Sciencia Acta Xaveriana
An International Science Journal
ISSN. 0976-1152

Volume 5
No. 2
pp. 19-30
September 2014

Maximum Independent Set Cover Pebbling Number of an m-ary Tree

A. Lourdusamy[1], C. Muthulakhmi @ Sasikala[2] and T. Mathivanan[3]

[1, 3] Department of Mathematics
St. Xavier's College,
Palayamkottai - 627 002.

[1] lourdusamy15@gmail.com

[3] tahit_van_man@yahoo.com

[2] Assistant Professor in mathematics,
Sri Paramakalyani College,
Alwarkurichi, India.

Abstract : A pebbling move is defined by removing two pebbles from some vertex and placing one pebble on an adjacent vertex. A graph is said to be cover pebbled if every vertex has a pebble on it after a series of pebbling moves. The maximum independent set cover pebbling number of a graph G is the minimum number, $\rho(G)$, of pebbles required so that any initial configuration of $\rho(G)$ pebbles can be transformed by a sequence of pebbling moves so that after the pebbling moves the set of vertices that contains pebbles form a maximum independent set S of G. In this paper, we determine the maximum independent set cover pebbling number of an m-ary tree.

Key words: Graph pebbling, cover pebbling, maximum independent set cover pebbling, m-ary tree.

1. Introduction

Given a graph G, distribute k pebbles on its vertices in some configuration, call it as C. Assume that G is connected in all cases. A pebbling move is defined by removing two pebbles from some vertex and placing one pebble on an adjacent vertex. [1] The pebbling number $\pi(G)$ is the minimum number of pebbles that are sufficient, so that for any initial configuration of $\pi(G)$ pebbles, it is possible to move a pebble to any root vertex v in G. [2] The cover pebbling number $\gamma(G)$ is defined as the minimum number of pebbles needed to place a pebble on every vertex of the graph using a sequence of pebbling moves, regardless of the initial configuration. A set S of vertices in a graph G is said to be an independent set (or an internally stable set) if no two vertices in the set S are adjacent. An independent set S is maximum if G has no independent set S' with $|S'| > |S|$.

We introduce the concept maximum independent set cover pebbling number in [4]. The maximum independent set cover pebbling number, $\rho(G)$, of a graph G, to be the minimum number of pebbles that are placed on $V(G)$ such that after a sequence of pebbling moves, the set of vertices with pebbles forms a maximum independent set S of G, regardless of their initial configuration. In this paper, we determine the maximum independent set cover pebbling number $\rho(G)$ for an m-ary tree.

Notation: $f(a)$ denotes the number of pebbles placed at the vertex a. Also $f(G)$ denotes the number of pebbles on the graph G.

2. Maximum independent set cover pebbling number of an m-ary tree

Definition 2.1. A complete m-ary tree, denoted by M_n, is a tree of height n with m^i vertices at distances i from the root. Each vertex of M_n has m children except for the set of m^n vertices that are at distance n away from the root, none of which have children. The root is denoted by R_n.

Or Simply a complete m -ary tree with height n , denoted by M_n, is an m -ary tree satisfying that v has m children for each vertex v not in the n th level.

Theorem 2.2. (i) $\rho(M_0) = 1$ (obvious).

(ii) $\rho(M_1) = 4m-3$ ($m \geq 3$) and if $m = 2$ then $\rho(M_1) = 6$. Since, for $m \geq 3$, $M_1 \equiv K_{1,m}$[4] and for $m = 2$, $M_1 \equiv P_3$, the path of length two[5].

(iii) $\rho(M_2) = 16m^2-12m+1$.

Proof of (iii). Note that M_2 contains m-M_1's as subtrees which are all connected to the root R_2 of M_2. Let $R_{11}, R_{12}, \ldots, R_{1m}$ be the root of the m-M_1's (say $M_{11}, M_{12}, \ldots, M_{1m}$). In general, M_n contains m-M_{n-1}'s as subtrees which are all connected to the root R_n of M_n. Let $R_{(n-1)1}, R_{(n-1)2}, \ldots, R_{(n-1)m}$ be the root of the m-$M_{(n-1)}$'s. Choose the rightmost vertex of this subtree, label it by v. Put $16m^2-12m$ pebbles on this vertex. Then we cannot cover the maximum independent set of M_2. Thus $\rho(M_2) \geq 16m^2-12m+1$.

Now consider the distribution of $16m^2-12m+1$ pebbles on the vertices of M_2. According to the distribution of these amounts of pebbles, we find the following cases:

Case 1 : $f(M_{1i}) \geq 4m-3$, where $1 \leq i \leq m$.

Clearly we are done if $f(R_2) \geq 1$. So assume that, $f(R_2) = 0$. This implies that $\sum_{i=1}^{m} f(M_{1i}) = 16m^2 - 12m + 1$. Any one of the m^2 paths (of length two) leading from the root R_2 to the bottom of M_2 must contain at least four pebbles and hence we are done, since any one the subtree contains at least $\left\lceil \frac{16m^2-12m+1}{m} \right\rceil \geq 16m-12+1$ pebbles.

Case 2: $f(M_{1i}) \leq 4m-4$, for all i $(1 \leq i \leq m)$

This implies that $f(R_2) \geq 16m^2-12m+1-m(4m-4)=12m^2-8m+1$. We need $2m(4m-3)+1$ pebbles at R_2. But $f(R_2)-2m(4m-3)-1>0$.

Case 3 : $f(M_{1i}) \geq 4m-3$ for some i $(1 \leq i \leq m)$.

Let $t \geq 1$ subtrees of M_2 contains at least 4m-3 pebbles. Note that, for every subtree (except one subtree that contains 4m-3 or more pebbles, we have 16m pebbles to cover its maximum independent set.

Let $f(M'_{1j}) = a_j$ where $a_j \leq 4m-4$. Thus, to cover the maximum independent set of the subtree M'_{1j}, we have another $16m-a_j$ pebbles somewhere on the graph. So, we can send $\left\lfloor \frac{16m-a_j}{4} \right\rfloor \geq 4m-\frac{a_j}{4}$ pebbles to the root R_2 and then we move $2m-\frac{a_j}{8}$ pebbles to the root R'_{1j} of M'_{1j}. Thus M'_{1j} contains $a_j+2m-\frac{a_j}{8}=2m+\frac{7}{8}a_j$. But these numbers of pebbles are enough to cover the maximum independent set of M'_{1j}, or the value of $2m+\frac{7}{8}a_j \geq 4m-3$, and hence we are done. So using $(m-t)(16m-a_j)-\sum_{i=1}^{t} a_j$ pebbles, we cover the maximum independent set of the (m-t) subtrees that contains a_j pebbles. So we have at least $(t-1)16m+4m+1$

pebbles on the t-subtrees plus R_2 that are all contains 4m-3 or more pebbles. If $f(R_2) \geq 1$ then we are done. Otherwise we can always move a pebble to R_2 using at most four pebbles from the remaining pebbles on the t-subtrees.

(iv) $\rho(M_3) = 64m^3 - 48m^2 + 4m - 15$ ($m \geq 3$).

Proof of (iv). Clearly, M_3 contains m-M_2's as subtrees which are all connected to the root R_3 of M_3. Consider the rightmost bottom vertex, say v, of M_3 and put $64m^3 - 48m^2 + 4m - 16$ pebbles on the vertex v. Then we cannot cover the maximum independent set of M_3. Thus $\rho(M_3) \geq 64m^3 - 48m^2 + 4m - 15$.

Now consider the distribution of $64m^3 - 48m^2 + 4m - 15$ pebbles on the vertices of M_3. According to the distribution of these amounts of pebbles, we find the following cases:

Case 1 : $f(M_{2i}) \geq \rho(M_2)$ where $1 \leq i \leq m$.

Clearly we are done if $f(R_2) = 0$, or 2 or $f(R_2) \geq 4$. So assume that $f(R_2) = 1$ or 3. This implies that, $\sum_{i=1}^{m} f(M_{2i}) \geq 64m^3 - 48m^2 + 4m - 18$ .pebbles. So, any one of the path (of length three) leading from the root R_3 to the bottom row of M_3 must contain at least eight pebbles. Thus we move a pebble to R_3 and hence we are done.

Case 2 : $f(M_{2i}) < \rho(M_2)$ where $1 \leq i \leq m$.

We need $2m\,\rho(M_2)+5$ pebbles on the root vertex R_3 of M_3. We have $\rho(M_3) - m\,\rho(M_2) + m$ pebbles on the root vertex R_3. But, $\rho(M_3) - m\,\rho(M_2) + m - (2m\,\rho(M_2)+5) \geq 0$. Since, $\rho(M_3) = 64m^3 - 48m^2 + 4m - 15$, $\rho(M_2) = 16m^2 - 12m + 1$ and $m \geq 3$.

Case 3 : $f(M_{2i}) \geq \rho(M_2)$ for some i ($1 \leq i \leq m$).

Let $t \geq 1$ subtrees contains $\rho(M_2)$ or more pebbles. Label those subtrees by M_{2i} ($1 \leq i \leq t$) and label the other subtrees by M'_{2j} ($1 \leq i \leq m-t$). Also, let $f(M'_{2j}) = a_j$ where $a_j < \rho(M_2)$. Note that, we have usually $(64m^2+16)(m-1)$ pebbles each to cover the maximum independent set of M_{2i}'s and M'_{2j}'s, except one subtree M_{2k} ($1 \leq k \leq t$) that contains $\rho(M_2)$ or more pebbles.

Since $a_j < \rho(M_2)$, we have another $64m^2+16-a_j$ pebbles that are in somewhere of the graph M_3 to cover the maximum independent set of M'_{2j}. So we can send $\left\lfloor \frac{64m^2+16-a_j}{8} \right\rfloor \geq 8m^2+2-\frac{a_j}{8}$ pebbles to the root R_3 and then we move $4m^2+1-\frac{a_j}{16}$ pebbles to the root R'_{2j} of M'_{2j}. Thus, M'_{2j} contains $4m^2+1+\frac{15}{16}a_j$ pebbles. But these number of pebbles are at least $\rho(M_2)$ or it is enough to cover the maximum independent set of M'_{2j} using the pebbles at R'_{2j} plus a_j pebbles. Thus the t-subtrees M_{2i} plus R_3 contains $(64m^2+16)(t-1)+ 16m^2-12m+1$ or more pebbles. We know that $f(M_{2i}) \geq \rho(M_2)$ where $1 \leq i \leq t$. Let $f(R_3)=1$ or 3 (Otherwise, we are done). We can move a pebble to R_3, using at most eight pebbles from the subtree that contains $16m^2-12m+9$ pebbles or more. And hence we are done.

(v) $\rho(M_4) = 256m^4 - 192m^3 + 16m^2 - 60m + 1$.

Proof of (v): Consider the rightmost bottom vertex, say v, of M_4, and put $256m^4 - 192m^3 + 16m^2 - 60m$ pebbles. Then we cannot cover the maximum independent set of M_4. Thus, $\rho(M_4) \geq 256m^4 - 192m^3 + 16m^2 - 60m + 1$.

Now consider the distribution of $256m^4 - 192m^3 + 16m^2 - 60m + 1$ pebbles on the vertices of M_4. According to the distribution of these amounts of pebbles, we find the following cases:

Case 1: $f(M_{3i}) \geq \rho(M_3)$ for all i $(1 \leq i \leq m)$.

Clearly we are done if $f(R_4) \geq 1$. So assume that $f(R_3) = 0$. This implies that $\sum_{i=1}^{m} f(M_{3i}) = \rho(M_4) = 256m^4 - 192m^3 + 16m^2 - 60m + 1$. So any one of the m^4 paths (of length four) leading from the root R_4 to the bottom row of M_4 contains at least sixteen 'extra' pebbles. Thus we can move a pebble to R_4 and hence we are done.

Case 2: $f(M_{3i}) < \rho(M_3)$ for all i $(1 \leq i \leq m)$.

We need $2m\rho(M_3) + 1$ pebbles on the root vertex R_4 of M_4. We have $\rho(M_4) - m[\rho(M_3) - 1]$ on the root vertex R_4. Since, $\rho(M_4) = 256m^4 - 192m^3 + 16m^2 - 60m + 1$, $\rho(M_3) = 64m^3 - 48m^2 + 4m - 15$ and $m \geq 3$, we get $f(R_3) \geq 2m\rho(M_3) + 1$ and hence we are done.

Case 3: $f(M_{3i}) \geq \rho(M_3)$ for some i.

Similar to Case (iii) of previous theorems; using the hints, from that $256m^3 + 64m$ pebbles, we can send $\left\lfloor \frac{256m^3 + 64m - a_j}{16} \right\rfloor \geq 16m^3 + 4m - \frac{a_j}{16}$ to the root R_4 of M_4.

Theorem 2.3: For a complete m-ary tree M_n $(n \geq 3)$, the maximum independent set cover pebbling number is given by,

$$\rho(M_n) = (m-1)P + Q + \gamma_n = S_{1,n} + S_{2,n} + S_{3,n}$$

where $P = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} m^{n-2k-1} 2^{2n-2k}$, $Q = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(2^{2i} + (m-1) \sum_{j=1}^{n-2i-1} m^{j-1} 2^{2i+2j} \right)$ and

$\gamma_n = \begin{cases} 2^n, if\ n\ is\ even \\ 0, if\ n\ is\ odd \end{cases}$.

Proof. Consider the rightmost vertex of M_n, say v, and put $\rho(M_n) - 1$ pebbles on the vertex v. Then we cannot cover a maximum independent set of M_n. Thus the lower bound follows.

We prove the upper bound of $\rho(M_n)$ by induction on n. For n=3 and n=4, this theorem is true by previous theorem (iv) and (v). So assume the result is true for the complete m-ary tree M_{n-1} ($n \geq 5$).

Consider the distribution of $\rho(M_n)$ pebbles on the vertices of M_n. According to the distribution of these amounts of pebbles, we find the following cases:

Case (1): $f(M_{(n-1)i}) < \rho(M_{n-1})$ for all i ($1 \leq i \leq m$).

We need, $2m\rho(M_{n-1}) + 5$ pebbles on the root R_n, to cover the maximum independent set of M_n. We have to prove that

$\rho(M_n) - m\rho(M_{n-1}) + m \geq 2m\rho(M_{n-1}) + 5$. It is enough to prove that,

$\rho(M_n) \geq 3m\rho(M_{n-1}) + 2$ (for $m \geq 3$).

$$\rho(M_n) \geq 3m\rho(M_{n-1}) + 2 \qquad \text{------ (1)}$$

From the 1st term, by considering k=0 we get,

$$\rho(M_n) \geq (m-1)(m^{n-1} 2^{2n}) \qquad \text{------ (2)}$$

$S_{1,n-1} = (m-1) \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} m^{n-2k-2}\, 2^{2n-2k-2}$

$$= (m-1)(m^{n-2}2^{2n-2})\sum_{k=0}^{\lceil\frac{n-2}{2}\rceil}\frac{1}{m^{2k}2^{2k}}$$

$$S_{1,n-1} \leq \frac{8}{7}[(m-1)(m^{n-2})(2^{2n-2})] \qquad \text{------ (3)}$$

$$S_{2,n-1} = \sum_{i=0}^{\lceil\frac{n-2}{2}\rceil}\left[2^{2i} + (m-1)\sum_{j=1}^{n-2i-2} m^{j-1}2^{2i+2j}\right]$$

$$= \sum_{i=0}^{\lceil\frac{n-2}{2}\rceil} 2^{2i} + (m-1)\sum_{i=0}^{\lceil\frac{n-2}{2}\rceil} 2^{2i}\sum_{j=1}^{n-2i-2} m^{j-1}2^{2j}$$

$$\leq \frac{(2^2)^{(\frac{n-2}{2}+1)}-1}{3} + (m-1)\sum_{i=0}^{\lceil\frac{n-2}{2}\rceil} 2^{2i}\left(\frac{m^{n-2i-2}}{m-1}\right)\left(\frac{4(4^{n-2i-2})}{3}\right)$$

$$\leq \frac{2^n}{3} + \frac{[4(m-1)(m]^{n-2})[(4]^{n-2})}{3(m-1)}\sum_{i=0}^{\lceil\frac{n-2}{2}\rceil} 2^{2i}m^{-2i}2^{-4i}$$

$$\leq \frac{2^n}{3} + \frac{[(m]^{n-2})[(4]^{n-1})}{3}\sum_{i=0}^{\lceil\frac{n-2}{2}\rceil}\frac{1}{m^{2i}2^{2i}}$$

$$S_{2,n-1} \leq \frac{2^n}{3} + \frac{4[(m]^{n-2})[(4]^{n-1})}{11} \qquad \text{------ (4)}$$

and $S_{3,n-1} = \begin{cases} o & if\ n\ is\ even \\ 2^{n-1} & if\ n\ is\ odd \end{cases}$ ------- (5)

Equation (2) through (5) show that (1) holds if,

$$(m-1)(m^{n-1}2^{2n}) \geq 3m\left[\frac{8}{7}(m-1)(m^{n-2}2^{2n-2}) + \frac{2^n}{3} + \frac{4[(m]^{n-2})[(4]^{n-1})}{11} + 2^{n-1}\right] + 2$$

$$(m-1)(m^{n-1}2^{2n}) \geq \frac{24}{7}(m-1)[(m^{n-1}4^{n-1})] + m2^n + \frac{12[(m]^{n-1})[(4]^{n-1})}{11} + [3m(2]^{n-1}$$

$$(m-1) \geq \frac{24(m-1)}{7(4)} + \frac{5(2^{n-1})}{m^{n-2}[(4]^n)} + \frac{12}{44} + \frac{2}{m^{n-1}[(4]^n)}$$

$$(m-1) - \frac{24(m-1)}{7(4)} - \frac{12}{44} \geq \frac{5(2^{n-1})}{m^{n-2}(4^n)} + \frac{2}{m^{n-1}[(4]^n)}$$

$$\frac{m-1}{7} - \frac{12}{44} \geq \frac{5}{m^{n-2}(2^{n+1})} + \frac{2}{m^{n-1}[(2]^{2n})}$$

which holds for $m \geq 3$ and $n \geq 5$. Also, $\rho(M_n) \geq 3m\rho(M_{n-1}) + 2$ for $n = 3$ and $n = 4$.

Case (2): $f\left(M_{(n-1)i}\right) \geq \rho(M_{n-1})$ for all i (1 ≤ i ≤ m).

Subcase 2.1: n is odd.

If $f(R_n) = 0, 2$ or $f(R_n) \geq 4$ then clearly we are done. So assume that $f(R_n) = 1$ or 3. Then, $\rho(M_n)$I 3 or more pebbles on the $m(M_{n-1})$'s. We know that, $\rho(M_n) \geq 3m\rho(M_{n-1}) + 2$ and $\rho(M_{n-1}) \geq (m-1)(m^{n-2})(2^{2n-2})$. We have, $\rho(M_n) - m\rho(M_{n-1})$ extra pebbles on the vertices of $V(M_n) - \{R_n\}$. Thus at least one subtree $M_{(n-1)i}$ contains $2\rho(M_{n-1}) \geq 2(m-1)(m^{n-2})(2^{2n})$ extra pebbles, so at least one of the m^{n-1} paths leading to the root R_n from the bottom of the subtree has at least 2^n pebbles and hence we are done.

Subcase 2.2: n is even.

If $f(R_n) \geq 1$ then we are done. So assume that $f(R_n) = 0$. Like, Subcase 2.1, at least one of the m^{n-1} paths has 2^n or more pebbles and hence we are done.

Case (3): $f\left(M_{(n-1)i}\right) \geq \rho(M_{n-1})$ for some i.

Let $t \geq 1$ subtrees contain $\rho(M_{n-1})$ or more pebbles. Label those subtrees by $M_{(n-1)i}\,(1 \leq i \leq t)$ and label the other subtrees by $M'_{(n-1)j}\,(1 \leq j \leq m-t)$. Also let $f\left(M'_{(n-1)j}\right) = a_j$ where $a_j < \rho(M_{n-1})$ and $1 \leq j \leq m-t$. Clearly we can supply at least one pebble to the root R_n of M_n for every 2^n extra pebbles on $M_{(n-1)i}\,(1 \leq i \leq t)$. Also, having one additional pebble in $M'_{(n-1)j}\,(1 \leq j \leq m-t)$ is equivalent to have at least one pebble on the root vertex R_n of M_n.

Note that, we have usually used P pebbles each to cover the maximum independent set of $M_{(n-1)i}\,(1 \leq i \leq t)$ and $M'_{(n-1)j}\,(1 \leq j \leq m-t)$, except one subtree, say $M_{(n-1)k}$, that contains $\rho(M_{n-1})$ or more pebbles. Since $a_j < \rho(M_{n-1})$, we have $P - a_j$ extra pebbles, that are in somewhere of the graph M_n, to cover the maximum independent set of $M'_{(n-1)j}$. So we can send $\sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} m^{n-2k-1} 2^{n-2k-1} - \frac{a_j}{2^{n+1}}$ pebbles to the root vertex $R'_{(n-1)j}$ of $M'_{(n-1)j}$. Thus $M'_{(n-1)j}$ contains $a_j + \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} m^{n-2k-1} 2^{n-2k-1} - \frac{a_j}{2^{n+1}}$ pebbles. But these amounts of pebbles are at least $\rho(M_{n-1})$ or it is enough to cover the maximum independent set of $M'_{(n-1)j}$, using the pebbles at $R'_{(n-1)j}$ plus a_j pebbles. Thus the t-subtrees $M_{2i}\,(1 \leq i \leq t)$ plus R_3 contains $(t-1)\sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} m^{n-2k-1} 2^{n-2k-1} + Q + \gamma_n$ or more pebbles. We know that $f\left(M_{(n-1)i}\right) \geq \rho(M_{n-1})$ where $1 \leq i \leq t$.

Subcase 3.1: n is odd.

Let $f(R_n) = 1$ or 3 (otherwise we are done easily). Then we can move a pebble to

R_n, using at most 2^n pebbles from the subtree that contains at least $\rho(M_{n-1}) + 2^n$ pebbles and hence we are done [since $\rho(M_{n-1}) \geq (m-1)(m^{n-2})(2^{2n-2})$].

Subcase 3.2 : n is even.

Let $f(R_n) = 0$ (otherwise we are done). Like the Subcase 3.1, we can move a pebble to R_n, using at most 2^n pebbles (from the subtree that contains $\rho(M_{n-1}) + 2^n$ pebbles or more).

References :

[1] F.R.K. Chung, Pebbling in hypercubes, SIAM J. Disc. Math 2(1989), 467-472.

[2] B.Crull, T.Cundiff, P.Feltman, G.H. Hurlbert, L.Pudwell, Z.Szaniszlo, Z.Tuza, The cover pebbling number of Graphs, (2004).

[3] G.Hurlbert, A survey of Graph Pebbling, Congressus Numerantium 139 (1999) 41-64.

[4] A. Lourdusamy, C. Muthulakshmi @ Sasikala and T. Mathivanan, Maximum independent set cover pebbling number of a Binary Tree, Sciencia Acta Xaveriana, Vol. 3(2) (2012) , 9-20.

[5] A. Lourdusamy, C. Muthulakshmi @ Sasikala, Maximum independent set cover pebbling number of a Star, International Journal of Mathematical Archive- 3(2), 2012, 616-618.

[6] A. Lourdusamy, C. Muthulakshmi @ Sasikala and T. Mathivanan, Maximum independent set cover pebbling number of complete graphs and paths, submitted for publication.