On Some Fractional Integral Inequalities Involving Generalized Riemann-Liouville Fractional Integral Operator

Mohamed Houas [a] *, Mohamed Bezziou [a]

[a] Laboratory FIMA, UDBKM, University of Khemis Miliana, Algeria.

ARTICLE INFO

ABSTRACT

In this paper, the generalized Riemann-Liouville fractional integral operator is used to generate some new fractional integral inequalities. By using the generalized Riemann-Liouville fractional integral operator, we also generate new classes of fractional integral inequalities using a family of n, $(n \geq 1)$ positive functions.

1. Introduction

Integral inequalities play a very important role in the theory of differential equations and applied mathematics. These inequalities have gained considerable popularity and importance during the past few decades due to their distinguished applications in numerical quadrature, transform theory, probability, and statistical problems. For details, we refer to [8, 9, 10, 11, 12, 14, 16, 18] and the references therein. Moreover, the study of fractional type inequalities is also of great importance. A detailed account of such fractional integral inequalities along with their applications can be found in the research contributions by many author see [1, 3, 4, 13, 19]. In the past several years, many author have studied on fractional integral inequalities using Riemann-Liouville, Hadamard fractional integral and $q-$fractional integral, see [2, 5, 6, 17]. In this paper we present some new fractional integral inequalities using generalized Riemann-Liouville fractional integral.

2. Preliminaries

Firstly, we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

---

*Email : houasmed@yahoo.fr

Definition 1 *A real valued function $f(t)$, $t \geq 0$ is said to be in the space $\mathbb{C}_\upsilon (0, \infty)$, $\upsilon \in \mathbb{R}$, if there exists a real number $p > \upsilon$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in \mathbb{C}([0, \infty[)$.*

Definition 2 *A function $f(t)$, $t > 0$ is said to be in the space $\mathbb{C}_\upsilon^n, n \in \mathbb{R}$, if $f^{(n)} \in \mathbb{C}_\upsilon$.*

Definition 3 *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[a, b]$ is defined as*

$$J_a^\alpha [f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha - 1} f(\tau) \, d\tau, \ \alpha > 0, \ a < t \leq b, \tag{2.1}$$

$$J_a^0 [f(t)] = f(t),$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha - 1} du$.

For the convenience of establishing the results, we give the following properties :

$$J_a^\alpha J_a^\beta [f(t)] = J_a^{\alpha + \beta} [f(t)], \tag{2.2}$$

and

$$J_a^\alpha J_a^\beta [f(t)] = J_a^\beta J_a^\alpha [f(t)]. \tag{2.3}$$

Definition 4 *Consider the space $L_{p,k}(a, b) \ (k \geq 0, \ 1 \leq p < \infty)$ of those real-valued Lebesgue measurable functions f on $[a, b]$ for which*

$$\|f\|_{L_{p,k}(a,b)} = \left(\int_a^b |f(x)|^p x^k dx \right)^{\frac{1}{p}} < \infty, \ k \geq 0, \ 1 \leq p < \infty. \tag{2.4}$$

Definition 5 *Consider the space $X_c^p(a, b) \ (c \in \mathbb{R}, \ 1 \leq p < \infty)$ of those real-valued Lebesgue measurable functions f on $[a, b]$ for which*

$$\|f\|_{X_c^p(a,b)} = \left(\int_a^b |x^c f(x)|^p \frac{dx}{x} \right)^{\frac{1}{p}} < \infty, \ c \in \mathbb{R}, \ 1 \leq p < \infty, \tag{2.5}$$

and for the case $p = \infty$

$$\|f\|_{X_c^\infty} = ess \sup_{a \leq x \leq b} [x^c f(x)], \ c \in \mathbb{R}. \tag{2.6}$$

In particular, when $c = \frac{k+1}{p} (k \geq 0, \ 1 \leq p < \infty)$ the space $X_c^p(a, b)$ coincides with the $L_{p,k}(a, b) -$space and also if we take $c = \frac{1}{p} (1 \leq p < \infty)$ the space $X_c^p(a, b)$ coincides with the classical $L^p(a, b) -$space.

Definition 6 *Let $f \in L_{1,k}[a, b]$. The generalized Riemann-Liouville fractional integral $J_a^{\alpha,k}$ of order $\alpha \geq 0$ and $k \geq 0$ is defined by*

$$J_a^{\alpha,k} f(t) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \left(t^{k+1} - \tau^{k+1} \right)^{\alpha - 1} \tau^k f(\tau) \, d\tau, \ \alpha > 0, \ a < t \leq b,$$
$$\tag{2.7}$$

$$J_a^{0,k} [f(t)] = f(t),$$

For more details on can consult [7, 10, 13].

3. Main Results

In this section, we prove some inequalities concerning the generalized Riemann-Liouville fractional integral.

Theorem 7 *Let f, h and g be three positive continuous functions on $[a, b]$, such that*

$$h(x) \frac{f(y)}{g(y)} + h(y) \frac{f(x)}{g(x)} \geq h(x) \frac{f(x)}{g(x)} + h(y) \frac{f(y)}{g(y)}; \ x, y \in [a, t], \ a < t \leq b . \tag{3.1}$$

Then the generalized fractional integral inequality

$$J_a^{\alpha,k} [g(t)] J_a^{\alpha,k} [f(t) h(t)] \leq J_a^{\alpha,k} [f(t)] J_a^{\alpha,k} [h(t) g(t)], \tag{3.2}$$

holds for all $a < t \leq b$, $\alpha > 0$, $k \geq 0$.

Proof. Suppose that f, h and g are positive and continuous functions on $[a, b]$ satisfying the condition (3.1). Then we define

$$\phi(x, y) := \varphi_\alpha^k(t, x)(f(y) g(x) h(x) + g(y) h(y) f(x)$$
$$\hspace{3cm} -g(y) h(x) f(x) - h(y) f(y) g(x)). \tag{3.3}$$

where,

$$\varphi_\alpha^k(t, x) := \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \left(t^{k+1} - x^{k+1} \right)^{\alpha-1} x^k, \tag{3.4}$$

It is clear that

$$\phi(x, y) \geq 0. \tag{3.5}$$

Integrating (3.5) with respect to x over (a, t), yields

$$0 \leq \int_a^t \phi(x, y) \, dx$$

$$= \int_a^t \varphi_\alpha^k(t, x)(f(y) g(x) h(x) + g(y) h(y) f(x)$$

$$-g(y) h(x) f(x) - h(y) f(y) g(x)) \, dx \tag{3.6}$$

$$= J_a^{\alpha,k} [g(t) h(t)] f(y) + J_a^{\alpha,k} [f(t)] g(y) h(y)$$

$$-J_a^{\alpha,k} [h(t) f(t)] g(y) - J_a^{\alpha,k} [g(t)] h(y) f(y).$$

Now, multiplying (3.6) by $\varphi_\alpha^k(t,y)$, $y \in (a,t)$, $a < t \leq b$ and integrating with respect to y over (a,t), we can write

$$0 \leq \int_a^t \int_a^t \varphi_\alpha^k(t,y) \, \phi(x,y) \, dxdy$$

$$= \int_a^t \int_a^t \varphi_\alpha^k(t,y) \, \varphi_\alpha^k(t,x) \, (f(y) \, g(x) \, h(x) + g(y) \, h(y) \, f(x)$$

$$-g(y) \, h(x) \, f(x) - h(y) \, f(y) \, g(x)) \, dxdy$$

$$= J_a^{\alpha,k} [g(t) \, h(t)] \int_a^t \varphi_\alpha^k(t,y) \, f(y) \, dy + J_a^{\alpha,k} [f(t)] \int_a^t \varphi_\alpha^k(t,y) \, g(y) \, h(y) \, dy$$

$$-J_a^{\alpha,k} [h(t) \, f(t)] \int_a^t \varphi_\alpha^k(t,y) \, g(y) \, dy - J_a^{\alpha,k} [g(t)] \int_a^t \varphi_\alpha^k(t,y) \, h(y) \, f(y) \, dy$$

$$= 2J_a^{\alpha,k} [g(t) \, h(t)] \, J_a^{\alpha,k} [f(t)] - 2J_a^{\alpha,k} [h(t) \, f(t)] \, J_a^{\alpha,k} [g(t)]$$

(3.7)

This implise that

$$J_a^{\alpha,k} [gh(t)] \, J_a^{\alpha,k} [f(t)] \geq J_a^{\alpha,k} [hf(t)] \, J_a^{\alpha,k} [g(t)] \tag{3.8}$$

The proof is completed.

Our the next result is the following theorem, in which we use two fractional positive parameters.

Theorem 8 *Let f, h and g be three positive continuous functions on $[a,b]$. Then, the following generalized fractional inequality*

$$J_a^{\alpha,k} [h(t) \, f(t)] \, J_a^{\beta,k} [g(t)] - J_a^{\alpha,k} [g(t)] \, J_a^{\beta,k} [h(t) \, f(t)] \tag{3.9}$$

$$\leq \quad J_a^{\alpha,k} [g(t) \, h(t)] \, J_a^{\beta,k} [f(t)] + J_a^{\alpha,k} [f(t)] \, J_a^{\beta,k} [g(t) \, h(t)] ,$$

is valid for all $a < t \leq b$, $\alpha > 0$, $\beta > 0$, $k \geq 0$.

Proof. Multiplying both sides of (3.5) by the quantity $\varphi_\beta^k(t,y)$, $y \in (a,t)$, $a < t \leq b$, then integrating the resulting inequality with respect to y over (a,t) we get

$$
\begin{aligned}
0 \quad &\leq \quad \int_a^t \int_a^t \varphi_\beta^k(t,y)\, \phi(x,y)\, dxdy \qquad\qquad (3.10)\\[2mm]
&= \quad \int_a^t \int_a^t \varphi_\beta^k(t,y)\, \varphi_\alpha^k(t,x)\, (f(y)\, g(x)\, h(x) + g(y)\, h(y)\, f(x)\\[2mm]
&\qquad - g(y)\, h(x)\, f(x) - h(y)\, f(y)\, g(x))\, dxdy\\[2mm]
&= \quad J_a^{\alpha,k}[g(t)\, h(t)] \int_a^t \varphi_\beta^k(t,y)\, f(y)\, dy + J_a^{\alpha,k}[f(t)] \int_a^t \varphi_\beta^k(t,y)\, g(y)\, h(y)\, dy\\[2mm]
&\qquad - J_a^{\alpha,k}[h(t)\, f(t)] \int_a^t \varphi_\beta^k(t,y)\, g(y)\, dy - J_a^{\alpha,k}[g(t)] \int_a^t \varphi_\beta^k(t,y)\, h(y)\, f(y)\, dy\\[2mm]
&= \quad J_a^{\alpha,k}[g(t)\, h(t)]\, J_a^{\beta,k}[f(t)] + J_a^{\alpha,k}[f(t)]\, J_a^{\beta,k}[g(t)\, h(t)]\\[2mm]
&\qquad - J_a^{\alpha,k}[h(t)\, f(t)]\, J_a^{\beta,k}[g(t)] - J_a^{\alpha,k}[g(t)]\, J_a^{\beta,k}[h(t)\, f(t)].
\end{aligned}
$$

This implies that

$$
J_a^{\alpha,k}[gh(t)]\, J_a^{\beta,k}[f(t)] + J_a^{\alpha,k}[f(t)]\, J_a^{\beta,k}[gh(t)] \qquad\qquad (3.11)
$$

$$
\geq \quad J_a^{\alpha,k}[hf(t)]\, J_a^{\beta,k}[g(t)] - J_a^{\alpha,k}[g(t)]\, J_a^{\beta,k}[hf(t)].
$$

Theorem 8 is thus proved.

Remark 9 *Applying Theorem 8 for $\alpha = \beta$, we obtain Theorem 7.*

Now, we shall propose a new generalization of integral inequalities using a family of n positive functions defined on $[a.b]$.

Theorem 10 *Let f, h and $g_i, i = 1, ..., n$ be positive and continuous functions on $[a,b]$. Then, the following fractional inequality*

$$
J_a^{\alpha,k}\left[\textstyle\prod_{i=1}^n g_i(t)\right] J_a^{\alpha,k}\left[h(t)\, f(t)\, \textstyle\prod_{i\neq q}^n g_i(t)\right]
$$

$$
\qquad\qquad (3.12)
$$

$$
\leq J_a^{\alpha,k}\left[f(t)\, \textstyle\prod_{i\neq q}^n g_i(t)\right] J_a^{\alpha,k}\left[h(t)\, \textstyle\prod_{i=1}^n g_i(t)\right],
$$

is valid for any $a < t \leq b$, $\alpha > 0$, $k \geq 0$.

Proof. Suppose that f, h and g_i, $i = 1, ..., n$ are positive continuous functions on $[a,b]$, then we can write

$$
h(x)\, \frac{f(y)}{g_q(y)} + h(y)\, \frac{f(x)}{g_q(x)} \geq h(x)\, \frac{f(x)}{g_q(x)} + h(y)\, \frac{f(y)}{g_q(y)}, \qquad\qquad (3.13)
$$

for any fixed $q \in \{1, ..., n\}$ and for any $x, y \in [a, t]$, $a < t \leq b$.

Denote

$$\phi_q(x, y) :=$$

$$\varphi_\alpha^k(t, x) \left(f(y) \prod_{i \neq q}^n g_i(y) h(x) \prod_{i=1}^n g_i(x) + h(y) \prod_{i=1}^n g_i(y) f(x) \prod_{i \neq q}^n g_i(x) \right. \tag{3.14}$$

$$\left. - \prod_{i=1}^n g_i(y) h(x) f(x) \prod_{i \neq q}^n g_i(x) - h(y) f(y) \prod_{i \neq q}^n g_i(y) \prod_{i=1}^n g_i(x) \right),$$

for all $x, y \in [a, t]$, $a < t \leq b$ and for any fixed integer $q \in \{1, ..., n\}$.

We have

$$\phi_q(x, y) \geq 0. \tag{3.15}$$

Now, integrating (3.15) with respect to x over (a, t), we obtain

$$0 \leq \int_a^t \phi_q(x, y) \, dx$$

$$= f(y) \prod_{i \neq q}^n g_i(y) J_a^{\alpha, k} \left[h(t) \prod_{i=1}^n g_i(t) \right]$$

$$+ h(y) \prod_{i=1}^n g_i(y) J_a^{\alpha, k} \left[f(t) \prod_{i \neq q}^n g_i(t) \right] \tag{3.16}$$

$$- \prod_{i=1}^n g_i(y) J_a^{\alpha, k} \left[h(t) f(t) \prod_{i \neq q}^n g_i(t) \right]$$

$$- h(y) f(y) \prod_{i \neq q}^n g_i(y) J_a^{\alpha, k} \left[\prod_{i=1}^n g_i(t) \right].$$

Next, multiplying both sides of (3.16) by $\varphi_\alpha^k(t, y)$, $y \in (a, t)$, integrating the resulting inequality with respect to y from a to t, we can write

$$0 \leq \int_a^t \int_a^t \varphi_\alpha^k(t, y) \phi_q(x, y) \, dx \, dy$$

$$= J_a^{\alpha, k} \left[f(t) \prod_{i \neq q}^n g_i(t) \right] J_a^{\alpha, k} \left[h(t) \prod_{i=1}^n g_i(t) \right]$$

$$+ J_a^{\alpha, k} \left[h(t) \prod_{i=1}^n g_i(t) \right] J_a^{\alpha, k} \left[f(t) \prod_{i \neq q}^n g_i(t) \right] \tag{3.17}$$

$$- J_a^{\alpha, k} \left[\prod_{i=1}^n g_i(t) \right] J_a^{\alpha, k} \left[h(t) f(t) \prod_{i \neq q}^n g_i(t) \right]$$

$$- J_a^{\alpha, k} \left[h(t) f(t) \prod_{i \neq q}^n g_i(t) \right] J_a^{\alpha, k} \left[\prod_{i=1}^n g_i(t) \right].$$

and consequently, we have

$$0 \leq 2 J_a^{\alpha, k} \left[f(t) \prod_{i \neq q}^n g_i(t) \right] J_a^{\alpha, k} \left[h(t) \prod_{i=1}^n g_i(t) \right] \tag{3.18}$$

$$- 2 J_a^{\alpha, k} \left[\prod_{i=1}^n g_i(t) \right] J_a^{\alpha, k} \left[h(t) f(t) \prod_{i \neq q}^n g_i(t) \right].$$

The proof is completed.

Using two fractional parameters, we obtain the following generalization of Theorem 10.

Theorem 11 *Let f, h and g_i, $i = 1, ..., n$ be positive continuous functions on $[a, b]$. Then, for any fixed $q \in \{1, ..., n\}$ and for all $a < t \leq b$, $\alpha > 0$, $\beta > 0$, $k \geq 0$, we have*

$$J_a^{\alpha,k} \left[\prod_{i=1}^n g_i(t)\right] J_a^{\beta,k} \left[h(t) f(t) \prod_{i \neq q}^n g_i(t)\right]$$

$$+ J_a^{\beta,k} \left[\prod_{i=1}^n g_i(t)\right] J_a^{\alpha,k} \left[h(t) f(t) \prod_{i \neq q}^n g_i(t)\right]$$

$$\leq J_a^{\alpha,k} \left[f(t) \prod_{i \neq q}^n g_i(t)\right] J_a^{\beta,k} \left[h(t) \prod_{i=1}^n g_i(t)\right] \tag{3.19}$$

$$+ J_a^{\beta,k} \left[f(t) \prod_{i \neq q}^n g_i(t)\right] J_a^{\beta,k} \left[h(t) \prod_{i=1}^n g_i(t)\right].$$

Proof. Multiplying both sides of (3.16) by $\varphi_\beta^k(t, y)$, $y \in (a, t)$, and integrating with respect to y from a to t, we obtain

$$0 \leq \int_a^t \int_a^t \varphi_\beta^k(t, y) \phi_q(x, y) \, dx dy \tag{3.20}$$

$$= \int_a^t \int_a^t \varphi_\beta^k(t, y) f(y) \prod_{i \neq q}^n g_i(y) \varphi_\alpha^k(t, x) h(x) \prod_{i=1}^n g_i(x) \, dx dy$$

$$+ \int_a^t \int_a^t \varphi_\beta^k(t, y) h(y) \prod_{i=1}^n g_i(y) \varphi_{\alpha,\theta}^k(t, x) f(x) \prod_{i \neq q}^n g_i(x) \, dx dy$$

$$- \int_a^t \int_a^t \varphi_\beta^k(t, y) \prod_{i=1}^n g_i(y) \varphi_\alpha^k(t, x) h(x) f(x) \prod_{i \neq q}^n g_i(x) \, dx dy$$

$$- \int_a^t \int_a^t \varphi_\beta^k(t, y) h(y) f(y) \prod_{i \neq q}^n g_i(y) \varphi_\alpha^k(t, x) \prod_{i=1}^n g_i(x) \, dx dy.$$

It follows that

$$0 \leq J_a^{\alpha,k} \left[f(t) \prod_{i \neq q}^n g_i(t)\right] J_a^{\beta,k} \left[h(t) \prod_{i=1}^n g_i(t)\right]$$

$$+ J_a^{\beta,k} \left[f(t) \prod_{i \neq q}^n g_i(t)\right] J_a^{\alpha,k} \left[h(t) \prod_{i=1}^n g_i(t)\right]$$

$$- J_a^{\alpha,k} \left[\prod_{i=1}^n g_i(t)\right] J_a^{\beta,k} \left[h(t) f(t) \prod_{i \neq q}^n g_i(t)\right] \tag{3.21}$$

$$- J_a^{\beta,k} \left[\prod_{i=1}^n g_i(t)\right] J_a^{\alpha,k} \left[h(t) f(t) \prod_{i \neq q}^n g_i(t)\right].$$

This completes the proof.

Remark 12 *If we take $\alpha = \beta$, in Theorem 11, we obtain Theorem 10.*

REFERENCES

[1] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, J. Inequal. Pure Appl. Math. 10(3) (2009) pp 1-12.

[2] V. L. Chinchane and D. B. Pachpatte, A note on some integral inequalities via Hadamard integral, Journal of Fractional Calculus and Applications. 4 (2013) 1-5.

[3] Z. Dahmani, New inequalities in fractional integrals, International Journal of Nonlinear Sciences. 9(4) (2010) pp 493-497.

[4] Z. Dahmani and N. Bedjaoui, New generalized integral inequalities, J. Advan. Res. Appl. Math. 3(4) (2011) pp 58–66.

[5] Z. Dahmani and A. Benzidane, New inequalities using $Q-$fractional theory, Bull. Math. Anal. Appl. 4 (2012) pp 190-196 .

[6] Z. Dahmani, New classes of integral inequalities of fractional order, Le Matematiche. (2014) pp 227–235.

[7] R. Gorenflo and F. Mainardi, Fractional calculus : integral and differentiable equations of fractional order, Springer, Verlag, Wien, 1997).

[8] M. Houas, Some inequalities for $k-$fractional continuous random variables, J. Advan. Res. In Dynamical and Control Systems. 7(4) (2015) pp 43-50.

[9] M. Houas, Some integral inequalities involving Saigo fractional integral operators. Accepeted.

[10] M. Houas, Some weighted integral inequalities for Hadamard fractional integral Operators. Accepeted.

[11] U. Katugampola, New approach to a generalized fractional integral, Appl. Math. Comput. 218(3) (2011) pp 860–865.

[12] P. Kumar, Inequalities involving moments of a continuous random variable defined over a finite interval, Computers and Mathematics with Applications. 48 (2004) pp 257-273.

[13] W. Liu, Q. A. Ngo and V. N. Huy, Several interesting integral inequalities, Journal of Math. Inequal. 3(2) (2009) pp 201–212.

[14] M. A. Noor, M. U. Awan and K. I. Noor, On some inequalities for relative semi-convex functions, Journal of Inequal and Appl. 332 (2013) pp 1-16 .

[15] S. G. Samko, A. A Kilbas, and O. I. Marichev, Fractional integrals and derivatives, Theory and Application, Gordan and Breach Science, New York, 1993.

[16] M. Z. Sarikaya and H. Yaldiz, Note on the Ostrowski type inequalities for fractional integrals. Vietnam. J. Math. 41(4), (2013) pp 2-6 .

[17] M. Z. Sarikaya and H. Yaldiz, New generalization fractional inequalities of Ostrowski-Gruss type, Lobachevskii Journal of Mathematics 34(4) (2013) pp 326-331.

[18] D. Ucar and A. Deniz, Generalizations of holder's inequalities on time scales, Journal of Math. Inequal. 9(1) (2015) pp 247–255.

[19] H. Yildirim and Z. Kirtay, Ostrowski inequality for generalized fractional integral and related inequalities, Malaya Journal of Matematik. 2(3) (2014) pp 322-329.