GESTÃO, FINANÇAS E CONTABILIDADE

FINANCIAL IMPLICATIONS OF THE "GAUSS METHOD"

IMPLICAÇÕES FINANCEIRAS DO "MÉTODO DE GAUSS"

IMPLICACIONES FINANCIERAS DEL "MÉTODO DE GAUSS"

DOI: 10.18028/2238-5320/rgfc.v6n2p179-188

Clovis José Daudt Lyra Darrigue de Faro
Pós-Doutor pela *Universität München*
Doutor em Engenharia Industrial (*Stanford University*)
Professor da Fundação Getulio Vargas (FGV)
Endereço: Praia de Botafogo, 190 – 11º andar – Botafogo
22.253-900 – Rio de Janeiro /RJ, Brasil
Email: cfaro@fgv.br

ABSTRACT
The Braziliam Judiciary has been determining that the popular system of constant payments, known as Tabela Price, based on compound interest, be substituted by an amortizacion scheme, named as "Method of Gauss", with constant payments based on simple interest. Moreover, the substitution being done while maintaining the numerical value of the contractual interest rate. Taking into account that a possible defensive strategy which may be used by the financial institutions, is not always feasible, it is also shown that the use of the so called "Method of Gauss" will introduce unsurmountable difficulties concerning the determination of outstanding debts. Since the methodology that has been proposed for the determination of outstanding debts, does not agree with procedures that are well known in the pertinent literature.
Key words: Debt Amortization – "Gauss Method"

RESUMO
Têm sido frequentes em nossos tribunais, sentenças determinando que contratos de financiamentos habitacionais substituam a popular Tabela Price, que se fundamenta no regime de juros compostos, por um sistema, também de prestações constantes e baseado no regime de juros simples, que tem sido chamado de "Método de Gauss". Com tal substituição sendo efetuada sem que seja alterado o valor numérico da taxa contratual de juros. Evidenciando que uma possível estratégia defensiva, por parte das instituições financeiras, no sentido de majorar as taxas contratuais de juros, nem sempre é factível, mostra-se também que a adoção do " Método de Gauss" ensejará insanáveis conflitos no que concerne à apuração de saldos devedores. Pois que a metodologia que tem sido proposta para a apuração de saldos devedores, é inconsistente com procedimentos consagrados na literatura pertinente.
Palavras-chave: Amortização - "Método de Gauss",

Recebido em 30.09.2015. Revisado por pares em 27.11.2015. Reformulações em 30.12.2015 e 28.03.2016. Recomendado para publicação em 22.04.2016. Publicado em 14.07.2016

Licensed under a Creative Commons Attribution 3.0 United States License

RESUMEN
Es frecuente en nuestros tribunales, sentencias determinando que contratos de financiamientos habitacionales sustituyan la popular Tabla Price, que se fundamenta en el régimen de intereses compuestos, sea substituido por un sistema, también de parcelas constantes, basadas no régimen de interesses simples, que ha sido llamado de “Método de Gauss”. Con tal substitución siendo efectuada sin que sea alterado el valor numérico de la tasa contractual de intereses. Evidenciando que una posible estrategia defensiva, por parte de las instituciones financieras, en el sentido de mejorar las tasas contractuales de intereses, ni siempre es factible, se muestra también que la adopción del “Método de Gauss” dejará insanables conflictos en lo que concierne a la apuración de los saldos deudores, puesto que la metodología que ha sido propuesta para la apuración de saldos deudores, es inconsistente como procedimientos consagrados en la literatura pertinente.
Palabras claves: Amortización-“Método de Gauss”

1- INTRODUCTION

In general, especially for housing loans, the financial institutions adopt the principles of the compound interest regime.

However, even though originally established in the Brazilian System of Housing Financing (SFH), created in 1964, the so called Tabela Price Scheme of Loan Amortization, which is based on compound interest, and is characterized by a sequence of constant payments, has been challenged in some Brazilian tribunals (cf. de Faro & Guerra, 2014).

Regarding the interpretation that Tabela Price necessarily implies the occurrence of anatocism (payment of interest on interest), it has been determined that Tabela Price be substituted with a simple interest procedure, coined “Method of Gauss” (cf. Antonick & Assunção, 2006 and Nogueira, 2013). Moreover, said substitution is made while maintaining the numerical value of the contractual interest rate.

Stressing the fact that it is not appropriate to associate the name of the great mathematician, Carl Friedrich Gauss (1777-1855), to the proposed procedure, our purpose is to show that, as with any other amortization scheme based on simple interest, the so called “Method of Gauss” is plagued by severe inconsistencies.

In particular, focusing attention on the case of two periods, for which it is possible to derive analytical solutions, it is shown that the determination of the outstanding debt leads to contradictory results.

2- STATEMENT OF THE PROBLEM

Considering the periodic interest rate i, suppose that a loan F has to be repaid by a sequence of n periodic payments of equal values.

If the rate i is of compound interest, the value of the constant payment p is given by the classical formula (cf. de Faro 2014b, p. 241):

$$p = F.\,i \,/ \left\{1-(1+i)^{-n}\right\} \tag{1}$$

On the other hand, the specification of simple interest would imply that the value of the constant payment $\hat{p}$ is (cf. Antonick & Assunção, 2006 and Nogueira, 2013, p. 150):

$$\hat{p} = 2F\,(1+n.i) \,/ \left\{n\left[2+i\,(n-1)\right]\right\} \tag{2}$$

Noting that, as pointed out by Nogueira (2013, p. 127-130), relation (2) dates back to at least the XVIII century (cf. Wilkie, 1794) when Gauss was only 17 years old, the justification of relations (1) and (2) is presented in the Appendix.

While it is clear that, if n = 1, we have $p = \hat{p} = F(1+i)$, it was shown in de Faro (2013) that $p > \hat{p}$ if $n \geq 2$.

Consequently, the substitution of Tabela Price by the "Method of Gauss," while maintaining the numerical value of the interest rate i, will imply, if $n > 1$, a loss to the financial institution providing the loan.

For instance, for a loan $F = R\$200,000.00$, with monthly payments over 5 years $(n = 60)$, with a contractual interest rate of 24% per year, in accordance to Tabela Price (which implies a monthly rate of 2%, compound interest), relation (1) would yield the monthly payment value of $p = R\$5,753.79,$.

On the other hand, keeping the monthly interest rate of 2%, the adoption of the "Method of Gauss" would imply that the monthly payment be reduced to $\hat{p} = R\$4,612.16$; according to relation (2). From the point of view of the financial institution providing the loan, the substitution of Tabela Price by the "Method of Gauss," would result in a loss of 19.84%.

3- A POSSIBLE FINANCIAL IMPLICATION

Given that financial institutions should be aware of the judicial rulings requiring the substitution of Tabela Price with the "Method of Gauss," maintaining the value of the contractual interest rate, it is likely that they would try to implement the following steps:

a) Given the value of the interest rate i, which expresses the financial institution's profitability, and the values of F and n, making use of relation (1) to determine the contractual value of the periodic payment p would be implemented.
b) From relation (2), taking the interest rate as the unknown, and making use of the value of p obtained in the previous step, the value $\hat{i}$ of the contractual interest rate would be determined. That is, the interest rate that will actually be written in the contract is:

$$\hat{i} = 2(F - n.p) / \{n[(n-1)p - 2F]\} \qquad (2')$$

Ultimately, it would be making use of the so called "Merchant's Rule" (cf. Ayres, 1963 and Butcher & Nesbitt, 1971), of which a version was also presented in de Faro (1969, p. 93-94).

c) Maintaining the values of F and n, the contract would specify the adoption of Tabela Price with the effective interest rate $\hat{i}$.

d) With such a procedure, if the judicial rulings stipulate that Tabela Price has to be substituted with the "Method of Gauss," the financial institution would protect its profitability.

As an illustration, consider the case of a loan of R$ 100,000.00, with 120 monthly payments.

If the financial institution has the objective of obtaining the monthly interest rate of 0.5%, it follows, from relation (1), that the value of the monthly payment has to be p = R$ 1,110.21.

Making use of this value of p, and of relation (2’), it follows that $\hat{i} = 0.0082$ p.m.; or, with greater precision, 0.8157% p.m. This would be the contractual value for use of Tabela Price.

Consequently, if the judicial ruling would require the substitution of Tabela Price with the “Method of Gauss,” relation (2), with F = R$ 100,000.00, n = 120, and i = 0.8157% p.m., would precisely yield the desired value of p = R$ 1,110.21.

Therefore, once the suggested procedure is adopted, the financial institution would maintain its desired profitability of 0.5% p.m.

4 – IS IT ALWAYS POSSIBLE TO AVOID LOSS OF PROFITABILITY?

As it will be shown here, the answer to the above question is not always in the affirmative.

Consider a loan F = R$ 100,000.00, which has to repaid with 10 annual payments, with an effective annual interest rate of 10%.

Under Tabela Price, it follows, from relation (1), that the value of each of the 10 annual payments would be:

$$p = 0.10 \times 100,000 \Big/ \left\{1-(1+0.10)^{-10}\right\} = \text{R\$}16,274.54$$

Hence, making use of relation (2’), in order to achieve its profitability, the contractual value of the interest rate would have to be written as:

$$\hat{i} = 2(100,000 - 10 \times 16,274.54) \Big/ \left\{10\left[(10-1)16,274.54 - 2 \times 100,000\right]\right\} = 0.2344$$

or 23.44% per year.

That is, as presented in Table I, we would have $\hat{i}/i = 2.3443$; which means that the actual value of the contractual interest rate would have to be more than double that of the desired profitability.

Table I also illustrates the results that would be obtained when the number of annual payments is successively increased.

Table I

Evolution of the ratio $\hat{i}/i$ when i = 10% p.y.

N (years)	***p*** (R$)	$\hat{i}$ (% p.y)	$\hat{i}/i$ --
10	16,274.54	23.44	2.3443
11	15,396.31	27.39	2.7393
12	14,676.33	32.90	3.2899
13	14,077.85	41.11	4.1110
14	13,574.62	54.67	5.4669

15	13,147.38	81.33	8.1331
16	12,781.66	157.86	15.7864
17	12,466.41	2,450.41	245.0407
18	12,193.02	does not exist	––

The values presented in Table I show that whenever the number of payments is increased, it is necessary to make successively larger increases to the contractual interest rate $\hat{i}$. For instance, the contractual value of $\hat{i}$ would have to be over 245 times greater than that of the required rate of 10% p.y.

Moreover, when n = 18 years, we would encounter an impossibility since, as shown in the Appendix, relation (2’) is not applicable whenever the value of p is greater than or equal to the ratio $2F/(n\text{-}1)$, which, in the case of our example, occurs when $p \geq 2 \times 100,000/(18-1) = R\$11,764.71$.

In Table II, considering the somewhat modest (by Brazilian standards) rate of 1% p.m., we have monthly payment results ranging from 5 to 14 years.

Thus, for a loan $F = R\$100,000.00$, which has to be repaid with monthly payments along n years, we have the corresponding results obtained from relations (1) and (2), as well those of the limit values $p*$, given by the following relation:

$$p* = 2F/(n-1) \tag{3}$$

Additionally, Table II depicts the corresponding results obtained from relation (2’).

Table II
Limits Values

n (years)	p (R$)	$\hat{p}$ (R$)	$p*$ (R$)	$\hat{i}$ (% p.m.)
5	2,224.44	2,059.20	3,389.83	1.62
7	1,765.27	1,548.04	2,409.64	2.15
10	1,434.71	1,149.34	1,680.67	4.11
11	1,367.79	1,061.98	1,523.72	5.86
12	1,313.42	988.01	1,398.62	10.16
13	1,268.67	924.52	1,290.32	37.40
14	1,231.43	869.34	1,197.60	impossible

Thus, even for a very short-term loan of 5 years, the contractual interest rate would have to be 62% higher than the desired rate of 1% p.m. Moreover, the contractual rate would even be higher when the number of years of the contract is increased.

Furthermore, the strategy of increasing the contractual interest rate is impossible whenever the number of years of the contract is greater than 13.

5 – **DETERMINATION OF THE OUTSTANDING DEBT**

Whichever the considered amortization scheme, a crucial issue is the determination of the outstanding debt.

For the case of the so called “Gauss Method,” their proponents (cf. Antonick & Assunção, 2006 and Nogueira, 2013, p.150), propose the following procedure.

Preliminarily, what has been named as “índice de ponderação” (weight-index), is defined as per the following relation:

$$I = 2i.F \Big/ \left\{ n\left[2 + i\left(n-1\right)\right] \right\} \tag{4}$$

with the interest component and the amortization component of the *k*-th payment being respectively given by:

$$\hat{J}_k = \left(n-k+1\right) I \tag{5}$$

and

$$\hat{A}_k = \hat{p}_k - \hat{J}_k \text{ , } k = 1, 2, ..., n \tag{6}$$

where $\hat{p}_k = \hat{p}$ for all *k*.

As a numerical illustration, consider a loan *F* = R$ 250,000.00, contracted according to Tabela Price at the nominal annual rate of 18% (which corresponding effective interest rate is 1.5% p.m), with monthly payments along 10 years. From relation (1), it follows that the monthly payment would be of R$ 4,504.63.

Supposing that a judicial ruling has determined the adoption of the “Method of Gauss,” maintaining the 1.5% p.m. effective interest rate, it follows from relation (2) that the monthly payment would be reduced to R$ 3,082.34, which amounts to a 31.57% reduction.

Consider now the situation where the debtor decides to liquidate his debt at the occasion of the 60th payment.

In general, according to the principles of the so called “Method of Gauss,” it follows from relations (2), (4), (5), and (6), that the outstanding debt, just after the *m*th payment has been made, can be given by:

$$\hat{S}_m = F\left\{1 - m\left[2 + i\left(m-1\right)\right] \Big/ \left\{ n\left[2 + i\left(n-1\right)\right] \right\}\right\}, m = 1, 2, \cdots, n \tag{7}$$

Thus, in our example, the value of the outstanding debt would be:

$$\hat{S}_{60} = 250,000\left\{1 - \frac{60\left[2 + 0.015\left(60-1\right)\right]}{120\left[2 + 0.015\left(120-1\right)\right]}\right\} = R\$154,722.59$$

However, certain truths cannot be ignored. That is, it is not possible to contradict certain principles, even more so when, besides being supported by the pertinent literature, they are in flagrant conflict with common sense.

Precisely for the case considered in our example, we have the following two classical procedures:

a) the retrospective method

According to this classical method (cf. Ayres, 1963, Butcher & Nesbitt, 1971 and Kellison, 2009), for any amortization scheme, the outstanding debt, just after the *m*th payment, is given by the difference between the value of the loan, increased with interest at the considered interest rate for *m* periods, and the accumulated value, also with interest, of the first *m* payments.

In the case of our example, the value of the outstanding debt, just after the occurrence of the 60th payment, would be:

$$S_{60}^{1} = 250,000\left(1+0.015\times 60\right)-\sum_{k=1}^{60} 3,082.34\left\{1+\left(60-k\right)\times 0.015\right\} = R\$\,208,223.47$$

Wherefore, we would have a stalemate, which, probably, would result in further judicial arguments.

b) the prospective method

Also a standard procedure in the financial literature (cf. Ayres, 1963, Butcher & Nesbitt, 1971 and Kellison, 2009), the prospective method determines that the outstanding debt just after the *m*th payment, is equal to the present value, considering the contractual value of the interest rate of the remaining *n*-*m* payments.

In the case of the considered example:

$$S_{60}^{2} = \sum_{k=1}^{60} 3,082.34/\left(1+0.015\times k\right) = R\$131,164.73$$

Once more, we would have conflicting results.

5.1 – **THE CASE OF TWO PERIODS**

In the previous section, making use of a numerical illustration, it was shown that the so called “Method of Gauss,” as in any amortization scheme based on simple interest, lacks what can be denoted as the property of financial consistency (cf. de Faro, 2014a). Namely, it implies that the prospective and retrospective methods have to produce equal results.

In this section, focusing attention in the case of $n = 2$ periods, for which analytical solutions can be readily derived, it will be proven that, for any positive rate of interest i, the so called “Method of Gauss” leads to financially inconsistent results.

Computing the value of the outstanding debt just after the occurrence of the first payment, it follows from (7) that:

$$\hat{S}_1 = F\left(1+i\right)/\left(2+i\right) \quad (8)$$

On the other hand, with the application of the retrospective method, we would have

$$S_1^1 = F\left(1+i\right)-\hat{p} \quad (9)$$

or

$$S_1^1 = F\left(1+i+i^2\right)/\left(2+i\right) \quad (9’)$$

While with the prospective method, we would have

$$S_1^2 = \hat{p}/\left(1+i\right) \quad (10)$$

or

$$S_1^2 = F\left(1+2i\right)/\left(2+3i+i^2\right) \qquad (10')$$

Hence, $S_1^2 > \hat{S}_1 > S_1^2$,

Therefore, we can conclude that the so called “Method of Gauss” is not financially consistent.

6 – **CONCLUSION**

On the grounds of the fallacy that Tabela Price implies in anatocism, the Brazilian Judicial System has been inadequately determining its substitution with the “Method of Gauss”.

However, besides inducing the undesirable effect of an increase of the contractual interest rate, a practice not always feasible, it was shown here that the “Method of Gauss” is not financially consistent.

That is, as with any amortization scheme with two or more payments, that are based on simple interest, the so called “Method of Gauss,” is not able to unequivocally resolve the question associated with the computation of the outstanding debt of a loan.

REFERENCES

ANTONICK, Luiz Roberto; ASSUNÇÃO, Márcio da Silva. Tabela Price e Anatocismo. **Revista de Administração da UNIMEP**, v.4, n.1, p.120-136, jan/abr, 2006.

AYRES JR., Frank. **Mathematics of Finance**. New York: Schaum, 1963.

BUTCHER, Marjorie V.; NESBITT, Cecil J. **Mathematics of Compound Interest**. *A*nn Arbor: Ulrich’s, 1971.

DE FARO, Clovis. **Matemática Financeira**. Rio de Janeiro: APEC, 1969.

DE FARO, Clovis. **Amortização de Dívidas e Prestações Constantes: Uma Análise Crítica**. Ensaio Econômico da EPGE, n. 746, 2013.

DE FARO, Clovis. Sistemas de Amortização: o Conceito de Consistência Financeira e Suas Implicações. Ensaio Econômico da EPGE, n. 751, 2014. To appear in **Revista de Economia e Administração**.

DE FARO, Clovis. **Matemática Financeira: uma Introdução à Análise de Risco**. São Paulo: Saraiva, 2014.

DE FARO, Clovis; GUERRA, Sergio. Proibição da Capitalização de Juros e o Poder Judiciário: Equívocos na Aplicação de Teorias Econômicas sobre Juros Simples e Compostos. **Revista de Direito Administrativo**, n. 266, p. 209-228, 2014.

KELLISON, Stephen G. **The Theory of Interest**, 3rd. New York: McGraw Hill Education, 2009.

NOGUEIRA, Jorge Meschiatti. **Tabela Price: Mitos e Paradigmas**. 3.ed. Campinas: Millenium, 2013.

WILKIE, David. **Theory of Interest**. Edinburgh, 1794.

Appendix

1 – *Constant payments according to Tabela Price*

With regard to the equation of value, which establishes the equivalence between the loan amount F and the sequence of n periodic payments, paid at the end of the relevant period and all equal to p, considering the periodic compound interest rate i, we have:

a) if the comparison date is time 0

$$F = p\sum_{k=1}^{n}\left(1+i\right)^{-k} \tag{A.1}$$

b) if the comparison date is time n

$$F\left(1+i\right)^{n} = p\sum_{k=1}^{n}\left(1+i\right)^{n-k} \tag{A.1’}$$

Taking into account the fact that, if compound interest is considered, the choice of the comparison date is irrelevant, it follows that:

$$p = i.F/\left\{1-\left(1+i\right)^{-n}\right\} \tag{A.2}$$

2 – *Constant payments considering simple interest*

For each of the two comparison dates that were previously considered, we now have:

a) if the comparison date is time 0

In this case, which should be considered as the most common, we have:

$$F = \sum_{k=1}^{n}\frac{\overline{p}}{1+k.i} \tag{A.3}$$

in which the value of the constant payment has been denoted as $\overline{p}$.

It should be pointed out that (and this is probably the reason why the advocates of simple interest do not adopt time 0 as the comparison date), it is not possible, even for a small number of payments, to derive a closed form solution for relation (A.3).

Taking into account that $p = \hat{p} = \overline{p}$ if $n = 1$, it was shown in de Faro (2013) that $p > \overline{p} > \hat{p}$ if $n \geq 2$.

With regard to the question of financial consistency, if we focus attention on the case of only two periods, for which $\overline{p} = F\left(1+3i+2i^2\right)/\left(2+3i\right)$, it follows that, just before the occurrence of the first payment, in accordance with the retrospective method, the outstanding debt is:

$$\overline{S}_1^1 = F\left(1+i\right)$$

While, in accordance with the prospective method, we would have:

$$\overline{S}_1^2 = \overline{p} + \overline{p}/\left(1+i\right)$$
$$= F\left(2+7i+7i^2+2i^3\right)/\left(2+5i+3i^2\right)$$

Therefore, as $\overline{S}_1^1 \neq \overline{S}_1^2$, the property of financial consistency would not be satisfied either.

b) if the comparison date is time *n*

In this case, the corresponding equation of value is

$$F = \sum_{k=1}^{n} \hat{p}\left\{1+i\left(n-k\right)\right\} \qquad \text{(A.4)}$$

from which relation (2) is easily derived in the text.

It is interesting to note that the solution of relation (A.4) makes use of the sum of the first *n* natural numbers, which according to Wikipedia, was known at least by Aryabhata, in 499 AD. Thus, even though the young Gauss, at the age of no more than 8 years, was able to produce, almost at once, the sum of the first 100 natural numbers, it is not appropriate to associate his name to a method of debt amortization that is plagued by severe inconsistencies.

It should be pointed out that, although it is easily seen that the value of $\hat{p}$ increases with the interest rate *i*, with

$$\lim_{i\to\infty} \hat{p} = \lim_{i\to\infty} \frac{2F\left(1+n.i\right)}{n\left[2+i\left(n-1\right)\right]}$$

which, making use of L' Hospital rule, is

$$\lim_{i\to\infty} \hat{p} = \frac{2F}{n}\cdot\lim_{i\to\infty}\frac{n}{n-1} = \frac{2F}{n-1} \qquad \text{(A.5)}$$

Also, as with $\overline{p}$, the value of $\hat{p}$ decreases when the number *n* of payments is increased. Accordingly, for large values of *n*, the values of both $\overline{p}$ and $\hat{p}$ are not sufficient to pay the first period of interest, referenced as *i.F*.