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ON MAJORIZATION TYPE RESULTS

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Abstract. In this paper, we state new results of majorization type using convex functions. We also generate $n-$exponential and logarithmic convexity of stated results. With the aid of $n-$exponential convexity, we generalize many established results.

Keywords: Convex functions; $n-$exponential convexity; majorization.

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1. Introduction

There is a certain intuitive appeal to the vague notion that the components of $n-$tuple $\mathbf{x}$ are less spread out, or more nearly equal, than are the components of $n-$tuple $\mathbf{y}$. The notion arises in a variety of contexts, and it can be made precise in a number of ways. But in remarkably many cases, the appropriate statement is that $\mathbf{x}$ majorizes $\mathbf{y}$ means that the sum of m largest entries of $\mathbf{y}$ does not exceed the sum of m largest entries of $\mathbf{x}$ for all $m \in \{1,2,\ldots,n-1\}$ with equality for $m = n$. A mathematical origin of majorization is illustrated by the work of Schur [19] on Hadamard's determinent inequality. Many mathematical characterization problems are

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known to have solutions that involve majorization. A complete and superb reference on the subject are the books [4] and [13].

The notion of majorization is defined as follows: For fixed $n \geq 2$, let

$$\mathbf{x} = (x_1, \ldots, x_n), \quad \mathbf{y} = (y_1, \ldots, y_n)$$

denote the $n-$tuples and

$$x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}, \quad y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]},$$

be their ordered components.

Definition 1.1. For $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$,

$$\mathbf{x} \prec \mathbf{y} \quad if \quad \begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, & k \in \{1, \ldots, n-1\}; \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \end{cases}$$

when $\mathbf{x} \prec \mathbf{y}$, is said to be majorized by $\mathbf{y}$ or $\mathbf{y}$ majorizes $\mathbf{x}$.

This notion and notation of majorization was introduce by Hardy *et al.* in [7].
The following weighted version of majorization theorem was given by Fuchs in [6] (see also [13, p. 580] and [18, p. 323]).

Proposition 1.2. Let $\mathbf{w} \in \mathbb{R}^n$ and let $\mathbf{x}, \mathbf{y}$ be two decreasing real n-tuples such that

$$\sum_{i=1}^{k} w_i x_i \leq \sum_{i=1}^{k} w_i y_i, \quad k \in \{1, \ldots, n-1\}$$
$$\text{and} \quad \sum_{i=1}^{n} w_i x_i = \sum_{i=1}^{n} w_i y_i.$$

Then for every continuous convex function $f : \mathbb{R} \to \mathbb{R}$, the following inequality holds

$$\sum_{i=1}^{n} w_i f(x_i) \leq \sum_{i=1}^{n} w_i f(y_i). \tag{1}$$

By using inequality (1) under the assumption of Proposition 1.2, we define the functional Λ_1 by

$$\Lambda_1(\mathbf{x}, \mathbf{y}, \mathbf{w}; f) = \sum_{i=1}^{n} w_i f(y_i) - \sum_{i=1}^{n} w_i f(x_i) \geq 0. \tag{A1}$$

We also give some integral inequalities related to majorization.

The following theorem is simple consequence of Theorem 12.14 in [15] (see also [18, p. 328]).

Proposition 1.3. Let $x, y : [a,b] \to \mathbb{R}$ be increasing continuous functions and let $H \in BV[a,b]$. Further, suppose that

$$\int_u^b x(t)dH(t) \leq \int_u^b y(t)dH(t), \quad u \in (a,b)$$
$$\int_a^b x(t)dH(t) = \int_a^b y(t)dH(t)$$

are valid. Then for every continuous convex function following inequality holds

$$\int_a^b f(x(t))dH(t) \leq \int_a^b f(y(t))dH(t). \tag{2}$$

Remark 1.4. Let $x, y : [a,b] \to \mathbb{R}$ be decreasing continuous functions such that

$$\int_b^u x(t)dH(t) \leq \int_b^u y(t)dH(t), \quad u \in (a,b)$$
$$\int_a^b x(t)dH(t) = \int_a^b y(t)dH(t),$$

where $H \in BV[a,b]$. Then inequality (2) holds for every continuous convex function $f : \mathbb{R} \to \mathbb{R}$.

Under the assumptions of Proposition 1.3, using (2) we define the following functional

$$\Lambda_2(x,y,H,f) = \int_a^b f(y(t))dH(t) - \int_a^b f(x(t))dH(t) \geq 0. \tag{A2}$$

For our next theorem, we recall a definition with some notion from [18, p. 330] as follows. Let $F, G : [0,\infty) \to \mathbb{R}$ be two increasing continuous functions which pass through the origin and define $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$.

Definition 1.5.

$$\bar{G} \prec \bar{F} \quad \text{if} \quad \begin{cases} \int_0^u \bar{G}(z)dz \leq \int_0^u \bar{F}(z)dz, & u \in (0,\infty) \\ \int_0^\infty \bar{G}(z)dz = \int_0^\infty \bar{F}(z)dz, \end{cases}$$

provided that integrals exist, when $\bar{G} \prec \bar{F}$, $\bar{G}$ is said to be majorized by $\bar{F}$ or $\bar{F}$ majorizes $\bar{G}$.

Boland and Proschan in [5] gave the following result.

Proposition 1.6. The inequality

$$\int_0^\infty f(t)dG(t) \geq \int_0^\infty f(t)dF(t) \tag{3}$$

holds for every convex function f if and only if $\bar{G} \prec \bar{F}$, provided that the integrals exist.

Also, under the assumption of Proposition 1.6 using (3) we define the following functional

$$\Lambda_3(F,G;f) = \int_0^\infty f(t)dG(t) - \int_0^\infty f(t)dF(t) \geq 0. \tag{A3}$$

Here we state some majorization type results for concave functions which are extracted from [16]. Proofs of these results can be seen in the same article. Later we will prove similar results for convex functions.

Proposition 1.7. Let g be a strictly increasing function from (a,b) to (c,d) and let $f \circ g^{-1}$ be a concave function on $[c,d]$. Let the $n-$tuple $\mathbf{x}$ and $\mathbf{y}$ with elements from (a,b) satisfy

$$\sum_{i=1}^{k} w_i g(x_i) \geq \sum_{i=1}^{k} w_i g(y_i), \ \ k = \{1,\ldots,n\}.$$

(a) If

(a_1) f is decreasing and

(a_2) the $n-$tuple $\mathbf{y}$ is decreasing, then

$$\sum_{i=1}^{k} w_i f(x_i) \leq \sum_{i=1}^{k} w_i f(y_i), \ \ k = \{1,\ldots,n\}.$$

(b) If

(b_1) $\sum_{i=1}^{n} w_i g(x_i) = \sum_{i=1}^{n} w_i g(y_i)$ *and*

(b_2) the $n-$tuple $\mathbf{y}$ is decreasing, then

$$\sum_{i=1}^{n} w_i g(x_i) \leq \sum_{i=1}^{n} w_i g(y_i),$$

(c) If

(c_1) f is increasing and

(c_2) the $n-$tuple $\mathbf{x}$ is increasing, then

$$\sum_{i=1}^{k} w_i f(x_i) \geq \sum_{i=1}^{k} w_i f(y_i), \quad k = \{1,\ldots,n\}.$$

(d) If

(d_1) $\sum_{i=1}^{n} w_i g(x_i) \geq \sum_{i=1}^{n} w_i g(y_i)$ *and*

(d_2) the $n-$tuple $\mathbf{x}$ is increasing, then

$$\sum_{i=1}^{n} w_i f(x_i) \geq \sum_{i=1}^{n} w_i f(y_i).$$

Proposition 1.8. Let x and y be integrable function on $[a,b]$ and let w be positive integrable function. Suppose that f is a strictly increasing function and $g \circ f^{-1}$ is concave. Suppose that x is decreasing and that

$$\int_u^b f(x(t))w(t)dt \geq \int_u^b f(y(t))w(t)dt, \quad \forall\, u \in [a,b].$$

(a) If

$$\int_a^b f(x(t))w(t)dt = \int_a^b f(y(t))w(t)dt,$$

then

$$\int_a^b g(x(t))w(t)dt \geq \int_a^b g(y(t))w(t)dt.$$

If y is increasing, the reverse inequality holds.
(b) If $g \circ f^{-1}$ is increasing, then

$$\int_u^b g(x(t))w(t)dt \geq \int_u^b g(y(t))w(t)dt, \quad a \leq u \leq b.$$

The present article is divided into four main sections. After introduction, the second section states new majorization type results for convex functions. These results are similar to the results stated in article [16] for concave functions and proved by using similar techniques stated in the same article. In the first and second section some positive linear functionals are constructed with the help of stated inequalities. For the sake of completeness, the Lagrange type and the Cauchy type mean value theorems for one functional are give in the third section. The last section generates the $n-$exponential and logarithmic convexity for majorization type results by using the class of continuous functions in linear functionals previously constructed. In the same section positive semi-definite matrices and Stolarsky type means are also constructed.

On one hand our article states new majorization type results for convex functions and on the other hand our main result related to $n-$exponential convexity generalizes most of the results stated in [9] and in turn it gives generalization of many results of the articles [2] and [10]. Proving techniques of our last main result are some how different form techniques used in [9].

2. New majorization type results for convex functions

Now we state and prove the results for convex functions similar to results stated in previous section.

Theorem 2.1. *Let g be a strictly increasing function from (a,b) to (c,d) and let $f \circ g^{-1}$ be a convex function on $[c,d]$. Let* **x** *and* **y** *be two $n-$tuples with elements from (a,b) satisfy*

$$\sum_{i=1}^{k} w_i g(x_i) \geq \sum_{i=1}^{k} w_i g(y_i), \;\; k = \{1,\ldots,n\}. \tag{4}$$

(a) If

(a_1) *f is decreasing and*

(a_2) *the $n-$tuple* **y** *is decreasing , then*

$$\sum_{i=1}^{k} w_i f(x_i) \leq \sum_{i=1}^{k} w_i f(y_i), \;\; k = \{1,\ldots,n\}. \tag{5}$$

(b) If

(b_1) $\sum_{i=1}^{n} w_i g(x_i) = \sum_{i=1}^{n} w_i g(y_i)$ *and*

(b_2) *the $n-$tuple* **y** *is decreasing, then*

$$\sum_{i=1}^{n} w_i f(x_i) \geq \sum_{i=1}^{n} w_i f(y_i),$$

(c) If

(c_1) *f is increasing and*

(c_2) *the $n-$tuple* **x** *is increasing, then*

$$\sum_{i=1}^{k} w_i f(x_i) \leq \sum_{i=1}^{k} w_i f(y_i), \quad k = \{1,\ldots,n\}.$$

(d) If

(d_1) $\sum_{i=1}^{n} w_i g(x_i) \geq \sum_{i=1}^{n} w_i g(y_i)$ *and*

(d_2) *the $n-$tuple* **x** *is increasing, then*

$$\sum_{i=1}^{n} w_i f(x_i) \le \sum_{i=1}^{n} w_i f(y_i).$$

Proof.

(a) Without loss of generality, it is easy to see that it is enough to prove the special case $g(x) = x$ by substitution

$$g(x_i) = a_i, \quad g(y_i) = b_i, \quad f(x_i) = f \circ g^{-1}(a_i) = \bar{f}(a_i). \tag{6}$$

Because of the convexity of $f(x)$

$$\bar{f}(u) - \bar{f}(v) \ge \bar{f}'_+(v)(u-v),$$

hence

$$\begin{aligned}\sum_{i=1}^{k} w_i(\bar{f}(a_i) - \bar{f}(b_i)) &\ge \sum_{i=1}^{k} w_i(a_i - b_i)\bar{f}'_+(b_i) \\ &= \bar{f}'_+(b_k)\sum_{i=1}^{k} w_i(a_i - b_i) \\ &+ \sum_{i=1}^{k-1}\left(\sum_{m=1}^{i} w_m(a_m - b_m)(\bar{f}'_+(b_i) - \bar{f}'_+(b_{i+1}))\right) \\ &\ge 0.\end{aligned}$$

The last inequality follows from $(a_1), (a_2), (4), (6)$ and the convexity of f, hence case (a) is proven.

(b) It easily follows from (a) by using identity

$$\sum_{i=1}^{n} w_i(\bar{f}(a_i) - \bar{f}(b_i)) \ge \sum_{i=1}^{n} w_i(a_i - b_i)\bar{f}'_+(b_i)$$

instead of

$$\sum_{i=1}^{k} w_i(\bar{f}(a_i) - \bar{f}(b_i)) \ge \sum_{i=1}^{k} w_i(a_i - b_i)\bar{f}'_+(b_i)$$

and using (b_1) and (b_2) instead of (a_1) and (a_2).

(c) It is direct consequence of (b).

(d) Its proof is similar to (c) so we omit the details.

By using inequality (5) under the assumption of Theorem 2.1, we define the functional λ_1 by

$$\lambda_1(\mathbf{x},\mathbf{y},\mathbf{w};f) = \sum_{i=1}^{n} w_i f(y_i) - \sum_{i=1}^{n} w_i f(x_i).$$

Now, we define the functional Λ_4 in terms of λ_1 by

$$\Lambda_4 = \begin{cases} \lambda_1, \text{if inequality (5) holds,} \\ -\lambda_1, \text{if reverse inequality in (5) holds.} \end{cases} \tag{A4}$$

Note that, whenever it is defined, Λ_4 is nonnegative.

Now we state some integral inequalities of majorization type for convex functions.

Theorem 2.2. *Let x and y be two integrable functions on $[a,b]$ and let w be a positive integrable function. Suppose that f is a strictly increasing function and $g \circ f^{-1}$ is convex. Suppose that x is decreasing and that*

$$\int_s^b f(x(t))w(t)dt \geq \int_s^b f(y(t))w(t)dt, \quad \forall s \in [a,b]$$

and

$$\int_a^b f(x(t))w(t)dt = \int_a^b f(y(t))w(t)dt.$$

(a) If y is a decreasing function, then

$$\int_a^b g(x(t))w(t)dt \leq \int_a^b g(y(t))w(t)dt, \tag{7}$$

if y is increasing, then reverse inequality holds.

(b) If $g \circ f^{-1}$ is increasing, then

$$\int_s^b g(x(t))w(t)dt \leq \int_s^b g(y(t))w(t)dt, \quad a \leq s \leq b.$$

Proof.

(a) Due to the convexity of the function f we can write

$$f(u) - f(v) \geq f^{'}(v)(u-v).$$

If we set

$$F(u) = \int_s^b [x(t)-y(t)]w(t)dt$$

for $a \leq s \leq b$, then

$$\begin{aligned}\int_a^b [f[y(t)] - f[x(t)]]w(t)dt &\geq \int_a^b [f^{'}[x(t)][y(t) - x(t)]w(t)dt \\ &= \int_a^b [f^{'}[x(t)]dF(t) \\ &= [f^{'}[x(t)]F(t)]_a^b - \int_a^b F(t)d(f^{'}(x(t))) \\ &= -\int_a^b F(t)f^{''}(x(t))x^{'}(t)dt \\ &\geq 0.\end{aligned}$$

(b) This part can be proved in similar way as (a) so we omit the details.

Again, under the assumption of Theorem 2.2 using (7) we define the functional λ_2 by

$$\lambda_2(x,y,w;g) = \int_a^b g(y(t))w(t)dt - \int_a^b g(x(t))w(t)dt \geq 0.$$

Now, we define the following functional Λ_5 in terms of λ_2 by

$$\Lambda_5 = \begin{cases} \lambda_2, \text{if inequality } (7) \text{ holds,} \\ -\lambda_2, \text{if reverse inequality } (7) \text{ holds.} \end{cases} \tag{A5}$$

Note that, whenever it is defined, Λ_5 is nonnegative.

3. Mean value theorems

Here we state Lagrange type and Cauchy type mean value theorems only for the functional Λ_3. For other functionals similar results can be found in articles [11] and [10]. Since proving technique is similar to mentioned results so we omit the details.

Theorem 3.1. *Let Λ_3 be a linear functional as defined in (A3) under the assumptions of Proposition 1.6 and let $f \in C^2(K)$, where K is a compact interval in $[0,\infty)$. Then there exist $\xi \in K$ such that*

$$\Lambda_3(F,G;f) = f^{''}(\xi)(F,G;f_0),$$

where $f_0(x) = \frac{x^2}{2}$.

Theorem 3.2. *Let Λ_3 be a linear functional functional as defined in under the assumptions of Proposition 1.6 and let $f,g \in C^{(2)}(K)$, where K is a compact interval in $[0,\infty)$. Then there exist*

$\xi \in K$ *such that*

$$\frac{\Lambda_3(F,G;f)}{\Lambda_3(F,G;f)} = \frac{f''(\xi)}{g''(\xi)},$$

provided that the denominator of the left-had side is nonzero.

Remark 3.3. If the inverse of $\frac{f''}{g''}$ exists, then from the above mean value theorem we get the following generalized mean

$$\xi = \left(\frac{f''}{g''}\right)^{-1}\left(\frac{\Lambda_3(F,G;f)}{\Lambda_3(F,G;g)}\right).$$

Remark 3.4. For the functionals $\Lambda_k,\ k \in \{1,2,4,5\}$ (as defined in $(A1),(A2),(A4)$ and $(A5)$) the results similar to Theorems 3.1 and 3.2 can be found in [10] and [11]. In the similar way, we can use Remark 3.3 for these functionals as well.

4. *n*-exponential convexity for majorization type results

4.1. *n*-exponentially convex functions

Bernstein [3] and Widder [20] independent introduced an important sub-class of convex functions, which is called class of exponentially convex functions on a given open interval and s-tudied properties of this newly defined class. Exponentially convex functions have many nice properties, e.g., these functions are analytic on their domain. These functions also provide us positive-semidefinite matrices. Moreover, they play an important role in studying the properties of Stolarsky and Cauchy means, such as monotonicity of these means etc. For further study of the class of exponentially convex functions we refer to [1], [8] and [14].

Pečarić and Perić in [17] introduced the notion of n-exponentially convex functions which is in fact generalization of the concept of exponentially convex functions.

In the present subsection, we discuss the same notion of n-Exponential convexity by describing related definitions and some important results with some remarks from [17].

Throughout this section I stands for an open interval in $\mathbb{R}$, $[x_0,x_1,x_2;f]$ represents second order divided difference of function f at distinct points x_0, x_1 and x_2, $Dom f$ represents domain of function f and a function f is $\log -$convex if $\log(f)$ is convex.

Definition 4.1. A function $f : I \to \mathbb{R}$ is *$n-$exponentially convex in the $J-$sense* if the inequality

$$\sum_{i,j=1}^{n} u_i u_j f\left(\frac{t_i+t_j}{2}\right) \geq 0$$

holds for each $t_i \in I$ and $u_i \in \mathbb{R},\ i \in \{1,\ldots,n\}$.

Definition 4.2. A function $f : I \to \mathbb{R}$ is *$n-$exponentially convex* if it is $n-$exponentially convex in the J-sense and continuous on I.

Remark 4.3. We can see from the definition that 1-exponentially convex functions in the $J-$sense are in fact nonnegative functions. Also, $n-$exponentially convex functions in $J-$sense are $k-$exponentially convex in the $J-$sense for every $k \in \mathbb{N}$ such that $k \leq n$.

Definition 4.4. A function $f : I \to \mathbb{R}$ is *exponentially convex in the $J-$sense*, if it is $n-$ exponentially convex in the $J-$sense for each $n \in \mathbb{N}$.

Remark 4.5. A function $f : I \to \mathbb{R}$ is exponentially convex in the $J-$sense, if it is $n-$ exponentially convex in the $J-$sense and continuous on I.

Proposition 4.6. If function $f : I \to \mathbb{R}$ is $n-$exponentially convex in the $J-$sense, then the matrix

$$\left[f\left(\frac{t_i+t_j}{2}\right)\right]_{i,j=1}^{m}$$

is positive-semidefinite. Particularly

$$det\left[f\left(\frac{t_i+t_j}{2}\right)\right]_{i,j=1}^{m} \geq 0$$

for each $m \in \mathbb{N}$ and $t_i \in I$ for $i \in \{1,\ldots,m\}$.

Corollary 4.7. *If function $f : I \to \mathbb{R}$ is exponentially convex, then the matrix*

$$\left[f\left(\frac{t_i+t_j}{2}\right)\right]_{i,j=1}^{m}$$

is positive-semidefinite. Particularly

$$det\left[f\left(\frac{t_i+t_j}{2}\right)\right]_{i,j=1}^{m} \geq 0$$

for each $m \in \mathbb{R}$ and $t_i \in I$ for $i \in \{1,\ldots,m\}$.

Corollary 4.8. *If function $f : I \to (0,\infty)$ is exponentially convex, then f is log-convex.*

Remark 4.9. A function $f: I \to (0,\infty)$ is $\log-$convex in $J-$sense if and only if the inequality

$$u_1^2 f(t_1) + 2u_1u_2 f\left(\frac{t_1+t_2}{2}\right) + u_2^2 f(t_2) \geq 0$$

holds for each $t_1, t_2 \in I$ and $u_1, u_2 \in \mathbb{R}$. It follows that a positive function is $\log-$convex in the $J-$sense if and only if it is 2-exponentially convex in the $J-$sense. Also, using basic convexity theory it follows that a positive function is $\log-$convex if and only if it is 2-exponentially convex.

Now we are ready to state main result of this section. Throughout this section I stands for open interval in $\mathbb{R}$. We will need the following result (see [18, p. 2]).

Proposition 4.10. A function $f: I \to \mathbb{R}$ is convex if the inequality

$$(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geq 0$$

holds for each $x_1, x_2, x_3 \in I$ such $x_1 < x_2 < x_3$.

Lemma 4.11. *If $f: I \to \mathbb{R}$ is a convex function and $x_1, x_2, y_1, y_2 \in I$ are such that $x_1 \leq y_1, x_2 \leq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}. \tag{8}$$

If the function f is concave, then the revers inequality in (8) holds.

Theorem 4.12. *Let $D_1 = \{f_t : t \in I\}$ be a class of continuous functions such that the function $t \to [z_0, z_1, z_2; f_t]$ is $n-$exponentially convex in the $J-$sense on I for any three mutually distinct points $z_0, z_1, z_2 \in Dom(f_t)$. Let Λ_k be the linear functionals for $k \in \{1,2,3\}$. The following statements are valid :*

- *(a) The function $t \to \Lambda_k(f_t)$ is $n-$exponentially convex function in the $J-$sense on I.*
- *(b) If the function $t \to \Lambda_k(f_t)$ is continuous on I, then the function $t \to \Lambda_k(f_t)$ is exponentially convex on I.*

Proof.

(a) Fix $k \in \{1,2,3\}$. Let us define the functional ω for $t_i \in I$, $\mu_i \in \mathbb{R}$, $i \in \{1,2,\ldots,n\}$ as follows

$$\omega = \sum_{i=1,2}^{n} \mu_i \mu_j f_{\frac{t_i+t_j}{2}}.$$

Since the function $t \to [z_0, z_1, z_2; f_t]$ is $n-$exponentially convex in the $J-$sense so

$$[z_0, z_1, z_2; \omega] = \sum_{i,j=1}^{n} \mu_i \mu_j [z_0, z_1, z_2; f_{\frac{t_i+t_j}{2}}] \geq 0,$$

which implies that ω is convex function on $Dom(f_t)$ and therefore $\Lambda_k(\omega) \geq 0$. Hence

$$\sum_{i,j=1}^{n} \mu_i \mu_j \Lambda_k (f_{\frac{t_i+t_j}{2}}) \geq 0.$$

We conclude that the function $t \to \Lambda_k(f_k)$ is an $n-$exponentially convex functions on I in $J-$sense .

(b) This part is easily follows from definition of $n-$exponentially convex function.

Corollary 4.13. *Let $D_2 = \{f_t : t \in I\}$ be a class of continuous functions such that the function $t \to [z_0, z_1, z_2; f_t]$ is an exponentially convex in the $J-$sense on I for any there mutually distinct points $z_0, z_1, z_2 \in Dom(f_t)$. Let Λ_k be the linear functionals for $k \in \{1, 2, 3\}$. Then the following statements are valid:*

(a) The function $t \to \Lambda_k(f_t)$ is exponentially convex in the $J-$sense on I.

(b) The function $t \to \Lambda_k(f_t)$ is continuous on I, then the function $t \to \Lambda_k(f_t)$ is exponentially convex on I.

(c) The matrix $\left[\Lambda_k \left(f^{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^{m}$ is positive semi definite particularly ,

$$det \left[\Lambda_k \left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^{m} \geq 0$$

for each $m \in \mathbb{N}$ and $t_i \in I$ where $i \in \{1, 2, \ldots, m\}$

Proof. Proof follows from Theorem 4.12 by using definition of exponential convexity and Corollary 4.7.

Corollary 4.14. *Let $D_3 = \{f_t : t \in I\}$ be a class of continuous functions such that the function $t \to [z_0, z_1, z_2; f_t]$ is 2-exponentially convex in the $J-$sense on I for any three mutually distinct points $z_0, z_1, z_2 \in Dom(f_t)$. Let Λ_k be the linear functional for $k \in \{1, 2, 3\}$. Then the following statements are valid:*

(a) If the following function $t \to \Lambda_k(f_t)$ is continuous on I, then it is 2-exponentially convex on I. If the function $t \to \Lambda_k(f_t)$ is additionally positive, then it is also log *−convex on I.*

Moreover, the following Lyapunov's inequality holds for $r < s < t;\ r,s,t \in I$.

$$[\Lambda_k(f_s)]^{t-r} \leq [\Lambda_k(f_r)]^{t-s} + [\Lambda_k(f_t)]^{s-r}$$

(b) If the function $t \to \Lambda_k(f_t)$ is positive and differentiable on I, then for every $s,t,u,v \in I$ such that $s \leq u$ and $t \leq v$, we have

$$\mu_{s,t}(\Lambda_k, D_3) \leq \mu_{u,v}(\Lambda_k, D_3),$$

where

$$\mu_{s,t}(\Lambda_k, D_3) = \begin{cases} \left(\frac{\Lambda(\varphi^{(s)})}{\Lambda(\varphi^{(t)})}\right)^{\frac{1}{s-t}} & ,s \neq t \\ exp\left(\frac{\frac{d}{ds}\Lambda(\varphi^{(s)})}{\Lambda(\varphi^{(s)})}\right) & ,s = t \end{cases} \tag{9}$$

Proof.

(a) As the function $t \to \Lambda_k(f_t)$ is $\log -$convex, i.e., $ln\Lambda_k(f_t)$ is convex, so by using Proposition 4.10, we have

$$[\Lambda_k(f_s)]^{t-r} \leq [\Lambda_k(f_r)]^{t-s} + [\Lambda_k(f_t)]^{s-r}.$$

(b) Since by (a) the function $t \to \Lambda_k(f_t)$ is $\log -$convex on I, that is, the function $t \to \Lambda_k(f_t)$ is convex on I, applying Lemma 4.11 for $s \leq u$ and $t \leq v$, we get

$$\frac{ln\Lambda_k(f_s) - ln\Lambda_k(f_t)}{s-t} \leq \frac{ln\Lambda_k(f_u) - ln\Lambda_k(f_v)}{u-v}$$

and therefore we conclude that

$$\mu_{s,t}(\Lambda_k, D_3) \leq \mu_{u,v}(\Lambda_k, D_3).$$

Remark 4.15. We note that the result from Theorem 4.12, Corollary 4.13 and Corollary 4.14 still hold when any two (three) points $z_0, z_1, z_2 \in [a,b]$ coincide for a family of differentiable (twice differentiable) functions f_t such that the function $t \to [z_0, z_1, z_2; f_t]$ is $n-$exponentially convex, exponentially convex and 2-exponentially convex in the $J-$sense, respectively.

Remark 4.16. The Corollary 4.13 and Corollary 4.14 gives us Theorem 2.5 and Theorem 2.2 of article [9]. Hence we may easily deduce many important results stated in articles [9, 2, 10] as consequences of our main result.

Theorem 4.17. *Let $D_4 = \{f_t : t \in I\}$ be a class of continuous functions such that the function $t \to [z_0, z_1, z_2; f_t \circ g^{-1}]$ is $n-$exponentially convex in the $J-$sense on I for any three mutually distinct points $z_0, z_1, z_2 \in [0, \infty)$ where the function f is increasing. Let Λ_k be the linear functional for $k \in \{4, 5\}$. The following statements are valid:*

(a) The function $t \to \Lambda_k(f_t)$ is $n-$exponentially convex function in the $J-$sense on I.

(b) If the function $t \to \Lambda_k(f_t)$ is continuous on I, then the function $t \to \Lambda_k(f_t)$ is exponentially convex on I.

Proof.

(a) Fix $k \in \{4, 5\}$. Let us define the functional ω for $t_i \in I,\ \mu_i \in \mathbb{R},\ i \in \{1, 2, \ldots, n\}$ as follows

$$\omega = \sum_{i=1,2}^{n} \mu_i \mu_j f_{\frac{t_i+t_j}{2}},$$

which implies that

$$\omega \circ g^{-1} = \sum_{i=1,2}^{n} \mu_i \mu_j f_{\frac{t_i+t_j}{2}} \circ g^{-1}.$$

Since the function $t \to [z_0, z_1, z_2; f_t \circ g^{-1}]$ is $n-$exponentially convex in the $J-$sense so

$$[z_0, z_1, z_2; \omega \circ g^{-1}] = \sum_{i,j=1}^{n} \mu_i \mu_j [z_0, z_1, z_2; f_{\frac{t_i+t_j}{2}} \circ g^{-1}] \geq 0.$$

Which implies that $\omega \circ g^{-1}$ is convex function on $[0, \infty)$ and therefore $\Lambda_k(\omega \circ g^{-1}) \geq 0$. Hence

$$\sum_{i,j=1}^{n} \mu_i \mu_j \Lambda_k(f_{\frac{t_i+t_j}{2}} \circ g^{-1}) \geq 0.$$

We conclude that the function $t \to \Lambda_k(f_k)$ is an $n-$exponentially convex functions on I in $J-$sense .

(b) This part is easily follows from definition of $n-$exponentially convex function.

Corollary 4.18. *Let $D_5 = \{f_t : t \in I\}$ be a class of continuous functions such that the function $t \to [z_0, z_1, z_2; f_t \circ g^{-1}]$ is an exponentially convex in the $J-$sense on I for any there mutually distinct points $z_0, z_1, z_2 \in [0, \infty)$ where the functional g is increasing. Let Λ_k be the linear functionals for $k \in \{4, 5\}$. Then the following statements are valid:*

(a) The function $t \to \Lambda_k(f_t)$ is exponentially convex in the $J-$sense on I.

(b) The function $t \to \Lambda_k(f_t)$ is continuous on I, then the function $t \to \Lambda_k(f_t)$ is exponentially convex on I.

(c) The matrix $\left[\Lambda_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^{m}$ is positive semi definite particularly ,

$$det\left[\Lambda_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^{m} \geq 0$$

for each $m \in \mathbb{N}$ and $t_i \in I$ where $i \in \{1,2,\dots,m\}$.

Proof. Proof follows directly from Theorem 4.17.

Corollary 4.19. *Let $D_6 = \{f_t : t \in I\}$ be a class of continuous functions such that the function $t \to [z_0,z_1,z_2; f_t \circ g^{-1}]$ is 2-exponentially convex in the $J-$sense on I for any three mutually distinct points $z_0,z_1,z_2 \in [0,\infty)$ were the functional g is increasing. Let Λ_k be the linear functional for $k \in \{4,5\}$. Then the following statements are valid:*

(a) If the following function $t \to \Lambda_k(f_t)$ is continuous on I, then it is 2-exponentially convex on I. If the function $t \to \Lambda_k(f_t)$ is additionally positive, then it is also $\log -$convex on I. Moreover, the following Lyapunov's inequality holds for $r < s < t;\ r,s,t \in I$.

$$(\Lambda_k(f_s))^{t-r} \leq (\Lambda_k(f_r))^{t-r}(\Lambda_k(f_t))^{s-r}.$$

(b) If the function $t \to \Lambda_k(f_t)$ is positive and differentiable on I, then for every $s,t,u,v \in I$ such that $s \leq u$ and $t \leq v$, we have

$$\mu_{s,t}(\Lambda_k, D_6) \leq \mu_{u,v}(\Lambda_k, D_6).$$

where $\mu_{s,t}$ is defined in (9).

Conflict of Interests

The authors declare that there is no conflict of interests.

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