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CONVERGENCE THEOREMS OF SOLUTIONS IN REFLEXIVE BANACH SPACES

SUN YOUNG CHO

Department of Mathematics, Gyeongsang National University, Jinju 660-701, Korea

Abstract. In this paper, fixed points of generalized asymptotically quasi-ϕ-nonexpansive mappings and equilibrium problems are investigated based on a monotone projection algorithm. Strong convergence theorems are established without the aid of compactness in the framework of reflexive Banach spaces.

Keywords: Asymptotically quasi-ϕ-nonexpansive mapping; Banach space; Equilibrium problem; Fixed point; Generalized projection.

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1. Introduction

Equilibrium problem, which was studied Blum and Oettli [1] in 1994, have had a great impact and influence in the development of several branches of pure and applied sciences, provides us a natural, novel and unified framework to study a wide class of problems arising in physics, economics, finance, transportation, network and structural analysis, elasticity and optimization; see [2-8] and the references therein. Recently, the equilibrium problem, which covers variational inequality problems, saddle point problem, variational inclusion problems, zero point problems, have been extended and generalized in many

E-mail address: ooly61@hotmail.com

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directions using novel and innovative techniques; see [9-13] and the references therein. It is known that the equilibrium problem is equivalent to a fixed point problem of nonlinear operators. Halpern iterative process (HIP) is an efficient and powerful process to study fixed points of nonlinear operators. Halpern iterative process generates a sequence $\{x_n\}$ in the following manner:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0,$$

where u is a fixed element, T is a nonlinear mapping and $\{\alpha_n\}$ is a control sequence. HIP was initially introduced in [14]. Halpern showed that the following conditions

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

are necessary in the sense if HIP is strongly convergent for all nonexpansive mappings, then $\{x_n\}$ must satisfy conditions (C1), and (C2). Due to restriction (C2), HIP is widely believed to have slow convergence though the rate of convergence has not be determined. Thus to improve the rate of convergence of HIP, one can not rely only on the process itself; instead, some additional step of iteration should be taken; see [15-20] and the references therein.

The purpose of this paper is to study the equilibrium problem in the terminology of Blum and Oettli [1] and fixed points of a family of nonlinear operators via a monotone projection algorithm. Strong convergence theorems are established without the aid of compactness in the framework of reflexive Banach spaces. The paper is organized as follows. In Section 2, we provide some necessary definitions, properties and lemmas. In Section 3, the main strong convergence theorems are established in the framework of real reflexive Banach spaces. In Sections 4, some applications are provided to support our main results.

2. Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \|x\|^2 = \langle x, x^* \rangle = \|x^*\|^2\}.$$

Let $B_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. E is said to be strictly convex if $\|x + y\| < 2$ for all $x, y \in B_E$ with $x \neq y$. It is said to be uniformly convex if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in B_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \|x + y\| \leq 2 - 2\delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. E is said to be smooth or is said to have a Gâteaux differentiable norm iff $\lim_{s\to\infty} \|sx + y\| - s\|x\|$ exists for each $x, y \in B_E$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in B_E$, the limit is uniformly obtained $\forall x \in B_E$. If the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm to weak* continuous on each bounded subset of E and single valued. It is also said to be uniformly smooth iff the above limit is attained uniformly for $x, y \in B_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that if E is uniformly smooth if and only if E^* is uniformly convex.

From now on, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. Recall that E is said to has the KKP if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $\|x_n\| \to \|x\|$ and $x_n \rightharpoonup x$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach spaces, then E has the KKP; see [21] and the references therein.

Let C be a nonempty subset of E and let $B : C \times C \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers, be a function. Recall the following equilibrium problem in the terminology of Blum and Oettli [1]. Find $\bar{x} \in C$ such that

$$B(\bar{x}\, y) \geq 0, \quad \forall y \in C. \tag{2.1}$$

We use $Sol(B)$ to denote the solution set of equilibrium problem (2.1). Given a mapping $A : C \to E^*$, let

$$B(x,y) := \langle Ax, y-x\rangle, \quad \forall x,y \in C.$$

Then $\bar{x} \in Sol(B)$ iff $\bar{x}$ is a solution of the following variational inequality. Find $\bar{x}$ such that

$$\langle A\bar{x}\, y-\bar{x}\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

The following restrictions are essential in this paper.

(R1) $B(x,x) = 0, \forall x \in C$;

(R2) $B(x,y) + B(y,x) \leq 0, \forall x,y \in C$;

(R3)

$$B(x,y) \geq \limsup_{t\to 0} B(tz + (1-t)x, y), \forall x,y,z \in C,$$

where $t \in (0,1)$;

(R4) for each $x \in C$, $y \mapsto B(x,y)$ is convex and weakly lower semi-continuous.

Let E be a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle, \quad \forall x,y \in E.$$

In a Hilbert space H, $\sqrt{\phi(x,y)} \equiv \|x-y\|$. Let D be a closed convex subset of H. For any $x \in H$, there exists an unique nearest point in D, denoted by $P_D x$, such that $\|x - P_D x\| \leq \|x-y\|$, for all $y \in D$. The operator P_D is called the metric projection from H onto D. It is known that P_D is firmly nonexpansive. In [22], a new operator $Proj$ was introduced based on the metric projection in the framework of Banach spaces. Recall that the generalized projection $Proj_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $Proj_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem $\phi(\bar{x},x) = \min_{y\in C} \phi(y,x)$. From the definition of ϕ, one has

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|y\| + \|x\|)^2, \quad \forall x,y \in E. \tag{2.3}$$

Let $T : C \to C$ be a mapping. Recall that a point $p \in C$ is said to be a fixed point of T iff $p = Tp$. In this paper, we use $Fix(T)$ to denote the fixed point set of T. Recall that a point p in C is said to be an asymptotic fixed point of T iff C contains a sequence $\{x_n\}$

which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{Fix}(T)$. Recall that T is said to be closed if for any sequence $\{x_n\}$ with $\lim_{n\to\infty} Tx_n = y'$ and $\lim_{n\to\infty} x_n = x'$, then $Tx' = y'$.

Recall that a mapping T is said to be relatively nonexpansive iff

$$\phi(p, Tq) \leq \phi(p, q), \quad \forall q \in C, \forall p \in \widetilde{Fix}(T) = Fix(T) \neq \emptyset.$$

T is said to be relatively asymptotically nonexpansive iff

$$\phi(p, T^n q) \leq k_n \phi(p, q), \quad \forall q \in C, \forall p \in \widetilde{Fix}(T) = Fix(T) \neq \emptyset, \forall n \geq 1,$$

where $\{k_n\} \subset [1, \infty)$ is a sequence such that $k_n \to 1$ as $n \to \infty$.

T is said to be quasi-ϕ-nonexpansive iff

$$\phi(p, Tq) \leq \phi(p, q), \quad \forall q \in C, \forall p \in Fix(T) \neq \emptyset.$$

Recall that a mapping T is said to be asymptotically quasi-ϕ-nonexpansive iff there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, T^n q) \leq k_n \phi(p, q), \quad \forall q \in C, \forall p \in Fix(T) \neq \emptyset, \forall n \geq 1.$$

Remark 2.1. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are more desirable than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive do not require the restriction $Fix(T) = \widetilde{Fix}(T)$.

T is said to be a generalized asymptotically quasi-ϕ-nonexpansive mapping iff there exist two sequences $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ and $\{\xi_n\} \subset [0, \infty)$ with $\xi_n \to 0$ as $n \to \infty$ such that

$$\phi(p, T^n q) \leq k_n \phi(p, q) + \xi_n, \quad \forall q \in C, \forall p \in F(T) \neq \emptyset, \forall n \geq 1.$$

Remark 2.2. The class of generalized asymptotically quasi-ϕ-nonexpansive mappings is a generalization of the class of generalized asymptotically quasi-nonexpansive mappings in the framework of Banach spaces.

In order to our main results, we also need the following lemmas.

Lemma 2.3 [1] *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $B : C \times C \to \mathbb{R}$ be a bifunction with (R1), (R2), (R3) and (R4). Let $r > 0$ and $x \in E$. There exists $z \in C$ such that $rB(z, y) \geq \langle z - y, Jz - Jx\rangle$, $\forall y \in C$.*

Lemma 2.4 [22] *Let E be a reflexive, strictly convex, and smooth Banach space and let C a convex and closed convex subset of E. Let $x \in E$. Then*

$$\phi(y, Proj_C x) + \phi(Proj_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.5 [22] *Let C be a convex closed subset of a smooth Banach space E and let $p \in E$. Then $q = Proj_C p$ if and only if $\langle q - y, Jp - Jq\rangle \geq 0$, $\forall y \in C$.*

Lemma 2.6. [15] *Let E be a reflexive, smooth and strictly convex Banach space such that both E and E^* have the KKP and let C be a convex and closed subset of E. Let T be a closed, generalized asymptotically quasi-ϕ-nonexpansive mapping on C. Then $Fix(T)$ is convex and closed.*

Lemma 2.7 [9] *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let B be a function with (R1), (R2), (R3) and (R4). Let $r > 0$ and $x \in E$. Define a mapping $K^{B,r}$ by*

$$K^{B,r}x = \{z \in C : rB(z, y) \geq \langle z - y, Jz - Jx\rangle, \quad \forall y \in C\}.$$

Then the following conclusions hold:

(1) *$K^{B,r}$ is a single-valued firmly nonexpansive-type mapping;*

(2) *$K^{B,r}$ is quasi-ϕ-nonexpansive and $Fix(K^{B,r}) = Sol(B)$;*

(3) *$\phi(q, K^{B,r}x) + \phi(K^{B,r}x, x) \leq \phi(q, x)$, $\forall q \in Fix(K^{B,r})$.*

3. Main results

Theorem 3.1. *Let E be a reflexive, smooth and strictly convex Banach space such that both E and E^* have the KKP and let C be a convex and closed subset of E. Let Λ be an index set and let B_i be a function with (R1), (R2), (R3) and (R4). Let T_i :*

$C \to C$ be a generalized asymptotically quasi-ϕ-nonexpansive mapping for every $i \in \Lambda$. Assume that T_i is closed and uniformly asymptotically regular on C for every $i \in \Lambda$ and $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, C_1 = \cap_{i\in\Lambda} C_{(1,i)}, x_1 = Proj_{C_1} x_0, \\ Jy_{(n,i)} = \alpha_{(n,i)} Jx_1 + (1-\alpha_{(n,i)}) JT_i^n x_n, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \phi(z, u_{(n,i)}) - \phi(z, x_n) \le \alpha_{(n,i)} D + \xi_{n,i}\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where $\{u_{(n,i)}\}$ is a sequence in C_n such that $r_{(n,i)} B_i(u_{(n,i)}, y) \ge \langle u_{(n,i)} - y, Ju_{(n,i)} - Jy_{(n,i)}\rangle$, $y \in C_n$, $D := \sup\{\phi(w, x_1) : p \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)\}$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_{(n,i)} = 0$, and $\{r_{(n,i)}\}$ is a real sequence in $[a_i, \infty)$, where $\{a_i\}$ is a positive real number sequence, for every $i \in \Lambda$. Then the sequence $\{x_n\}$ converges strongly to $Proj_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)} x_1$.

Proof. The proof is split into seven steps.

Step 1. Prove that $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)$ is closed and convex.

Using Lemmas 2.6 and 2.7, we find that $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)$ is closed and convex so that the generalization projection onto the set is well defined.

Step 2. Prove that C_n is closed and convex.

To show C_n is closed and convex, it suffices to show that, for each fixed but arbitrary $i \in \Lambda$, $C_{(n,i)}$ is closed and convex. This can be proved by induction on n. It is obvious that $C_{(1,i)} = C$ is closed and convex. Assume that $C_{(m,i)}$ is closed and convex for some $m \ge 1$. Let For $z_1, z_2 \in C_{(m+1,i)}$, we see that $z_1, z_2 \in C_{(m,i)}$. It follows that $z = tz_1 + (1-t)z_2 \in C_{(m,i)}$, where $t \in (0,1)$. Notice that $\phi(z_1, u_{(m,i)}) \le \phi(z_1, x_n) + \alpha_{(m,i)} D + \xi_{n,i}$ and $\phi(z_2, u_{(m,i)}) \le \phi(z_2, x_n) + \alpha_{(m,i)} D + \xi_{n,i}$. The above inequalities are equivalent to $2\langle z_1, Jx_m - Ju_{(m,i)}\rangle \le \|x_m\|^2 - \|u_{(m,i)}\|^2 + \alpha_{(m,i)} D + \xi_{n,i}$ and $2\langle z_2, Jx_m - Ju_{(m,i)}\rangle \le \|x_m\|^2 - \|u_{(m,i)}\|^2 + \alpha_{(m,i)} D + \xi_{n,i}$. Multiplying t and $(1-t)$ on the both sides of the inequalities above, respectively yields that and

$$2\langle z, Jx_m - Ju_{(m,i)}\rangle \leq \|x_m\|^2 - \|u_{(m,i)}\|^2 + \alpha_{(m,i)}D + \xi_{n,i}.$$

That is, $\phi(z, u_{(m,i)}) \leq \phi(z, x_n) + \alpha_{(m,i)}D + \xi_{n,i}$, where $z \in C_{(m,i)}$. This finds that $C_{(m+1,i)}$ is closed and convex. We conclude that $C_{(n,i)}$ is closed and convex. This in turn implies that $C_n = \cap_{i\in\Lambda} C_{(n,i)}$ is closed, and convex. This implies that $\Pi_{C_n} x_1$ is well defined.

Step 3. Prove $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i) \subset C_n$.

Since $\lim_{n\to\infty} k_{(n,i)} = 1$, we may assume that $k_{(n,i)} \leq 1 + \frac{\alpha_{(n,i)}}{1-\alpha_{(n,i)}}$ for $\forall n \geq 1$. Note that $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i) \subset C_1 = C$ is clear. Suppose that $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i) \subset C_{(m,i)}$ for some positive integer m. For any $w \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i) \subset C_{(m,i)}$, we see that

$$\begin{aligned}
&\phi(w, u_{(m,i)})\\
&\leq \phi(w, y_{(m,i)})\\
&= \phi(w, J^{-1}(\alpha_{(m,i)}Jx_1 + (1-\alpha_{(m,i)})JT_i^m x_m))\\
&= \|w\|^2 - 2\langle w, \alpha_{(m,i)}Jx_1 + (1-\alpha_{(m,i)})JT_i^m x_m\rangle\\
&\quad + \|\alpha_{(m,i)}Jx_1 + (1-\alpha_{(m,i)})JT_i^m x_m\|^2\\
&\leq \|w\|^2 - 2\alpha_{(m,i)}\langle w, Jx_1\rangle - 2(1-\alpha_{(m,i)})\langle w, JT_i^m x_m\rangle\\
&\quad + \alpha_{(m,i)}\|x_1\|^2 + (1-\alpha_{(m,i)})\|T_i^m x_m\|^2\\
&\leq \phi(w, x_m) + \alpha_{(m,i)}D + \xi_{(m,i)},
\end{aligned} \tag{3.1}$$

where $D := \sup\{\phi(w, x_1) : w \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)\}$. This shows that $w \in C_{(m+1,i)}$. This implies that $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i) \subset C_{(n,i)}$. Hence, $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i) \subset \cap_{i\in\Lambda} C_{(n,i)}$. This completes the proof that $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i) \subset C_n$.

Step 4. Prove $\{x_n\}$ is bounded

Since $x_n = Proj_{C_n} x_1$, we find from Lemma 2.5 that $\langle x_n - z, Jx_1 - Jx_n\rangle \geq 0$, for any $z \in C_n$. Since $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i) \subset C_n$, we find that

$$\langle x_n - w, Jx_1 - Jx_n\rangle \geq 0, \quad \forall w \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i). \tag{3.2}$$

Using Lemma 2.4, one sees that

$$\begin{aligned}\phi(x_n, x_1) &\leq \phi(Proj_{\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(B_i)}x_1, x_1) - \phi(Proj_{\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(B_i)}x_1, x_n)\\ &\leq \phi(Proj_{\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(B_i)}x_1, x_1).\end{aligned}$$

This implies that $\{\phi(x_n, x_1)\}$ is bounded. Hence, $\{x_n\}$ is also bounded. Since the space is reflexive, we may assume that $x_n \rightharpoonup \bar{x}$.

Step 5. Prove $\bar{x} \in \cap_{i\in\Lambda}Fix(T_i)$.

Since C_n is closed and convex, we find that $\bar{x} \in C_n$. This implies from $x_n = Proj_{C_n}x_1$ that $\phi(x_n, x_1) \leq \phi(\bar{x}, x_1)$. On the other hand, we see from the weakly lower semicontinuity of the norm that

$$\begin{aligned}\phi(\bar{x}, x_1) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx_1\rangle + \|x_1\|^2\\ &\leq \liminf_{n\to\infty}(\|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2)\\ &= \liminf_{n\to\infty}\phi(x_n, x_1)\\ &\leq \limsup_{n\to\infty}\phi(x_n, x_1)\\ &\leq \phi(\bar{x}, x_1),\end{aligned}$$

which implies that $\lim_{n\to\infty}\phi(x_n, x_1) = \phi(\bar{x}, x_1)$. Hence, we have $\lim_{n\to\infty}\|x_n\| = \|\bar{x}\|$. Since E has the KKP one has $x_n \to \bar{x}$ as $n \to \infty$. Since $x_n = Proj_{C_n}x_1$, and $x_{n+1} = Proj_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, one sees $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. This shows that $\{\phi(x_n, x_1)\}$ is nondecreasing. Since it is also bounded, one sees that $\lim_{n\to\infty}\phi(x_n, x_1)$ exists. It follows that $\phi(x_{n+1}, x_n) + \phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. This implies that

$$\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0. \tag{3.3}$$

Since $\phi(x_{n+1}, y_{(n,i)}) \leq \phi(x_{n+1}, x_n) + \alpha_{(n,i)}D + \xi_{(n,i)}$, Hence, we have

$$\lim_{n\to\infty}\phi(x_{n+1}, y_{(n,i)}) = 0. \tag{3.4}$$

It follows that $\lim_{n\to\infty}(\|x_{n+1}\| - \|y_{(n,i)}\|) = 0$. This implies that $\lim_{n\to\infty}\|y_{(n,i)}\| = \|\bar{x}\|$. On the other hand, we have

$$\lim_{n\to\infty}\|Jy_{(n,i)}\| = \lim_{n\to\infty}\|y_{(n,i)}\| = \|\bar{x}\| = \|J\bar{x}\|. \tag{3.5}$$

This implies that $\{Jy_{(n,i)}\}$ is bounded. Since both E and E^* are reflexive, we may assume that $Jy_{(n,i)} \rightharpoonup y^{(*,i)} \in E^*$. Since E is reflexive, we see $J(E) = E^*$. This shows that there exists an element $y^i \in E$ such that $Jy^i = y^{(*,i)}$. It follows that $\phi(x_{n+1}, y_{(n,i)}) = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_{(n,i)}\rangle + \|Jy_{(n,i)}\|^2$. Therefore, one has

$$0 \geq \|\bar{x}\|^2 - 2\langle \bar{x}, y^{(*,i)}\rangle + \|y^{(*,i)}\|^2 = \|\bar{x}\|^2 - 2\langle \bar{x}, Jy^i\rangle + \|y^i\|^2 = \phi(\bar{x}, y^i) \geq 0.$$

That is, $\bar{x} = y^i$, which in turn implies that $y^{(*,i)} = J\bar{x}$. It follows that $Jy_{(n,i)} \rightharpoonup J\bar{x} \in E^*$. Since E^* has the KKP, we obtain from (3.5) that $\lim_{n\to\infty} Jy_{(n,i)} = J\bar{x}$. Since $\lim_{n\to\infty} \alpha_{(n,i)} = 0$ for every $i \in \Lambda$, one has $\lim_{n\to\infty} \|Jy_{(n,i)} - JT_i^n x_n\| = 0$. Note that

$$\|J\bar{x} - JT_i^n x_n\| \leq \|Jy_{(n,i)} - J\bar{x}\| + \|Jy_{(n,i)} - JT_i^n x_n\|,$$

one has $JT_i^n x_n \to J\bar{x}$ as $n \to \infty$, for every $i \in \lambda$. Since J^{-1} is demi-continuous, we have $T_i^n x_n \rightharpoonup \bar{x}$, for every $i \in \Lambda$. Since $|\|T_i^n x_n\| - \|\bar{x}\|| \leq \|J(T_i^n x_n) - J\bar{x}\|$, one has $\|T_i^n x_n\| \to \|\bar{x}\|$, as $n \to \infty$, for every $i \in \Lambda$. Since E has the Kadec-Klee property, one obtains $\lim_{n\to\infty} \|T_i^n x_n - \bar{x}\| = 0$. On the other hand, we have

$$\|T_i^{n+1} x_n - \bar{x}\| \leq \|T_i^{n+1} x_n - T_i^n x_n\| + \|T_i^n x_n - \bar{x}\|.$$

In view of the uniformly asymptotic regularity of T_i, one has $\lim_{n\to\infty} \|T_i^{n+1} x_n - \bar{x}\| = 0$, that is, $T_i T_i^n x_n - \bar{x} \to 0$ as $n \to \infty$. Since every T_i is closed, we find that $T_i\bar{x} = \bar{x}$ for every $i \in \Lambda$.

Step 6. Prove $\bar{x} \in \cap_{i\in\Lambda} Sol(B_i)$.

Since $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1}$, we find that $\phi(x_{n+1}, u_{(n,i)}) \leq \phi(x_{n+1}, x_n) + \alpha_{(n,i)} D + \xi_{(n,i)}$. It follows from (3.3) that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_{(n,i)}) = 0. \tag{3.6}$$

Hence, we have $\lim_{n\to\infty}(\|x_{n+1}\| - \|u_{(n,i)}\|) = 0$. This implies that $\lim_{n\to\infty} \|u_{(n,i)}\| = \|\bar{x}\|$. On the other hand, we have

$$\lim_{n\to\infty} \|Ju_{(n,i)}\| = \lim_{n\to\infty} \|u_{(n,i)}\| = \|\bar{x}\| = \|J\bar{x}\|. \tag{3.7}$$

This implies that $\{Ju_{(n,i)}\}$ is bounded. Since both E and E^* are reflexive, we may assume that $Ju_{(n,i)} \rightharpoonup u^{(*,i)} \in E^*$. Since E is reflexive, we see $J(E) = E^*$. This shows that there exists an element $u^i \in E$ such that $Ju^i = u^{(*,i)}$. It follows that $\phi(x_{n+1}, u_{(n,i)}) = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_{(n,i)}\rangle + \|Ju_{(n,i)}\|^2$. Therefore, one has

$$0 \geq \|\bar{x}\|^2 - 2\langle \bar{x}, u^{(*,i)}\rangle + \|u^{(*,i)}\|^2 = \|\bar{x}\|^2 - 2\langle \bar{x}, Ju^i\rangle + \|u^i\|^2 = \phi(\bar{x}, u^i) \geq 0.$$

That is, $\bar{x} = u^i$, which in turn implies that $u^{(*,i)} = J\bar{x}$. It follows that $Ju_{(n,i)} \rightharpoonup J\bar{x} \in E^*$. Since E^* has the KKP, we obtain from (3.6) that $\lim_{n\to\infty} Ju_{(n,i)} = J\bar{x}$. Hence, we have $\lim_{n\to\infty} \|Ju_{(n,i)} - Jy_{(n,i)}\| = 0$. Since $\langle y - u_{(n,i)}, Ju_{(n,i)} - Jy_{(n,i)}\rangle + r_{(n,i)}B_i(u_{(n,i)}, y) \geq 0$, $\forall y \in C_n$, we see that

$$\|y - u_{(n,i)}\|\|Ju_{(n,i)} - Jy_{(n,i)}\| \geq r_{(n,i)}B_i(y, u_{(n,i)}), \quad \forall y \in C_n.$$

In view of $(R4)$, one has $B_i(y, \bar{x}) \leq 0$. For $0 < t_i < 1$, define $y_{(t,i)} = t_i y + (1 - t_i)\bar{x}$. It follows that $y_{(t,i)} \in C$, which yields that $B_i(y_{(t,i)}, \bar{x}) \leq 0$. It follows from the $(R1)$ and $(R4)$ that

$$0 = B_i(y_{(t,i)}, y_{(t,i)}) \leq t_i B_i(y_{(t,i)}, y) + (1 - t_i)B_i(y_{(t,i)}, \bar{x}) \leq t_i B_i(y_{(t,i)}, y).$$

That is, $B_i(y_{(t,i)}, y) \geq 0$. Letting $t_i \downarrow 0$, we obtain from $(R3)$ that $B_i(\bar{x}, y) \geq 0$, $\forall y \in C$. This implies that $\bar{x} \in Sol(B_i)$ for every $i \in \Lambda$. This shows that $\bar{x} \in \cap_{i\in\Lambda} Sol(B_i)$.

Step 7. Prove $\bar{x} = Proj_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)} x_1$.

Letting $n \to \infty$ in (3.2), we see that

$$\langle \bar{x} - w, Jx_1 - J\bar{x}\rangle \geq 0, \quad \forall w \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i).$$

In view of Lemma 2.5, we find that that $\bar{x} = Proj_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)} x_1$. This completes the proof.

Remark 3.2. Theorem 3.1 improve the corresponding results in [9,11,15,18-20] from a finite family of nonlinear mappings to an uncountable infinitely family of nonlinear mapping. And the algorithm is more efficient since $u_{(n,i)}$ is searched monotonicially in C_n instead of always in C. Theorem 3.1 does not require that the framework of the space is both uniformly convex and uniformly smooth, which is a standard assumption in most of related work. The typical example of the space in Theorem 3.1 is a reflexive, strictly convex and smooth Musielak-Orlicz space; see [19] and the references therein.

From Theorem 3.1, we also have the following result.

Corollary 3.3. *Let E be a reflexive, smooth and strictly convex Banach space such that both E and E^* have the KKP and let C be a convex and closed subset of E. Let Λ be an index set and let B_i be a function with (R1), (R2), (R3) and (R4). Assume that $\cap_{i\in\Lambda} Sol(B_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_{(1,i)} = C, C_1 = \cap_{i\in\Lambda} C_{(1,i)}, x_1 = Proj_{C_1} x_0, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \phi(z, u_{(n,i)}) - \phi(z, x_n) \leq 0\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

where $\{u_{(n,i)}\}$ is a sequence in C_n such that $r_{(n,i)} B_i(u_{(n,i)}, y) \geq \langle u_{(n,i)} - y, Ju_{(n,i)} - Jx_n \rangle$, $y \in C_n$, $\{\alpha_{(n,i)}\}$ is a real sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_{(n,i)} = 0$, and $\{r_{(n,i)}\}$ is a real sequence in $[a_i, \infty)$, where $\{a_i\}$ is a positive real number sequence, for every $i \in \Lambda$. Then the sequence $\{x_n\}$ converges strongly to $Proj_{\cap_{i\in\Lambda} Sol(B_i)} x_1$.

4. Applications

First, we give some deduced results in the framework of Hilbert spaces.

Theorem 4.1. *Let E be a Hilbert space and let C be a convex and closed subset of E. Let Λ be an index set and let B_i be a function with (R1), (R2), (R3) and (R4). Let $T_i : C \to C$ be a generalized asymptotically quasi-nonexpansive mapping for every $i \in \Lambda$. Assume that T_i is closed and uniformly asymptotically regular on C for every $i \in \Lambda$ and $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated* *in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, C_1 = \cap_{i\in\Lambda} C_{(1,i)}, x_1 = P_{C_1} x_0, \\ y_{(n,i)} = \alpha_{(n,i)} x_1 + (1-\alpha_{(n,i)}) T_i^n x_n, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \|u_{(n,i)} - z\|^2 - \|z - x_n\|^2 \le \alpha_{(n,i)} D + \xi_{n,i}\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, x_{n+1} = P_{C_{n+1}} x_1, \end{cases}$$

where $\{u_{(n,i)}\}$ *is a sequence in* C_n *such that* $r_{(n,i)} B_i(u_{(n,i)}, y) \ge \langle u_{(n,i)} - y, u_{(n,i)} - y_{(n,i)}\rangle$, $y \in C_n$, $D := \sup\{\|w - x_1\|^2 : p \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)\}$, $\{\alpha_{(n,i)}\}$ *is a real sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_{(n,i)} = 0$*, and* $\{r_{(n,i)}\}$ *is a real sequence in* $[a_i, \infty)$*, where* $\{a_i\}$ *is a positive real number sequence, for every* $i \in \Lambda$*. Then the sequence* $\{x_n\}$ *converges strongly to* $P_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)} x_1$.

Proof. In the framework of Hilbert spaces, one has $\sqrt{\phi(x,y)} = \|x - y\|$, $\forall x, y \in E$. The generalized projection is reduced to the metric projection and the generalized asymptotically-ϕ-nonexpansive mapping is reduced to the generalized asymptotically quasi-nonexpansive mapping. Using Theorem 3.1, we find the desired conclusion immediately.

Corollary 4.2. *Let* E *be a Hilbert space and let* C *be a convex and closed subset of* E*. Let* Λ *be an index set and let* B_i *be a function with (R1), (R2), (R3) and (R4). Assume that* $\cap_{i\in\Lambda} Sol(B_i)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, C_1 = \cap_{i\in\Lambda} C_{(1,i)}, x_1 = P_{C_1} x_0, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \|u_{(n,i)} - z\|^2 - \|z - x_n\|^2 \le \alpha_{(n,i)} D + \xi_{n,i}\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, x_{n+1} = P_{C_{n+1}} x_1, \end{cases}$$

where $\{u_{(n,i)}\}$ *is a sequence in* C_n *such that* $r_{(n,i)} B_i(u_{(n,i)}, y) \ge \langle u_{(n,i)} - y, u_{(n,i)} - x_n\rangle$, $y \in C_n$, $\{\alpha_{(n,i)}\}$ *is a real sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_{(n,i)} = 0$*, and* $\{r_{(n,i)}\}$ *is a* *real sequence in* $[a_i, \infty)$, *where* $\{a_i\}$ *is a positive real number sequence, for every* $i \in \Lambda$. *Then the sequence* $\{x_n\}$ *converges strongly to* $P_{\cap_{i\in\Lambda}Sol(B_i)}x_1$.

Let $A : C \to E^*$ be a single valued monotone operator which is continuous along each line segment in C with respect to the weak* topology of E^* (hemicontinuous). Recall the the following variational inequality. Finding a point $x \in C$ such that $\langle x - y, Ax\rangle \leq 0$, $\forall y \in C$. The symbol $Nc(x)$ stand for the normal cone for C at a point $x \in C$; that is, $Nc(x) = \{x^* \in E^* : \langle x - y, x^*\rangle \geq 0,\ \forall y \in C\}$. From now on, we use $VI(C, A)$ to denote the solution set of the variational inequality.

Theorem 4.3. *Let* E *be a reflexive, smooth and strictly convex Banach space such that both* E *and* E^* *have the KKP and let* C *be a convex and closed subset of* E. *Let* Λ *be an index set and let* $A_i : C \to E^*$ *be a single valued, monotone and hemicontinuous operator. Let* B_i *be a function with (R1), (R2), (R3) and (R4). Assume that* $\cap_{i\in\Lambda}VI(C, A_i) \bigcap \cap_{i\in\Lambda}Sol(B_i)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated in the following process. d*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, x_1 = Proj_{C_1:=\cap_{i\in\Delta}C_{(1,i)}}x_0, \\ z_{(n,i)} = VI(C, A_i + \frac{1}{r_i}(J - Jx_n)), \\ Jy_{(n,i)} = (1 - \alpha_{(n,i)})Jz_{n,i} + \alpha_{(n,i)}Jx_1, \quad n \geq 1, \\ C_{(n+1,i)} = \{w \in C_{(n,i)} : \phi(w, x_n) \geq \phi(w, u_{n,i})\}, \\ C_{n+1} = \cap_{i\in\Lambda}C_{(n+1,i)}, x_{n+1} = Proj_{C_{n+1}}x_0, \quad \forall n \geq 1, \end{cases}$$

where $\{u_{(n,i)}\}$ *is a sequence in* C_n *such that* $r_{(n,i)}B_i(u_{(n,i)}, y) \geq \langle u_{(n,i)} - y, Ju_{(n,i)} - Jy_{(n,i)}\rangle$, $y \in C_n$, $\{\alpha_{(n,i)}\}$ *is a real sequence in* $(0, 1)$ *such that* $\lim_{n\to\infty} \alpha_{(n,i)} = 0$, *and* $\{r_{(n,i)}\}$ *is a real sequence in* $[a_i, \infty)$, *where* $\{a_i\}$ *is a positive real number sequence, for every* $i \in \Lambda$. *Then the sequence* $\{x_n\}$ *converges strongly to* $Proj_{\cap_{i\in\Lambda}VI(C,A_i)\bigcap\cap_{i\in\Lambda}Sol(B_i)}x_1$.

Proof. First, we define a new operator M_i by

$$M_i x = \begin{cases} A_i x + Nc(x), & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Hence, M_i is maximal monotone and $M_i^{-1}(0) = VI(C, A_i)$ [9], where $M_i^{-1}(0)$ stands for the zero point set of M. For each $r_i > 0$, and $x \in E$, we see that there exists an unique x_{r_i} in the domain of M_i such that $Jx \in Jx_{r_i} + r_i M_i(x_{r_i})$, where $x_{r_i} = (J + r_i M_i)^{-1} Jx$. Notice that $z_{n,i} = VI(C, \frac{1}{r_i}(J - Jx_n) + A_i)$, which is equivalent to $\langle z_{n,i} - y, r_i A_i z_{n,i} + (Jz_{n,i} - Jx_n)\rangle \leq 0$, $\forall y \in C$, that is, $\frac{1}{r_i}\big(Jx_n - Jz_{n,i}\big) \in Nc(z_{n,i}) + A_i z_{n,i}$. This implies that $z_{n,i} = (J + r_i M_i)^{-1} Jx_n$. From [23], we find that $(J + r_i M_i)^{-1} J$ is closed quasi-ϕ-nonexpansive with $Fix((J + r_i M_i)^{-1} J) = M_i^{-1}(0)$. Using Theorem 3.1, we find the desired conclusion immediately.

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