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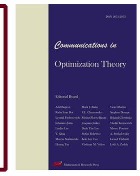

ULTRA METRIC SPACES AND COUPLED COINCIDENCE POINT THEOREMS

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Abstract. In this paper, coincidence and fixed points of nonlinear operators are investigated in a spherically complete ultra metric space. The results presented in this paper improve the corresponding results in the literature.

Keywords: Ultra metric space; Coupled coincidence point; Common coupled fixed point.

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1. Introduction-preliminaries

Fixed point theory is a beautiful mixture of topology, analysis and geometry. Over the last 60 years the theory of fixed points has been revealed as a very powerful and important tool in the study of many real world applications. In particular fixed point techniques have been applied in such diverse fields as economics, engineering, game theory, and physics. Recently, many authors studied fixed points of contractive mappings in the different framework of spaces. In [1], Roovij introduced the concept of ultra metric spaces. Later, many authors studied fixed points theorems for different nonlinear mappings on a spherically complete ultra metric spaces; see [3-8] and the references therein.

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In this paper, we consider the couple fixed points, introduced by Karapinar [2], for mappings on ultra metric spaces. Coupled coincidence point and common coupled fixed point theorems are established.

Next, we recall the following definitions.

Definition 1.1. [1] Let (X,d) be a metric space. If the metric d satisfies strong triangle inequality

$$d(x,y) < \max\{d(x,z),d(z,y)\}\ \forall x,y,z \in X,$$

then d is called an ultra metric on X and (X,d) is called an ultra metric space.

Example. Let $X \neq \theta$, $d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$ Then (X,d) is called an ultra metric space.

Definition 1.2. [1] An ultra metric space is said to be spherically complete if every shrinking collection of balls in X has a non empty intersection.

Definition 1.3. [2] Let (X,d) be a ultra metric space. An element (x,y) is said to be a couple fixed point of the mapping $F : X \times X \longrightarrow X$ if $F(x,y) = x$ and $F(y,x) = y$.

Definition 2.4. [2] An element $(x,y) \in X \times X$ is called a coupled coincidence point of the mapping $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if $F(x,y) = gx$ and $F(y,x) = gy$.

Definition 1.5. [2] An element $(x,y) \in X \times X$ is called a common coupled fixed point of the mapping $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if $F(x,y) = gx = x$ and $F(y,x) = gy = y$.

Definition 1.6. [2] Let X be a nonempty set. Then we say that the mapping $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are commutative if $gF(x,y) = F(gx,gy)$.

2. Main results

Theorem 2.1. *Let (X,d) be a spherically complete ultra metric space. If the mapping $F :$ $X \times X \longrightarrow X$ and $g : X \longrightarrow X$, for all $x,y,u,v \in X, x \neq u, y \neq v$ satisfy*

$$F(X \times X) \subseteq g(X)$$

and

$$d(F(x,y),F(u,v)) < \max\{d(gx,gu),d(gy,gv)\},$$

then there exist $x,y \in X$ *such that* $F(x,y) = gx$ *and* $F(y,x) = gy$*, that is,* F *and* g *have a unique coupled coincidence point. Further if* F *and* g *are commutes, then* F *and* g *have a unique common coupled fixed point, that is, there exists a unique pair* $(x,y) \in X \times X$ *such that* $F(x,y) = gx = x$ *and* $F(y,x) = gy = y$*.*

Proof. Let $B_a = (ga, \max_{u\in X}\{d(F(a,u),ga)\})$ denote the closed sphere centered at ga with the radius $\max_{u\in X}\{d(F(a,u),ga)\}$ and let A be the collection of these spheres for all $a \in X$. Then the relation

$$B_a \preceq B_b \text{ iff } B_b \subseteq B_a$$

is a partial order on A. Let A_1 be a totally ordered sub family of A. From the spherical completeness of X, we have $\bigcap_{B_a\in A_1} B_a = B_y \neq \emptyset$. Let $gb \in B_y$ and $B_a \in A_1$. Then $gb \in B_a$. Hence

$$d(gb,ga) \leq \max_{u\in X}\{d(F(a,u),ga)\}.$$

If $a = b$ then $B_a = B_b$. Assume that $a \neq b$. Note that $\forall x \in B_b$,

$$\begin{aligned} d(x,gb) &\leq \max_{v\in X}\{d(F(b,v),gb)\} \\ &\leq \max_{v\in X}\{\max\{d(F(b,v),F(a,v)),d(F(a,v),ga),d(ga,gb)\}\} \\ &< \max_{v\in X}\{d(gb,ga),d(F(a,v),ga),d(ga,gb)\} \\ &\leq \max_{u\in X}\{d(F(a,u)ga)\} \end{aligned}$$

and

$$d(x,ga) \leq \max\{d(x,gb),d(gb,ga)\} \leq \max_{u\in X}\{d(F(a,u),ga)\}.$$

So $x \in B_a$ and $B_b \subseteq B_a$ for any $B_a \in A_1$. Thus B_b is an upper bound in A for the family A_1 and hence by Zorn's Lemma, A has a maximal element, say B_z, for some $z \in Z$. We claim that $F(z,y) = gz, F(y,z) = gy$. Suppose $F(z,y) \neq gz$. Since

$$F(x,y) \in F(X \times X) \subseteq g(X),$$

there exists $w \in X$ such that $F(z,y) = gw$. Clearly $z \neq w$.

$$d(gw,gz) = d(F(z,y),gz) \leq \max_{c\in X}\{d(F(z,c),gz)\}.$$

We have $gw \in B_z$. Let $\forall x \in B_w$,

$$\begin{aligned} d(x, gw) &\leq \max_{s \in X} \{d(F(w,s), gw)\} \\ &\leq \max_{s \in X} \{\max\{d(F(w,s), F(z,s)), d(F(z,s), gz), d(gz, gw)\}\} \\ &< \max_{s \in X} \{d(gw, gz), d(F(z,s), gz), d(gz, gw)\} \\ &\leq \max_{s \in X} \{d(F(z,s)gz)\} \end{aligned}$$

and

$$d(x, gz) \leq \max\{d(x, gw), d(gw, gz)\} \leq \max_{c \in X} \{d(F(z,c), gz)\}.$$

So $x \in B_z$ and $B_w \subseteq B_z$. If $B_w = B_z$, then $w = z$. This is a contradiction. So we have $B_w \subsetneqq B_z$, and this contradicts the maximality of B_z. Therefore $F(z,y) = gz$. Similar, we have $F(y,z) = gy$. It means that (z,y) is a coupled coincidence point of F. If (z^*, y^*) is another coupled coincidence point of F, then

$$d(gz, gz^*) = d(F(z,y), F(z^*, y^*)) < \max\{d(gz, gz^*), d(gy, gy^*)\} = d(gy, gy^*),$$

$$d(gy, gy^*) = d(F(y,z), F(y^*, z^*)) < \max\{d(gy, gy^*), d(gz, gz^*)\} = d(gz, gz^*).$$

This is a contradiction. Hence, $(z,y) = (z^*, y^*)$.

Further assume that F and g commutes. Let $p = gz, q = gy$. Then

$$gp = g(gz) = g(F(z,y)) = F(gz, gy) = F(p,q),$$

and

$$gq = g(gy) = g(F(y,z)) = F(gy, gz) = F(q,p).$$

Hence the pair (p,q) is also a coupled coincidence point of F and g. Thus we have $gp = gz$ and $gq = gy$. Then $gp = p, gq = q$. Therefore $p = gp = F(p,q)$ and $q = gq = F(q,p)$. Thus (p,q) is a common coupled fixed point of F and g. Let (s,t) be any common coupled fixed point of F and g. Then

$$\begin{aligned} s &= gs = F(s,t), \\ t &= gt = F(t,s). \end{aligned}$$

Since the pair (s,t) is a coupled coincidence point of F and g, we have $gs = gp, gt = gq$. So $s = gs = gp = p$ and $t = gt = gq = q$. This completes the proof.

Theorem 2.2. *Let (X,d) be a spherically complete ultra metric space. If the mapping $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ for all $\forall x,y,u,v \in X$, satisfy*

$$F(X \times X) \subseteq g(X)$$
$$d(F(x,y),F(u,v)) < \max\{d(d(F(x,y),gx),d(F(u,v),gu)\},$$

then F and g have a unique coupled coincidence point. Further if F and g are commutes, then F and g have a unique common coupled fixed point, that is, there exits a unique pair $(x,y) \in X \times X$ such that $F(x,y) = gx = x$ and $F(y,x) = gy = y$.

Proof. Let $B_a = (ga, \max_{u\in X}\{d(F(a,u),ga)\})$ denote the closed sphere centered at ga with the radius $\max_{u\in X}\{d(F(a,u),ga)\}$ and let A be the collection of these spheres for all $a \in X$. Then the relation $B_a \preceq B_b$ iff $B_b \subseteq B_a$ is a partial order on A. Let A_1 be a totally ordered sub family of A. From the spherical completeness of X , we have $\bigcap_{B_a \in A_1} B_a = B_y \neq \emptyset$. Let $gb \in B_y$ and $B_a \in A_1$. Then $gb \in B_a$. It follows that $d(gb,ga) \leq \max_{u\in X}\{d(F(a,u),ga)\}$. If $a = b$ then $B_a = B_b$. Assume that $a \neq b$. Note that $\forall x \in B_b$,

$$\begin{aligned}
d(x,gb) &\leq \max_{v\in X}\{d(F(b,v),gb)\} \\
&\leq \max_{v\in X}\{\max\{d(F(b,v),F(a,v)),d(F(a,v),ga),d(ga,gb)\}\} \\
&< \max\{\max_{v\in X}\{d(F(b,v),gb),d(F(a,v),ga),d(F(a,v),ga),d(ga,gb)\}\} \\
&\leq \max_{u\in X}\{d(F(a,u)ga)\}
\end{aligned}$$

and $d(x,ga) \leq \max\{d(x,gb),d(gb,ga)\} \leq \max_{u\in X}\{d(F(a,u),ga)\}$. So $x \in B_a$ and $B_b \subseteq B_a$ for any $B_a \in A_1$. Thus B_b is an upper bound in A for the family A_1 and hence by Zorn's Lemma, A has a maximal element, say B_z, for some $z \in Z$. We claim that $F(z,y) = gz, F(y,z) = gy$. Suppose $F(z,y) \neq gz$. Since $F(x,y) \in F(X \times X) \subseteq g(X)$, there exists $w \in X$ such that $F(z,y) = gw$. Clearly $z \neq w$ $d(gw,gz) = d(F(z,y),gz) \leq \max\{d(F(z,c),gz)\}$. We have $gw \in B_z$. Let $\forall x \in B_w$,

$$\begin{aligned}
d(x,gw) &\leq \max_{s\in X}\{d(F(w,s),gw)\} \\
&\leq \max_{s\in X}\{\max\{d(F(w,s),F(z,s)),d(F(z,s),gz),d(gz,gw)\}\} \\
&< \max\{\max_{s\in X}\{d(F(w,s),gw),d(F(z,s),gz),d(gz,gw)\}\} \\
&\leq \max_{s\in X}\{d(F(z,s),gz)\}
\end{aligned}$$

and $d(x,gz) \leq \max\{d(x,gw),d(gw,gz)\} \leq \max_{c\in X}\{d(F(z,c),gz)\}$. So $x \in B_z$ and $B_w \subseteq B_z$. If $B_w = B_z$, then $w = z$. This is a contradiction. So we have $B_w \subsetneqq B_z$, and this contradicts the maximality of B_z. Therefore $F(z,y) = gz$. Similar, we have $F(y,z) = gy$. It means that (z,y) is a coupled coincidence point of F. If (z^*,y^*) is another coupled coincidence point of F, then

$$\begin{aligned} d(gz,gz^*) &= d(F(z,y),F(z^*,y^*)) \\ &< \max\{d(F(z,y),gz),d(F(z^*,y^*),gz^*)\} \\ &= \max\{d(gz,gz),d(gz,gz^*) \\ &= 0, \\ d(gy,gy^*) &= d(F(y,z),F(y^*,z^*)) \\ &< \max\{d(F(y,z),gy),d(F(y^*,z^*),gy^*)\} \\ &= \max\{d(gy,gy),d(gy,gy^*) \\ &= 0. \end{aligned}$$

Hence, $(z,y) = (z^*,y^*)$. Further assume that F and g commutes. Let $p = gz, q = gy$. Then

$$gp = g(gz) = g(F(z,y)) = F(gz,gy) = F(p,q),$$

and

$$gq = g(gy) = g(F(y,z)) = F(gy,gz) = F(q,p),$$

Hence the pair (p,q) is also a coupled coincidence point of F and g. Thus, we have $gp = gz$ and $gq = gy$. Then $gp = p, gq = q$. Therefore $p = gp = F(p,q)$ and $q = gq = F(q,p)$. Thus (p,q) is a common coupled fixed point of F and g. Let (s,t) be any common coupled fixed point of F and g, then $s = gs = F(s,t)$, $t = gt = F(t,s)$. Since the pair (s,t) is a coupled coincidence point of F and g, we have $gs = gp, gt = gq$. So $s = gs = gp = p$ and $t = gt = gq = q$. This completes the proof.

Corollary 2.3. *Theorems 2.2 still holds if the inequality is replaced by*

$$d(F(x,y),F(u,v)) < \max\{d(F(x,y),gu),d(F(u,v),gx)\},\ \forall x,y,u,v \in X.$$

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