction

By 'relevant logics', we mean logics where thinning:

$$\alpha \to (\beta \to \alpha)$$

is not provable. One way to obtain sequent calculi of thinning–less logics is to drop the structural rule of thinning, from Gentzen's systems LJ and LK [**8**]. But then, the distribution of conjunction over disjunction would not be provable. Really, the

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structural rule of thinning is crucial in the proof of the distribution law:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha\vdash\alpha}{\alpha,\beta\vdash\alpha}\,{\scriptstyle(\text{thinning})}\qquad\dfrac{\beta\vdash\beta}{\alpha,\beta\vdash\beta}\,{\scriptstyle(\text{thinning})}}{\alpha,\beta\vdash\alpha\wedge\beta}\,{\scriptstyle(\vdash\wedge)}}{\alpha,\beta\vdash(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)}\,{\scriptstyle(\vdash\vee)}\qquad\begin{array}{c}\vdots\\ \alpha,\gamma\vdash(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)\end{array}}{\alpha,\beta\vee\gamma\vdash(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)}\,{\scriptstyle(\vee\vdash)}}{\alpha\wedge(\beta\vee\gamma),\alpha\wedge(\beta\vee\gamma)\vdash(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)}\,{\scriptstyle(\wedge\vdash)}}{\alpha\wedge(\beta\vee\gamma)\vdash(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)}\,{\scriptstyle(\text{contraction})}}{\vdash(\alpha\wedge(\beta\vee\gamma))\to.(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)}\,{\scriptstyle(\vdash\to)}
$$

The problem of formulating sequent calculi where the distribution of $\wedge$ over $\vee$ is provable in the absence of Gentzen's thinning is solved by Professors Dunn [**6**] and Minc [**12**], by introducing two types of sequences of formulae.

2. Two types of sequences of formulae

It is well–known that, in Gentzen's calculi LJ and LK [**8**], the sequent $\alpha_1,\ldots,\alpha_n\vdash\beta$ has the same informal meaning as the formula $\alpha_1\wedge\ldots\wedge\alpha_n\to\beta$. This means that, in those calculi, a sequence of formulae from an antecedent of a sequent, represents the conjunction of those formulae.

However, the effect of the absence of thinning in relevant logics (we have the same effect in the absence of contraction) is that the classical connectives split into dual pairs. Thus, in relevant logics, we have two different conjunctions, extensional conjunction $\wedge$ and intensional conjunction $\circ$, with the following properties: the formulae $\alpha\wedge\beta\to\alpha$ and $\alpha\wedge\beta\to\beta$ are valid, but the formulae $\alpha\circ\beta\to\alpha$ and $\alpha\circ\beta\to\beta$ are not, and on the other hand, the formula $(\alpha\circ\beta\to\gamma)\to(\alpha\to.\beta\to\gamma)$ is valid, but the formula $(\alpha\wedge\beta\to\gamma)\to(\alpha\to.\beta\to\gamma)$ is not.

With two different conjunctions, two kinds of sequences in an antecedent of a sequent can be defined, in sequent calculi of relevant logics: *extensional* sequences to stand in for extensional conjunction $\wedge$ and *intensional* sequences to stand in for intensional conjunction $\circ$. Two different punctuation marks are to be used to denote them: usually, semicolons are used for intensional, and commas for extensional sequences. Due to that, e.g. sequents $\alpha_1,\ldots,\alpha_n\vdash\beta$ and $\alpha_1;\ldots;\alpha_n\vdash\beta$ are two different sequents, such that the former corresponds to the formula $\alpha_1\wedge\ldots\wedge\alpha_n\to\beta$ and the later to the formula $\alpha_1\circ\cdots\circ\alpha_n\to\beta$.

Intensional and extensional sequences must be allowed to be nested within one another, otherwise the Cut Elimination Theorem would not be provable. Therefore, Professor Dunn gives the following definition of an antecedent of a sequent in sequent calculi of relevant logics [**7**]: an antecedent is either an intensional sequence of formulae, or an intensional sequence of extensional sequences of formulae, or etc, or the same thing but with 'intensional' and 'extensional' interchanged.

Usually, Greek capitals are used to denote an antecedent of a sequent. With square brackets (e.g with $\Gamma[\Pi]$) a specific occurrence of a sequence (Π) within a sequence (Γ) is emphasized.

With intensional and extensional sequences, we can define two types of structural rules, also: extensional and intensional ones. As for the structural rule of thinning on the left, we may have intensional and extensional thinning:

$$\frac{\Gamma[\Sigma] \vdash \gamma}{\Gamma[\Sigma;\Pi] \vdash \gamma}\ (\text{KI}) \qquad \frac{\Gamma[\Sigma] \vdash \gamma}{\Gamma[\Sigma,\Pi] \vdash \gamma}\ (\text{KE})$$

(for now, we analyze positive, i.e. negation–less, relevant logics only, and their corresponding single–conclusion sequent calculi). On the other hand, we may also have two different contractions, intensional and extensional one:

$$\frac{\Gamma[\Pi;\Pi] \vdash \gamma}{\Gamma[\Pi] \vdash \gamma}\ (\text{WI}) \qquad \frac{\Gamma[\Pi,\Pi] \vdash \gamma}{\Gamma[\Pi] \vdash \gamma}\ (\text{WE}).$$

We shall see that in sequent calculi of relevant logics, some of those structural rules are forbidden, some of them are needed and some of them are allowed.

In relevant logics, the formula:

$$(\alpha \circ \beta \to \gamma) \to (\alpha \to .\beta \to \gamma)$$

is valid therefore the inference:

$$\frac{\alpha;\beta \vdash \alpha}{\alpha \vdash \beta \to \alpha}$$

is correct, unlike the inference:

$$\frac{\alpha,\beta \vdash \alpha}{\alpha \vdash \beta \to \alpha}$$

which is not correct, since the formula:

$$(\alpha \wedge \beta \to \gamma) \to (\alpha \to .\beta \to \gamma)$$

is not valid. Consequently, the following derivation:

$$\frac{\alpha \vdash \alpha}{\alpha,\beta \vdash \alpha}\ (\text{KE})$$

is harmless (it doesn't lead to irrelevance, since from $\alpha,\beta \vdash \alpha$ it is not possible to derive $\alpha \vdash \beta \to \alpha$, from where thinning would be derivable), unlike the derivation:

$$\frac{\alpha \vdash \alpha}{\alpha;\beta \vdash \alpha}\ (\text{KI})$$

with which we would have the following proof of thinning:

$$\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha;\beta \vdash \alpha}\ (\mathrm{KI})}{\alpha \vdash \beta \to \alpha}\ (\vdash\to)}{\vdash \alpha \to (\beta \to \alpha)}\ (\vdash\to)$$

So, to disable the inference of $\alpha \to (\beta \to \alpha)$, it is enough to forbid the structural rule of thinning in intensional sequences. On the other hand, thinning in extensional sequences is not only harmless, it is needed, together with contraction in extensional sequences, for the proof of the distribution law. The proof of the distribution law in relevant logics is the same as the Gentzen's proof above, except that instead of Gentzen's rules of thinning and contraction, we use extensional thinning and contraction rules.

Therefore, sequent calculi for relevant logics are without the intensional thinning rule and they are with the extensional thinning rule and the extensional contraction rule. The intensional contraction rule is allowed: it is absent in sequent calculi of contraction–less relevant logics, only.

3. Problems with the structural rule of cut

Another problem in formulating sequent calculi of relevant logics is how to disable the inference of the modal fallacy, in the presence of the rule of cut. Namely, Professors Dunn [**6**],[**7**] and Minc [**12**] also point out that an empty left–hand side, in the left premise of the cut rule, can lead to irrelevance. Really, the modal fallacy would be derivable, then:

$$\dfrac{\dfrac{\vdash \beta \to \beta \qquad \dfrac{\beta \to \beta \vdash \beta \to \beta}{\alpha, \beta \to \beta \vdash \beta \to \beta}\ (\mathrm{KE})}{\alpha \vdash \beta \to \beta}\ (\mathrm{cut})}{\vdash \alpha \to (\beta \to \beta)}\ (\vdash\to)$$

To disable the inference of the modal fallacy, Professor Dunn defines the cut rule of the following form (see [**7**]):

$$\dfrac{\Pi \vdash \varphi \qquad \Gamma[\varphi] \vdash \gamma}{\Gamma[\Pi] \vdash \gamma}\ (\mathrm{cut})$$

where $\Gamma[\Pi]$ is the result of replacing arbitrarily many occurrences of φ in $\Gamma[\varphi]$ by Π if Π is non–empty, and otherwise by $'t'$. With this cut, the modal fallacy is not provable:

$$\dfrac{\vdash \beta \to \beta \qquad \dfrac{\beta \to \beta \vdash \beta \to \beta}{\alpha, \beta \to \beta \vdash \beta \to \beta}\,(\mathrm{KE})}{\dfrac{\alpha, t \vdash \beta \to \beta}{\dfrac{\cdots}{\vdash \alpha \to (\beta \to \beta)}\,?}}\,(\mathrm{cut})$$

but, the presence of $'t'$, causes another problem.

Namely, Hilbert–style formulations of relevant logics, contain the rule *modus ponens* (see e.g. [**1**], §27.1):

$$\frac{\alpha \qquad \alpha \to \beta}{\beta}.$$

To prove the equivalence between sequent calculi and their Hilbert–style formulations, we need to prove the admissibility of *modus ponens*, i.e. we need to prove that whenever sequents $\vdash \alpha$ and $\vdash \alpha \to \beta$ are both derivable in our sequent calculi, so is $\vdash \beta$. With the above cut, this is not straightforward. Really, with this cut we have the proof of $t; t \vdash \beta$:

$$\dfrac{\vdash \alpha \qquad \dfrac{\vdash \alpha \to \beta \qquad \dfrac{\alpha \vdash \alpha \qquad \beta \vdash \beta}{\alpha; \alpha \to \beta \vdash \beta}\,(\to\vdash)}{\alpha; t \vdash \beta}\,(\mathrm{cut})}{t; t \vdash \beta}\,(\mathrm{cut})$$

but not the proof of $\vdash \beta$. To obtain the proof of $\vdash \beta$, after the elimination of cut, the additional techniques are needed to eliminate the constant $'t'$.

As for contraction–less relevant logics, similar technique is used by Giambrone [**9**] and Brady [**4**] in gentzenizations of RW_+ and RW.

4. Giambrone's gentzenization of RW_+

RW_+ is the positive fragment of RW. The Hilbert–type formulation of RW can be given by the following group of axioms and rules (RW is exactly the system R in [**1**], p. 341., without the contraction axiom R4: $(\alpha \to .\alpha \to \beta) \to .\alpha \to \beta)$):

$Ax1.$ $\alpha \to \alpha$
$Ax2.$ $(\alpha \to \beta) \to .\beta \to \gamma \to .\alpha \to \gamma$
$Ax3.$ $\alpha \to .(\alpha \to \beta) \to \beta$
$Ax4.$ $\alpha \wedge \beta \to \alpha$
$Ax5.$ $\alpha \wedge \beta \to \beta$
$Ax6.$ $(\alpha \to \beta) \wedge (\alpha \to \gamma) \to .\alpha \to .\beta \wedge \gamma$
$Ax7.$ $\alpha \to \alpha \vee \beta$
$Ax8.$ $\beta \to \alpha \vee \beta$
$Ax9.$ $(\alpha \to \gamma) \wedge (\beta \to \gamma) \to .(\alpha \vee \beta) \to \gamma$
$Ax10.$ $\alpha \wedge (\beta \vee \gamma) \to (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$

$Ax11.\ (\alpha \to\sim \beta) \to .\beta \to\sim \alpha$
$Ax12.\ \sim\sim \alpha \to \alpha$

$R'.\quad \dfrac{\alpha \qquad \alpha \to \beta}{\beta}$ (modus ponens) $\qquad R''.\quad \dfrac{\alpha \qquad \beta}{\alpha \wedge \beta}$ (adjunction)

To obtain a sequent calculus for RW_+, Giambrone first formulates a cut–free sequent calculus $LRW_+^{\circ t}$ for the logic $RW_+^{\circ t}$ ($RW_+^{\circ t}$ is the system RW_+ with the postulates for $\circ$ and $'t'$), where sequents are never allowed to have empty left–hand sides, they have $'t'$ there instead. Then he expands $LRW_+^{\circ t}$ to include sequents with empty antecedents, which are harmless in the absence of the rule of cut. In this expanded system, he eliminates $'t'$ and from this modified system, he obtains the sequent calculus for RW_+, by the conservative extension theorem for $\circ$.

In the next section we shall give another formulation of the rule of cut, for positive relevant logics, where the constant $'t'$ is not needed, and we propose another gentzenization of RW_+, where this cut is used.

5. Cut without $'t'$, in positive relevant logics

To disable the inference of the modal fallacy, we suggest the cut rule of the following forms:

Cut:

$$\frac{\Pi \vdash \varphi \qquad \Gamma[\varphi] \vdash \gamma}{\Gamma[\Pi] \vdash \gamma}\ (\text{cut}-\text{i}) \qquad\qquad \frac{\vdash \varphi \qquad \varphi \vdash \gamma}{\vdash \gamma}\ (\text{cut}-\text{iii})$$

$$\frac{\vdash \varphi \qquad \Gamma[\varphi;\Pi] \vdash \gamma}{\Gamma[\Pi] \vdash \gamma}\ (\text{cut}-\text{ii}),$$

provided Π is non–empty.

In (cut-i), $\Gamma[\Pi]$ is the result of replacing exactly one occurrence of φ in $\Gamma[\varphi]$ by Π; in (cut-ii) the single occurrence of φ in $\Gamma[\varphi;\Pi]$ is replaced by an empty sequence; similarly in (cut-iii).

With this cut, the modal fallacy is not provable:

$$\dfrac{\dfrac{\beta \vdash \beta}{\vdash \beta \to \beta}\ (\to\ \text{r}) \qquad \dfrac{\beta \to \beta \vdash \beta \to \beta}{\alpha, \beta \to \beta \vdash \beta \to \beta}\ (\text{KE})}{\dfrac{\alpha \vdash \beta \to \beta}{\vdash \alpha \to (\beta \to \beta)}\ (\to\ \text{r})}\ (\text{cut}-?)$$

since $\vdash \beta \to \beta$ and $\alpha, \beta \to \beta \vdash \beta \to \beta$ cannot be the upper sequents of any form of the proper cut.

We use this cut to formulate a sequent calculus GRW_+° of RW_+° (see [**11**]):

Axiom:

$\alpha \vdash \alpha$

Structural rules (Π and Σ are non–empty):

extensional contraction:

$$\frac{\Gamma[\Pi, \Pi] \vdash \gamma}{\Gamma[\Pi] \vdash \gamma}\ \text{(WE)}$$

extensional thinning:

$$\frac{\Gamma[\Sigma] \vdash \gamma}{\Gamma[\Pi, \Sigma] \vdash \gamma}\ \text{(KE)}$$

cut is as above.

Operational rules:

$$\frac{\Gamma_1 \vdash \alpha \qquad \Gamma_2[\beta] \vdash \gamma}{\Gamma_2[\Gamma_1; \alpha \to \beta] \vdash \gamma}\ (\to\ \text{l}) \qquad \frac{\alpha; \Gamma \vdash \beta}{\Gamma \vdash \alpha \to \beta}\ (\to\ \text{r})$$

$$\frac{\Gamma[\alpha; \beta] \vdash \gamma}{\Gamma[\alpha \circ \beta] \vdash \gamma}\ (\circ\ \text{l}) \qquad \frac{\Gamma_1 \vdash \alpha \qquad \Gamma_2 \vdash \beta}{\Gamma_1; \Gamma_2 \vdash \alpha \circ \beta}\ (\circ\ \text{r})$$

$$\frac{\Gamma[\alpha] \vdash \gamma}{\Gamma[\alpha \wedge \beta] \vdash \gamma} \quad \frac{\Gamma[\beta] \vdash \gamma}{\Gamma[\alpha \wedge \beta] \vdash \gamma}\ (\wedge\ \text{l}) \qquad \frac{\Gamma \vdash \alpha \qquad \Gamma \vdash \beta}{\Gamma \vdash \alpha \wedge \beta}\ (\wedge\ \text{r})$$

$$\frac{\Gamma[\alpha] \vdash \gamma \qquad \Gamma[\beta] \vdash \gamma}{\Gamma[\alpha \vee \beta] \vdash \gamma}\ (\vee\ \text{l}) \qquad \frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta} \quad \frac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \vee \beta}\ (\vee\ \text{r})$$

Unlike Giambrone's gentzenization of RW_+°, which is based on intensional and extensional *sequences* of formulae, our calculus is based on intensional and extensional *multisets* (lists without order) of formulae. Therefore, the structural rule of interchange, which is available equally for extensional and intensional sequences in Giambrone's gentzenization, is implicit in GRW_+°.

In [**11**] we prove that GRW_+° is exactly the sequent calculus corresponding to RW_+°, and we prove the Cut–elimination theorem and the Subformula property for GRW_+°. After the cut is eliminated, the sequent calculus for RW_+ can be obtained from the system GRW_+°, by removing the rules for $\circ$. The reason why we have to consider the system with $\circ$ first, is that we need $\circ$ to define the interpretation of intensional multisets in terms of formulae, which is needed in the proof of the equivalence between Hilbert–style formulation and sequent calculus formulation of the logic.

It should be mentioned that although we can avoid the use of $'t'$ in sequent calculus formulation of RW_+ (and R_+, but we do not elaborate on this here) the use of $'t'$ remains crucial in sequent calculus for TW_+ and in sequent calculi for other weaker, permutation–less, relevant logics such as B_+, E_+ and even $T_\to$, where $'t'$ precludes intensional structures from becoming scrambled, see e.g. [**3**] (there is a sequent calculus for $T_\to$ without $'t'$ in [**2**], however the one with $'t'$ is much easier to use).

6. How to add negation?

In this section we analyze sequent calculus formulations of the logic RW. Brady in [**4**] establishes the first gentzenization of this logic, proceeding in exactly the same way as Giambrone in [**9**]. Namely, he first formulates the sequent calculus $LRW^{\circ t}$ for $RW^{\circ t}$, where sequents with empty left–hand sides are not allowed. After the proof of the Cut–elimination theorem in $LRW^{\circ t}$, he expands this system to include sequents with empty antecedents. From this expanded system, $L'RW^{\circ t}$, he obtains the sequent calculus $L'RW$ for RW, by the conservative extension theorems for $\circ$ and $'t'$. However, the first problem, i.e. the problem of adding negation to Giambrone's system and formulating the system $LRW^{\circ t}$, was not easy to solve.

It is well–known that to enable the proof of the formula $\sim\sim \alpha \to \alpha$, Gentzen allows multiple–conclusion sequents. With multiple–conclusion sequents, the rule of cut should be modified, to allow multiple conclusions. To avoid the change of Dunn's rule of cut and to enable the inference of the formula $\sim\sim \alpha \to \alpha$ in a sequent calculus with single–conclusion sequents, only, Brady uses signed formulae, $T\,\alpha$ and $F\,\alpha$ instead of just formulae α, with logical rules for both types of signed formulae. For example, the proof of $\sim\sim \alpha \to \alpha$ in $LRW^{\circ t}$ is:

$$\dfrac{\dfrac{\dfrac{T\alpha \vdash T\alpha}{F \sim \alpha \vdash T\alpha}\,(F\sim\vdash)}{T \sim\sim \alpha \vdash T\alpha}\,(T\sim\vdash) \qquad \dfrac{\dfrac{F\alpha \vdash F\alpha}{F\alpha \vdash T \sim \alpha}\,(\vdash T\sim)}{F\alpha \vdash F \sim\sim \alpha}\,(\vdash F\sim)}{\vdash T \sim\sim \alpha \to \alpha}\,(\vdash T\to)$$

Brady's rule $(\vdash T \to)$:

$$\frac{\Gamma; T\alpha \vdash T\beta \qquad \Gamma; F\beta \vdash F\alpha}{\Gamma \vdash T\alpha \to \beta}$$

is, unusually, two–premise rule. This is needed for the elimination of cut, to pair the rule $(T \to\vdash)$:

$$\frac{\Gamma \vdash T\alpha \qquad \Pi[T\beta] \vdash S\gamma}{\Pi[T\alpha \to \beta; \Gamma] \vdash S\gamma} \qquad \frac{\Gamma \vdash T\alpha \qquad \Pi[T\beta] \vdash}{\Pi[T\alpha \to \beta; \Gamma] \vdash}$$

$$\frac{\Gamma \vdash F\beta \qquad \Pi[F\alpha] \vdash S\gamma}{\Pi[T\alpha \to \beta; \Gamma] \vdash S\gamma} \qquad \frac{\Gamma \vdash F\beta \qquad \Pi[F\alpha] \vdash}{\Pi[T\alpha \to \beta; \Gamma] \vdash}$$

By formulating $L'RW$, Brady answered the question: "How do we gentzenize RW, by enlarging Giambrone's systems, with the rules for negation?". However, it is clear that $L'RW$ is not a simple system. Moreover, it is formulated with the help of $'t'$. Our motive, in gentzenizing RW, is to avoid the use of $'t'$. As the consequence, we obtain the calculus GRW, which has a much neater presentation than $L'RW$.

The calculus GRW is based on right-handed sequents, with ordinary, unsigned formulae. The postulates of GRW are:

Axiom:

$\vdash \alpha^*; \alpha$

Structural rules (Π and Σ are non–empty):

extensional contraction:

$$\frac{\vdash \Gamma[\Pi, \Pi]}{\vdash \Gamma[\Pi]}\ (\mathrm{WE})$$

extensional thinning:

$$\frac{\vdash \Gamma[\Sigma]}{\vdash \Gamma[\Pi, \Sigma]}\ (\mathrm{KE})$$

cut:

$$\frac{\vdash \Gamma[\alpha] \qquad \vdash \alpha^*; \Pi}{\vdash \Gamma[\Pi]}\ (\mathrm{cut{-}i}) \qquad \frac{\vdash \Gamma[\alpha^*] \qquad \vdash \alpha; \Pi}{\vdash \Gamma[\Pi]}\ (\mathrm{cut{-}ii})$$

Operational rules:

$$\frac{\vdash \Gamma_1; \alpha \qquad \vdash \Gamma_2[\beta^*]}{\vdash \Gamma_2[\Gamma_1; (\alpha \to \beta)^*]}\ (\to_1^*) \qquad \frac{\vdash \Gamma_1[\alpha] \qquad \vdash \Gamma_2; \beta^*}{\vdash \Gamma_1[\Gamma_2; (\alpha \to \beta)^*]}\ (\to_2^*) \qquad \frac{\vdash \Gamma[\alpha^*; \beta]}{\vdash \Gamma[\alpha \to \beta]}\ (\to)$$

$$\frac{\vdash \Gamma[\alpha]}{\vdash \Gamma[(\sim \alpha)^*]}\ (\sim^*) \qquad \frac{\vdash \Gamma[\alpha^*]}{\vdash \Gamma[\sim \alpha]}\ (\sim)$$

$$\frac{\vdash \Gamma[\alpha^*]}{\vdash \Gamma[(\alpha \wedge \beta)^*]} \quad \frac{\vdash \Gamma[\beta^*]}{\vdash \Gamma[(\alpha \wedge \beta)^*]}\ (\wedge^*) \qquad \frac{\vdash \Gamma[\alpha] \qquad \vdash \Gamma[\beta]}{\vdash \Gamma[\alpha \wedge \beta]}\ (\wedge)$$

$$\frac{\vdash \Gamma[\alpha^*] \qquad \vdash \Gamma[\beta^*]}{\vdash \Gamma[(\alpha \vee \beta)^*]}\ (\vee^*) \qquad \frac{\vdash \Gamma[\alpha]}{\vdash \Gamma[\alpha \vee \beta]} \quad \frac{\vdash \Gamma[\beta]}{\vdash \Gamma[\alpha \vee \beta]}\ (\vee)$$

The notion of a formula with $*$ is defined in [**10**], where the Cut Theorem and other significant theorems are briefly discussed. (Informally, α^* can be understood as the negation of α.)

The proof of $\vdash \sim\sim \alpha \to \alpha$ in GRW is:

$$\frac{\dfrac{\dfrac{\vdash \alpha^*; \alpha}{\vdash \sim \alpha; \alpha}\ (\sim)}{\vdash (\sim\sim \alpha)^*; \alpha}\ (\sim^*)}{\vdash \sim\sim \alpha \to \alpha}\ (\to)$$

The proof of the distribution law in GRW is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \alpha^*;\alpha}{\vdash (\alpha^*,\beta^*);\alpha}\,(\mathrm{KE}) \qquad \dfrac{\vdash \beta^*;\beta}{\vdash (\alpha^*,\beta^*);\beta}\,(\mathrm{KE})}{\vdash (\alpha^*,\beta^*);\alpha\wedge\beta}\,(\wedge)}{\vdash (\alpha^*,\beta^*);(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)}\,(\vee) \qquad \begin{matrix}\vdots\\ \vdash (\alpha^*,\gamma^*);(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)\end{matrix}}{\vdash (\alpha^*,(\beta\vee\gamma)^*);(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)}\,(\vee)}{\vdash ((\alpha\wedge(\beta\vee\gamma))^*,(\alpha\wedge(\beta\vee\gamma))^*);(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)}\,(\wedge^*)}{\vdash (\alpha\wedge(\beta\vee\gamma))^*;(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)}\,(\mathrm{WE})}{\vdash (\alpha\wedge(\beta\vee\gamma))\to.(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)}\,(\to)$$

Modus ponens is derivable rule in GRW:

$$\dfrac{\vdash \alpha \qquad \dfrac{\vdash \alpha\to\beta \qquad \dfrac{\vdash \alpha^*;\alpha \qquad \vdash \beta^*;\beta}{\vdash \alpha^*;(\alpha\to\beta)^*;\beta}\,(\to^*)}{\vdash \alpha^*;\beta}\,(\text{cut-i})}{\vdash \beta}\,(\text{cut}-\text{i})$$

but the modal fallacy is not:

$$\dfrac{\dfrac{\dfrac{?}{\vdash \alpha^*;\beta^*;\beta}\,(\text{non}-\text{cut})}{\vdash \alpha^*;\beta\to\beta}\,(\to)}{\vdash \alpha\to(\beta\to\beta)}\,(\to)$$

7. Conclusion and future research

This paper is the summary of the results given in [**9**], [**4**], [**11**] and [**10**], concerning gentzenizations of distributive contraction–less relevant logics RW_+ and RW. We explain the motivation of Professors Dunn, Minc, Giambone and Brady, for the use of the truth constant $'t'$, but also we propose other solutions where the use of $'t'$ is not needed. It should be mentioned that there are other gentzenizations of those logic, e.g. Brady [**5**] establishes sequent calculi of the large class of major relevant logics from B through to R, including RW, all with distribution, by formulating cut-free left-handed calculi. However, the use of $'t'$ and the use of sequents based on signed formulae $T\,\alpha$ and $F\,\alpha$, indicate that, at least some of them can probably be gentzenized by simpler calculi.

We hope that our procedure can be applied in gentzenizations of R and R_+, but this we will leave for the future research.

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