ISSN № 9125 0912. ВІСНИК ДНУ. Серія "Моделювання". № 8. 2012. Вип. 4. С. 63–82

УДК 517.95

ON ATTAINABILITY OF OPTIMAL SOLUTIONS FOR LINEAR ELLIPTIC EQUATIONS WITH UNBOUNDED COEFFICIENTS

P. I. Kogut*, O. P. Kupenko**

* *Dnipropetrovsk National University, Kozakov str., 18/14, DNU,
49010 Dnipropetrovsk, Ukraine, E-mail: p.kogut@i.ua,*
** *National Mining University, K. Marks av. 19,
49005, Dnipropetrovsk, Ukraine*

We study an optimal boundary control problem (OCP) associated to a linear elliptic equation $-\mathrm{div}\,(\nabla y + A(x)\nabla y) = f$ describing diffusion in a turbulent flow. The characteristic feature of this equation is the fact that, in applications, the stream matrix $A(x) = [a_{ij}(x)]_{i,j=1,\dots,N}$ is skew-symmetric, $a_{ij}(x) = -a_{ji}(x)$, measurable, and belongs to L^2-space (rather than L^∞). An optimal solution to such problem can inherit a singular character of the original stream matrix A. We show that optimal solutions can be attainable by solutions of special optimal boundary control problems.

Key words: diffusion equations, boundary control, variational convergence, fictitious control, direct method in the Calculus of Variations.

1. Introduction

In this paper we deal with an optimal control problem (OCP) for linear diffusion elliptic equation with unbounded coefficients in the main part of elliptic operator. The characteristic feature of this problem is the fact that, in applications, the stream matrix $A(x) = [a_{ij}(x)]_{i,j=1,\dots,N}$ is skew-symmetric, $a_{ij}(x) = -a_{ji}(x)$, measurable, and belongs to L^2-space (rather than L^∞). The existence, uniqueness, and variational properties of the weak solution to such boundary value problems usually are drastically different from the corresponding properties of solutions to the elliptic equations with L^∞-matrices in coefficients. In most cases, the situation can change dramatically for the stream matrices with unremovable singularity in matrix A. Typically, in such cases, boundary value problem may admit infinitely many weak solutions which can be divided into two classes: approximable and non-approximable solutions. Following Zhikov [11], a function $y = y(u)$ is called an approximable solution to the above boundary value problem if it can be attained by weak solutions to the similar boundary value problem with L^∞-approximated matrix A. However, this type of solutions do not exhaust the set of all weak solutions to the above problem. There is another type of weak solutions, which cannot be approximated by the weak solutions of regularized problems. Usually, such solutions are called non-variational [11].

The aim of this work is to study the existence of optimal controls to this class of OCPs and propose the scheme of their approximation. We give the sufficient conditions for elements of the stream matrix $A \in L^2(\Omega;\mathbb{S}^N)$ which guarantee that non-variational optimal solutions can be attained through special sequence of optimal solutions to corresponding OCPs in perforated domains with fictitious boundary controls on the holes.

The main technical difficulty, which is related with the study of the asymptotic behaviour of OCPs in perforated domains as $\varepsilon \to 0$, deals with the identification of the limit $\lim_{\varepsilon\to 0}\left\{\left\langle v_\varepsilon^0, y_\varepsilon^0\right\rangle_{H^{-\frac{1}{2}}(\Gamma_\varepsilon);H^{\frac{1}{2}}(\Gamma_\varepsilon)}\right\}_{\varepsilon>0}$ of two weakly convergent sequences, where $\{y_\varepsilon^0\}_{\varepsilon>0}$ is a sequence of optimal states, $\{v_\varepsilon^0\}_{\varepsilon>0}$ is a sequence of fictitious controls on the holes, Γ_ε is a boundary of hole for every $\varepsilon > 0$. Due to the special properties of the skew-symmetric matrix $A \in L^2(\Omega;\mathbb{S}^N)$ (those we call matrices of the funnel type) we show that this limit can be recovered in an explicit form and it is not equal to the product of the corresponding weak limits.

2. Notation and Preliminaries

Let Ω be a bounded open connected subset of $\mathbb{R}^N$ $(N \geq 3)$ with Lipschitz boundary. We assume that Ω contains the origin and its boundary consists of two disjoint parts $\partial\Omega = \Gamma_D \cup \Gamma_N$. Let the sets Γ_D and Γ_N have positive $(N-1)$-dimensional measures. Let χ_E be the characteristic function of a subset $E \subset \Omega$, i.e. $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \notin E$. For any subset $E \subset \Omega$ we denote by $|E|$ its N-dimensional Lebesgue measure $\mathcal{L}^N(E)$.

Let $C_0^\infty(\mathbb{R}^N;\Gamma_D) = \left\{\varphi \in C_0^\infty(\mathbb{R}^N) \;:\; \varphi = 0 \text{ on } \Gamma_D\right\}$. We define the Banach space $H_0^1(\Omega;\Gamma_D)$ as the closure of $C_0^\infty(\mathbb{R}^N;\Gamma_D)$ with respect to the norm $\|y\| = \left(\int_\Omega \|\nabla y\|^2_{\mathbb{R}^N}\,dx\right)^{1/2}$. Let $H^{-1}(\Omega;\Gamma_D)$ be the dual space to $H_0^1(\Omega;\Gamma_D)$.

For any vector field $v \in L^2(\Omega;\mathbb{R}^N)$, the divergence of v can be defined as an element $\operatorname{div} v$ of the space $H^{-1}(\Omega)$ if v and $\operatorname{div} v$ are related by the formula [5]

$$\langle \operatorname{div} v, \varphi\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} = -\int_\Omega (v, \nabla\varphi)_{\mathbb{R}^N}\,dx, \quad \forall\,\varphi \in C_0^\infty(\Omega). \tag{2.1}$$

Here, $H^{-1}(\Omega)$ is the dual space to the classical Sobolev space $H_0^1(\Omega)$.

We denote by $\mathbb{S}^N$ the set of all skew-symmetric matrices $C = [c_{ij}]_{i,j=1}^N$, i.e., C is a square matrix whose transpose is also its negative. Thus, if $C \in \mathbb{S}^N$ then $c_{ij} = -c_{ji}$ and, hence, $c_{ii} = 0$. The set $\mathbb{S}^N$ can be identified with the Euclidean space $\mathbb{R}^{\frac{N(N-1)}{2}}$. Let $L^2(\Omega)^{\frac{N(N-1)}{2}} = L^2(\Omega;\mathbb{S}^N)$ be the space of measurable square-integrable functions whose values are skew-symmetric matrices.

Let

$$(\mathrm{OCP}_\varepsilon): \qquad \min\left\{I_\varepsilon(u,y) \;:\; (u,y) \in \Xi_\varepsilon\right\}, \tag{2.2}$$

be a parameterized OCP, where ε is a small parameter, $I_\varepsilon : \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \to \overline{\mathbb{R}}$ is a cost functional, $\mathbb{Y}_\varepsilon$ is a space of states, $\mathbb{U}_\varepsilon$ is a space of controls, and

$$\Xi_\varepsilon \subset \{(u_\varepsilon, y_\varepsilon) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon \ : \ u \in U_\varepsilon, \ I_\varepsilon(u,y) < +\infty\}$$

is a set of all admissible pairs linked by some state equation. Hereinafter, we associate to every OCP (2.2) the corresponding constrained minimization problem:

$$(\mathrm{CMP}_\varepsilon) \ : \qquad \left\langle \inf_{(u,y)\in\,\Xi_\varepsilon} I_\varepsilon(u,y) \right\rangle. \tag{2.3}$$

Since the sequence of constrained minimization problems (2.3) lives in variable spaces $\mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon$, we assume that there exists a Banach space $\mathbb{U} \times \mathbb{Y}$ with respect to which a τ-convergence in the scale $\{\mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ is defined.

In order to study the asymptotic behavior of a family of $(\mathrm{CMP}_\varepsilon)$, the passage to the limit in (2.3) as the small parameter ε tends to zero has to be realized. The expression "passing to the limit" means that we have to find a kind of "limit cost functional" I and "limit set of constraints" Ξ with clearly defined structure such that the limit object $\left\langle \inf_{(u,y)\in\Xi} I(u,y) \right\rangle$ could be interpreted as some OCP.

Following the scheme of the direct variational convergence [7], we adopt the following definition for the convergence of minimization problems in variable spaces.

Definition 2.1. A problem $\left\langle \inf_{(u,y)\in\Xi} I(u,y) \right\rangle$ is a variational limit of sequence (2.3) as $\varepsilon \to 0$

$$\left(\text{in symbols,} \quad \left\langle \inf_{(u,y)\in\,\Xi_\varepsilon} I_\varepsilon(u,y) \right\rangle \xrightarrow[\varepsilon\to 0]{\mathrm{Var}} \left\langle \inf_{(u,y)\in\Xi} I(u,y) \right\rangle \right)$$

if and only if the following conditions are satisfied:

(d) The space $\mathbb{U} \times \mathbb{Y}$ possesses the weak τ-approximation property with respect to the scale of spaces $\{\mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$, that is, for every $\delta > 0$ and every pair $(u,y) \in \mathbb{U} \times \mathbb{Y}$, there exist a pair $(u^*, y^*) \in \mathbb{U} \times \mathbb{Y}$ and a sequence $\{(u_\varepsilon, y_\varepsilon) \in \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon\}_{\varepsilon>0}$ such that

$$\|u - u^*\|_{\mathbb{U}} + \|y - y^*\|_{\mathbb{Y}} \le \delta \quad \text{and} \quad (u_\varepsilon, y_\varepsilon) \xrightarrow{\ \tau\ } (u^*, y^*) \quad \text{in} \ \ \mathbb{U}_\varepsilon \times \mathbb{Y}_\varepsilon. \tag{2.4}$$

(dd) If sequences $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and $\{(u_k, y_k)\}_{k\in\mathbb{N}}$ are such that $\varepsilon_k \to 0$ as $k \to \infty$, $(u_k, y_k) \in \Xi_{\varepsilon_k}$ $\forall\, k \in \mathbb{N}$, and $(u_k, y_k) \xrightarrow{\ \tau\ } (u,y)$ in $\mathbb{U}_{\varepsilon_k} \times \mathbb{Y}_{\varepsilon_k}$, then

$$(u,y) \in \Xi; \quad I(u,y) \le \liminf_{k\to\infty} I_{\varepsilon_k}(u_k, y_k). \tag{2.5}$$

(ddd) For every $(u,y) \in \Xi \subset \mathbb{U} \times \mathbb{Y}$ and any $\delta > 0$, there are a constant $\varepsilon^0 > 0$ and a sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ (called a (Γ, δ)-realizing sequence) such that

$$(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon, \ \forall\, \varepsilon \le \varepsilon^0, \quad (u_\varepsilon, y_\varepsilon) \xrightarrow{\ \tau\ } (\widehat{u}, \widehat{y}) \ \text{ in } \ \mathbb{Y}_\varepsilon, \tag{2.6}$$

$$\|u - \widehat{u}\|_{\mathbb{U}} + \|y - \widehat{y}\|_{\mathbb{Y}} \le \delta, \tag{2.7}$$

$$I(u,y) \ge \limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon, y_\varepsilon) - \widehat{C}\delta, \tag{2.8}$$

with some constant $\widehat{C} > 0$ independent of δ.

Then the following result takes place [7].

Theorem 2.1. *Assume that the constrained minimization problem*

$$\Big\langle \inf_{(u,y)\in\Xi_0} I_0(u,y)\Big\rangle \tag{2.9}$$

is the variational limit of sequence (2.3) in the sense of Definition 2.1 and this problem has a unique solution $(u^0,y^0)\in\Xi_0$. For every $\varepsilon>0$, let $(u^0_\varepsilon,y^0_\varepsilon)\in\Xi_\varepsilon$ be a minimizer of I_ε on the corresponding set Ξ_ε. If the sequence $\{(u^0_\varepsilon,y^0_\varepsilon)\}_{\varepsilon>0}$ is relatively compact with respect to the τ-convergence in $\mathbb{U}_\varepsilon\times\mathbb{Y}_\varepsilon$, then

$$(u^0_\varepsilon,y^0_\varepsilon) \xrightarrow{\ \tau\ } (u^0,y^0) \quad in \quad \mathbb{U}_\varepsilon\times\mathbb{Y}_\varepsilon, \tag{2.10}$$

$$\inf_{(u,y)\in\,\Xi_0} I_0(u,y) = I_0\left(u^0,y^0\right) = \lim_{\varepsilon\to 0} I_\varepsilon(u^0_\varepsilon,y^0_\varepsilon) = \lim_{\varepsilon\to 0}\inf_{(u_\varepsilon,y_\varepsilon)\in\,\Xi_\varepsilon} I_\varepsilon(u_\varepsilon,y_\varepsilon). \tag{2.11}$$

3. Setting of the Optimal Control Problem

Let $f\in H^{-1}(\Omega;\Gamma_D)$ be a given distribution and let $A\in L^2\big(\Omega;\mathbb{S}^N\big)$ be a given matrix. The optimal control problem we consider in this paper is to minimize the discrepancy between a given distribution $y_d\in H^1_0(\Omega,\Gamma_D)$ and a solution y of the boundary value problem for the stationary diffusion equation in turbulent flow (see [2])

$$-\mathrm{div}\,\big(\nabla y + A(x)\nabla y\big) = f \quad \text{in} \quad \Omega, \tag{3.1}$$

$$y=0 \text{ on } \Gamma_D, \qquad \partial y/\partial\nu_A = u \text{ on } \Gamma_N. \tag{3.2}$$

by choosing an appropriate boundary control $u\in L^2(\Gamma_N)$. Here,

$$\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{N}\Big(\delta_{ij}+a_{ij}(x)\Big)\frac{\partial y}{\partial x_j}\cos(\nu,x_i),$$

δ_{ij} is the Kronecker's delta, $\cos(n,x_i)$ is the i-th directing cosine of ν, and ν is the outward unit normal vector at Γ_N to Ω.

More precisely, we are concerned with the following OCP

$$\text{Minimize } I(u,y) = \|y-y_d\|^2_{H^1_0(\Omega;\Gamma_D)} + \|u\|^2_{L^2(\Gamma_N)} \tag{3.3}$$

$$\text{subject to the constraints (3.1)–(3.2) and } u\in L^2(\Gamma_N). \tag{3.4}$$

The characteristic feature of this problem is the fact that the matrix A is only measurable and belongs to the space $L^2\big(\Omega;\mathbb{S}^N\big)$ (rather than the space of bounded matrices $L^\infty\big(\Omega;\mathbb{S}^N\big)$). In order to make a precise meaning of the OCP setting, we begin with the following concept.

Definition 3.1. We say that a function $y = y(A, f, u)$ is a weak solution to boundary value problem (3.1)–(3.2) for a fixed control $u \in L^2(\Gamma_N)$ and given distributions $f \in H^{-1}(\Omega; \Gamma_D)$ and $A \in L^2\big(\Omega; \mathbb{S}^N\big)$ if $y \in H_0^1(\Omega; \Gamma_D)$ and the integral identity

$$\int_\Omega \big(\nabla\varphi, \nabla y + A(x)\nabla y\big)_{\mathbb{R}^N}\, dx = \langle f, \varphi\rangle_{H^{-1}(\Omega;\Gamma_D);H_0^1(\Omega;\Gamma_D)} + \int_{\Gamma_N} u\varphi \, d\mathcal{H}^{N-1} \quad (3.5)$$

holds for any $\varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)$.

Remark 3.1. Note that by Hölder inequality this definition makes sense for any matrix $A \in L^2\big(\Omega; \mathbb{S}^N\big)$. At the same time, the matrix $I+A(x)$ defines an unbounded bilinear form on $L^2(\Omega; \mathbb{R}^N)$. This motivates an introduction of the following set.

Definition 3.2. We say that an element $y \in H_0^1(\Omega; \Gamma_D)$ belongs to the set D if

$$\left|\int_\Omega \big(\nabla\varphi, A(x)\nabla y\big)_{\mathbb{R}^N}\, dx\right| \le c(y) \left(\int_\Omega |\nabla\varphi|^2_{\mathbb{R}^N}\, dx\right)^{1/2} \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D) \quad (3.6)$$

with some constant $c(y)$ depending on y.

Note that having set

$$[y, \varphi] = \int_\Omega \big(\nabla\varphi, A(x)\nabla y\big)_{\mathbb{R}^N}\, dx, \quad \forall\, y \in D,\ \forall\, \varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D),$$

we can define the bilinear form $[y, \varphi]$ for all $\varphi \in H_0^1(\Omega; \Gamma_D)$ using the rule

$$[y, \varphi] = \lim_{\varepsilon \to 0} [y, \varphi_\varepsilon], \quad (3.7)$$

where $\{\varphi_\varepsilon\}_{\varepsilon>0} \subset C_0^\infty(\mathbb{R}^N; \Gamma_D)$ and $\varphi_\varepsilon \to \varphi$ strongly in $H_0^1(\Omega; \Gamma_D)$. In this case the value $[v, v]$ is finite for every $v \in D$, although the "integrand" $\big(\nabla v(x), A(x)\nabla v(x)\big)_{\mathbb{R}^N}$ can be not integrable in general.

Proposition 3.1. Let $u \in L^2(\Gamma_N)$ be a given control. If $y \in H_0^1(\Omega; \Gamma_D)$ is a weak solution to the boundary value problem (3.1)–(3.2) in the sense of Definition 3.1, then $y \in D$.

Proof. In order to verify this assertion it is enough to rewrite integral identity (3.5) in the form

$$[y, \varphi] = -\int_\Omega \big(\nabla y, \nabla\varphi\big)_{\mathbb{R}^N}\, dx + \langle f, \varphi\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} + \langle u, \varphi\rangle_{H^{-\frac{1}{2}}(\Gamma_N);H^{\frac{1}{2}}(\Gamma_N)} \quad (3.8)$$

and apply the Hölder inequality, the Trace Sobolev Theorem (see [9]), and the compactness of the embeddings $H^{\frac{1}{2}}(\Gamma_N) \hookrightarrow L^2(\Gamma_N) \hookrightarrow H^{-\frac{1}{2}}(\Gamma_N)$ to the right-hand side of (3.8). As a result, we have

$$\Big|[y, \varphi]\Big| \le \Big(\|y\|_{H_0^1(\Omega;\Gamma_D)} + \|f\|_{H^{-1}(\Omega;\Gamma_D)} + C(\Omega)\|u\|_{L^2(\Gamma_N)}\Big) \|\varphi\|_{H_0^1(\Omega;\Gamma_D)}.$$

□

Remark 3.2. Due to Proposition 3.1, Definition 3.1 can be reformulated as follows: y is a weak solution to problem (3.1)–(3.2) if and only if $y \in D$ and

$$\int_\Omega \left(\nabla y, \nabla \varphi\right)_{\mathbb{R}^N} dx + [y,\varphi] \\ = \langle f, \varphi\rangle_{H^{-1}(\Omega;\Gamma_D);H_0^1(\Omega;\Gamma_D)} + \int_{\Gamma_N} u\varphi \, d\mathcal{H}^{N-1} \quad \forall\, \varphi \in H_0^1(\Omega;\Gamma_D). \quad (3.9)$$

Moreover, as immediately follows from (3.9), every weak solution $y \in D$ to the problem (3.1)–(3.2) satisfies the energy equality

$$\|y\|^2_{H_0^1(\Omega;\Gamma_D)} + [y,y] = \langle f, y\rangle_{H^{-1}(\Omega;\Gamma_D);H_0^1(\Omega;\Gamma_D)} + \int_{\Gamma_N} uy\, d\mathcal{H}^{N-1}. \quad (3.10)$$

It is well know that boundary value problem (3.1)–(3.2) is ill-posed, in general. It means that there exists a matrix $A \in L^2\big(\Omega;\mathbb{S}^N\big)$ such that for every admissible control $u \in L^2(\Gamma_N)$ the corresponding state $y \in H_0^1(\Omega;\Gamma_D)$ may be not unique. It is clear that in this case, it is not possible to write $y = y(u)$. To avoid this situation, we adopt the following notion.

Definition 3.3. We say that (u, y) is an admissible pair to OCP (3.3)–(3.4) if $u \in L^2(\Gamma_N)$, $y \in D \subset H_0^1(\Omega;\Gamma_D)$, and (u, y) are related by integral identity (3.9). We denote by Ξ the set of all admissible pairs for OCP (3.3)–(3.4).

We say that a pair $(u^0, y^0) \in L^2(\Gamma_N)\times D$ is optimal for problem (3.3)–(3.4) if

$$(u^0, y^0) \in \Xi \quad \text{and} \quad I(u^0,y^0) = \inf_{(u,y)\in\,\Xi} I(u,y).$$

As follows from the definition of the bilinear form $[y,\varphi]$, the value $[y,y]$ is not constant-sign for all $y \in D$. Hence, energy equality (3.10) does not allow us to derive any a priory estimate in H_0^1-norm for the weak solutions. In spite of this, the following result indicates that OCP (3.3)–(3.4) is well-posed (see [8]).

Theorem 3.1. *Assume that OCP* (3.3)–(3.4) *is regular, i.e.* $\Xi \neq \emptyset$. *Then this problem has a unique solution for each* $f \in H^{-1}(\Omega;\Gamma_D)$, $A \in L^2\big(\Omega;\mathbb{S}^N\big)$, *and* $y_d \in H_0^1(\Omega;\Gamma_D)$.

Proof. Since the original problem is regular and the cost functional is bounded below on Ξ, it follows that there exists a minimizing sequence $\{(u_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi$ for OCP (3.3)–(3.4); that is,

$$I(u_k,y_k) \xrightarrow[k\to\infty]{} I_{\min} \equiv \inf_{(u,y)\in\,\Xi} I(u,y) \geq 0.$$

Hence, $\sup_{k\in\mathbb{N}} I(u_k,y_k) \leq C$, where the constant C is independent of k. Since

$$\sup_{k\in\mathbb{N}} \left[\|y_k\|^2_{H_0^1(\Omega;\Gamma_D)} + \|u_k\|^2_{L^2(\Gamma_N)}\right] \leq 2\|y_d\|^2_{H_0^1(\Omega;\Gamma_D)} + 2\sup_{k\in\mathbb{N}} I(u_k,y_k) \leq 2(C_1+C),$$

it follows that passing to a subsequence if necessary, we may assume that

$$u_k \rightharpoonup u^0 \ \text{ in } \ L^2(\Gamma_N), \quad y_k \rightharpoonup y^0 \ \text{ in } \ H_0^1(\Omega;\Gamma_D), \quad I(u^0,y^0) < +\infty.$$

Using the fact that

$$[y_k,\varphi] = \int_\Omega \left(\nabla\varphi, A(x)\nabla y_k\right)_{\mathbb{R}^N} dx = -\int_\Omega \left(A\nabla\varphi, \nabla y_k\right)_{\mathbb{R}^N} dx = -[\varphi, y_k]$$

and $A\varphi \in L^2(\Omega;\mathbb{R}^N)$ for any $\varphi \in C_0^\infty(\Omega;\Gamma_D)$, we can pass to the limit in integral identity (3.5) with $u = u_k$ and $y = y_k$ as $k \to \infty$. As a result, we obtain: the pair (u^0, y^0) is related by identity (3.5). Hence, $y^0 \in D$ by Proposition 3.1. Thus, (u^0, y^0) is an admissible pair to problem (3.3)–(3.4). Using the property of lower semicontinuity for I with respect to the product of the weak topologies for $L^2(\Gamma_N)$ and $H^1(\Omega)$, we get

$$0 \le I(u^0,y^0) \le \lim_{k\to\infty} I(u_k,y_k) = I_{\min}.$$

Thus, the pair (u^0, y^0) is optimal for problem (3.3)–(3.4). The uniqueness of the optimal pair is the direct consequence of the strict convexity of the cost functional I on $L^2(\Gamma_N) \times H_0^1(\Omega;\Gamma_D)$. The proof is complete. □

4. On attainability of non-variational solutions to OCP (3.3)–(3.4)

We begin this section with some auxiliary results and notions. Let ε be a small parameter. Assume that the parameter ε varies within a strictly decreasing sequence of positive real numbers which converge to 0. When we write $\varepsilon > 0$, we consider only the elements of this sequence.

Let for every $\varepsilon > 0$, $T_\varepsilon : \mathbb{R} \to \mathbb{R}$ be the truncation function defined by

$$T_\varepsilon(s) = \max\left\{\min\left\{s, \varepsilon^{-1}\right\}, -\varepsilon^{-1}\right\}. \tag{4.1}$$

The following property of T_ε is well-known (see [6]). Let $f \in L^2(\Omega)$ be an arbitrary function. Then we have:

$$T_\varepsilon(f) \in L^\infty(\Omega)\ \forall\, \varepsilon > 0 \quad \text{and} \quad T_\varepsilon(f) \to f \text{ strongly in } \ L^2(\Omega). \tag{4.2}$$

Let $A \in L^2\left(\Omega;\mathbb{S}^N\right)$ be a stream matrix. For a given sequence $\{\varepsilon > 0\}$ we define the cut-off operators $\mathbf{T}_\varepsilon : \mathbb{S}^N \to \mathbb{S}^N$ as follows $\mathbf{T}_\varepsilon(A) = [T_\varepsilon(a_{ij})]_{i,j=1}^N$ for every $\varepsilon > 0$. We associate with such operators the following set of subdomains $\left\{\Omega_\varepsilon\right\}_{\varepsilon>0}$ of Ω

$$\Omega_\varepsilon = \Omega \setminus Q_\varepsilon, \quad \forall\, \varepsilon > 0, \tag{4.3}$$

where

$$Q_\varepsilon = \text{closure}\left\{x \in \Omega \ : \ \|A(x)\|_{\mathbb{S}^N} := \max_{i,j=1,\dots,N} |a_{ij}(x)| \ge \varepsilon^{-1}\right\}. \tag{4.4}$$

Definition 4.1. We say that a stream matrix $A \in L^2(\Omega; \mathbb{S}^N)$ is of the funnel-type if there exists a strictly decreasing sequence of positive real numbers $\{\varepsilon\}$ converging to 0 such that the corresponding collection of sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$, defined by (4.3), possesses the following properties:

(i) Ω_ε are open connected subsets of Ω with Lipschitz boundaries for which there exists a positive value $\delta > 0$ such that

$$\partial\Omega \subset \partial\Omega_\varepsilon \quad \text{and} \quad \operatorname{dist}(\Gamma_\varepsilon, \partial\Omega) > \delta, \quad \forall\, \varepsilon > 0,$$

where $\Gamma_\varepsilon = \partial\Omega_\varepsilon \setminus \partial\Omega$.

(ii) The surface measure of the boundaries of holes $Q_\varepsilon = \Omega \setminus \Omega_\varepsilon$ is small enough in the following sence:

$$\mathcal{H}^{N-1}(\Gamma_\varepsilon) = o(\varepsilon) \quad \forall\, \varepsilon > 0. \tag{4.5}$$

(iii) For any element $h \in D$ there is a constant $c(h)$ depending on h and independent of ε such that

$$\left| \int_{\Omega\setminus\Omega_\varepsilon} \left(\nabla\varphi, A(x)\nabla h\right)_{\mathbb{R}^N} dx \right| \le c(h)\sqrt{\frac{|\Omega\setminus\Omega_\varepsilon|}{\varepsilon}} \left(\int_{\Omega\setminus\Omega_\varepsilon} |\nabla\varphi|^2_{\mathbb{R}^N}\, dx \right)^{1/2} \tag{4.6}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)$.

Thus, if A is of the funnel-type then each of the sets Ω_ε is locally located on one side of its Lipschitz boundary $\partial\Omega_\varepsilon$. Moreover, in this case the boundary $\partial\Omega_\varepsilon$ can be divided into three parts $\partial\Omega_\varepsilon = \Gamma_D \cup \Gamma_N \cup \Gamma_\varepsilon$. Observe also that condition (ii) of Definition 4.1 excludes the appearance of self-similar domains Ω_ε with some fractal behavior of the boundaries. Moreover, if $A \in L^\infty(\Omega; \mathbb{S}^N)$ then estimate (4.6) is obvious.

Remark 4.1. As immediately follows from Definition 4.1, the sequence $\{\Omega_\varepsilon\}_{\varepsilon>0}$ is monotonically expanding, i.e., $\Omega_{\varepsilon_k} \subset \Omega_{\varepsilon_{k+1}}$ for all $\varepsilon_k > \varepsilon_{k+1}$, and perimeters of Q_ε tend to zero as $\varepsilon \to 0$. Moreover, because of the initial supposition, we have

$$\frac{1}{\varepsilon^2}\,|\Omega\setminus\Omega_\varepsilon| \le \int_{\Omega\setminus\Omega_\varepsilon} \|A(x)\|^2_{\mathbb{S}^N}\, dx, \ \forall\, \varepsilon > 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \|A\|_{L^2(\Omega\setminus\Omega_\varepsilon;\mathbb{S}^N)} = 0.$$

This entails the property: $|\Omega\setminus\Omega_\varepsilon| = o(\varepsilon^2)$ and, hence, $\lim_{\varepsilon\to 0}|\Omega_\varepsilon| = |\Omega|$. Besides, in view of assumption (ii) of Definition 4.1, which plays an important role in our further analysis, we have the following estimate

$$\frac{\varepsilon\mathcal{H}^{N-1}(\Gamma_\varepsilon)}{|\Omega\setminus\Omega_\varepsilon|} = O(1). \tag{4.7}$$

It should be also stressed that as obvious consequence of condition (4.5), we have the following one: the support of singularities of the funnel type matrices, i.e., the limit of the boundaries Γ_ε in the Hausdorff topology, may have non-zero capacity in general [4].

Remark 4.2. As follows from [3], the funnel-type property implies the so-called strong connectedness of the sets $\{\Omega_\varepsilon\}_{\varepsilon>0}$ which means the existence of extension operators P_ε from $H^1_0(\Omega_\varepsilon;\Gamma_D)$ to $H^1_0(\Omega;\Gamma_D)$ such that for some positive constant C independent of ε,

$$\|\nabla(P_\varepsilon y)\|_{L^2(\Omega;\mathbb{R}^N)} \le C\,\|\nabla y\|_{L^2(\Omega_\varepsilon;\mathbb{R}^N)}, \quad \forall\, y\in H^1_0(\Omega_\varepsilon;\Gamma_D). \tag{4.8}$$

As a direct consequence of Definition 4.1, we have the following result.

Proposition 4.1. Assume that $A\in L^2\big(\Omega;\mathbb{S}^N\big)$ is of the funnel-type. Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a sequence of perforated subdomains of Ω associated with matrix A and let $\{\chi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ be the sequence of their characteristic functions. Then

$$\chi_{\Omega_\varepsilon}\to\chi_\Omega \quad \text{strongly in} \quad L^2(\Omega). \tag{4.9}$$

Proof. As immediately follows from Definition 4.1, the sequence $\{\chi_{\Omega_\varepsilon}\}_{\varepsilon>0}$ is monotonically increasing, i.e., $\chi_{\Omega_{\varepsilon_k}}\le\chi_{\Omega_{\varepsilon_{k+1}}}$ almost everywhere in Ω provided $\varepsilon_k>\varepsilon_{k+1}$. Taking into account the following equality for the cut-off operators

$$\|\mathbf{T}_\varepsilon(A(x))\|_{\mathbb{S}^N}=\chi_{\Omega_\varepsilon}(x)\|A(x)\|_{\mathbb{S}^N}+(1-\chi_{\Omega_\varepsilon}(x))\varepsilon^{-1}, \quad \forall\,\varepsilon>0.$$

and condition $(4.2)_2$, we may suppose, within a subsequence, that

$$\Big(\chi_{\Omega_\varepsilon}(x)\|A(x)\|_{\mathbb{S}^N}+(1-\chi_{\Omega_\varepsilon}(x))\varepsilon^{-1}\Big)\to\|A(x)\|_{\mathbb{S}^N} \text{ a.e. in } \Omega \text{ as } \varepsilon\to0,$$
$$\text{and}\quad |\Omega\setminus\Omega_\varepsilon|:=\mathcal{L}^N(\Omega\setminus\Omega_\varepsilon)\overset{\text{by Remark 4.1}}{=}o(\varepsilon^2)\to0 \quad\text{as}\quad \varepsilon\to0.$$

Hence, in view of the monotonicity property of $\{\chi_{\Omega_\varepsilon}\}_{\varepsilon>0}$, we finally obtain (see [7])

$$\chi_{\Omega_\varepsilon}\to\chi_\Omega \quad \text{a.e. in}\ \ \Omega, \quad \text{and, hence,}\ \ \chi_{\Omega_\varepsilon}\to\chi_\Omega \text{ strongly in } \ L^1(\Omega).$$

Since the strong convergence of characteristic functions in $L^1(\Omega)$ implies their strong convergence in $L^2(\Omega)$, this concludes the proof. □

Definition 4.2. We say that a sequence $\big\{y_\varepsilon\in H^1_0(\Omega_\varepsilon;\Gamma_D)\big\}_{\varepsilon>0}$ is weakly convergent in variable space $H^1_0(\Omega_\varepsilon;\Gamma_D)$ if there exists an element $y\in H^1(\Omega;\Gamma_D)$ such that

$$\lim_{\varepsilon\to0}\int_{\Omega_\varepsilon}(\nabla y_\varepsilon,\nabla\varphi)_{\mathbb{R}^N}\,dx=\int_\Omega(\nabla y,\nabla\varphi)_{\mathbb{R}^N}\,dx, \quad \forall\,\varphi\in C^\infty_0(\Omega;\Gamma_D)$$

Remark 4.3. Since

$$\int_\Omega(\nabla y,\nabla\varphi)_{\mathbb{R}^N}\,dx=\lim_{\varepsilon\to0}\int_{\Omega_\varepsilon}(\nabla y_\varepsilon,\nabla\varphi)_{\mathbb{R}^N}\,dx=\lim_{\varepsilon\to0}\int_\Omega(\nabla(P_\varepsilon y_\varepsilon),\nabla\varphi)_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx$$
$$\overset{\text{by (4.9) and (4.8)}}{=}\int_\Omega(\nabla y^*,\nabla\varphi)_{\mathbb{R}^N}\,dx, \quad \forall\,\varphi\in C^\infty_0(\Omega;\Gamma_D),$$

where $y^*\in H^1_0(\Omega;\Gamma_D)$ is a weak limit in $H^1_0(\Omega;\Gamma_D)$ of the extended functions $\big\{P_\varepsilon y_\varepsilon\in H^1_0(\Omega;\Gamma_D)\big\}_{\varepsilon>0}$, it follows that the weak limit in the sense of Definition 4.2 does not depend on a choice of extension operators $P_\varepsilon: H^1_0(\Omega_\varepsilon;\Gamma_D)\to H^1_0(\Omega;\Gamma_D)$ with properties (4.8).

Let us consider the following sequence of regularized OCPs associated with domains Ω_ε

$$\left\{ \left\langle \inf_{(u,v,y)\in\Xi_\varepsilon} I_\varepsilon(u,v,y) \right\rangle, \quad \varepsilon \to 0 \right\}, \tag{4.10}$$

where

$$I_\varepsilon(u,v,y) := \|y - y_d\|^2_{H^1_0(\Omega_\varepsilon;\Gamma_D)} + \|u\|^2_{L^2(\Gamma_N)} + \frac{1}{\varepsilon^\alpha}\|v\|^2_{H^{-\frac{1}{2}}(\Gamma_\varepsilon)}, \tag{4.11}$$

$$\Xi_\varepsilon = \left\{ (u,v,y) \;\middle|\; \begin{array}{c} -\mathrm{div}\left(\nabla y + A\nabla y\right) = f \quad \text{in} \quad \Omega_\varepsilon, \\ y = 0 \text{ on } \Gamma_D, \qquad \partial y/\partial \nu_A = u \text{ on } \Gamma_N, \\ \partial y/\partial \nu_A = v \text{ on } \Gamma_\varepsilon, \\ u \in L^2(\Gamma_N),\ v \in H^{-\frac{1}{2}}(\Gamma_\varepsilon),\ y \in H^1_0(\Omega_\varepsilon;\Gamma_D). \end{array} \right\} \tag{4.12}$$

Here, $y_d \in H^1_0(\Omega,\Gamma_D)$ and $f \in L^2(\Omega)$ are given functions, ν is the outward normal unit vector at Γ_N and Γ_ε to Ω_ε, $v \in H^{-\frac{1}{2}}(\Gamma_\varepsilon)$ is considered as a fictitious control, and α is a positive number such that

$$\varepsilon^{-\alpha}\mathcal{H}^{N-1}(\Gamma_\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad (\text{see } (4.5)). \tag{4.13}$$

Using the fact that $A \in L^\infty(\Omega_\varepsilon;\mathbb{S}^N)$ for every $\varepsilon > 0$, we arrive at the following obvious result.

Theorem 4.1. *For every $\varepsilon > 0$ there exists a unique minimizer $(u^0_\varepsilon, v^0_\varepsilon, y^0_\varepsilon) \in \Xi_\varepsilon$ to the problem $\left\langle \inf_{(u,v,y)\in\Xi_\varepsilon} I_\varepsilon(u,v,y)\right\rangle$.*

In order to study the asymptotic behavior of the sequences of admissible solutions $\left\{(u_\varepsilon, v_\varepsilon, y_\varepsilon) \in L^2(\Gamma_N) \times H^{-\frac{1}{2}}(\Gamma_\varepsilon) \times H^1_0(\Omega_\varepsilon;\Gamma_D)\right\}_{\varepsilon>0}$ in the scale of variable spaces, we adopt the following concept.

Definition 4.3. We say that a sequence $\{(u_\varepsilon, v_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ weakly converges to $(u,y) \in L^2(\Gamma_N) \times H^1_0(\Omega;\Gamma_D)$ in the scale of spaces

$$\left\{L^2(\Gamma_N) \times H^{-\frac{1}{2}}(\Gamma_\varepsilon) \times H^1_0(\Omega_\varepsilon;\Gamma_D)\right\}_{\varepsilon>0}, \tag{4.14}$$

if

$$u_\varepsilon \rightharpoonup u \ \text{ in } \ L^2(\Gamma_N), \quad y_\varepsilon \rightharpoonup y \ \text{ in } \ H^1_0(\Omega_\varepsilon;\Gamma_D), \tag{4.15}$$

$$\text{and} \quad \sup_{\varepsilon>0} \frac{1}{\mathcal{H}^{N-1}(\Gamma_\varepsilon)} \|v_\varepsilon\|^2_{H^{-\frac{1}{2}}(\Gamma_\varepsilon)} < +\infty. \tag{4.16}$$

Now we are in a position to state the main result of this section.

Theorem 4.2. *Let $A \in L^2\big(\Omega;\mathbb{S}^N\big)$ be a funnel-type stream matrix such that*

$$\textit{the equality} \quad [y,y] = 0 \quad \textit{does not hold in} \ \ D. \tag{4.17}$$

Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *be a sequence of perforated subdomains of* Ω *associated with matrix* A. *Then problem* $\big\langle \inf_{(u,y)\in\Xi} I(u,y)\big\rangle$, *where* $y_d \in H_0^1(\Omega,\Gamma_D)$ *and* $f \in L^2(\Omega)$ *are given functions, is a variational limit of sequence* (4.10)–(4.12) *as the parameter* ε *tends to zero.*

Remark 4.4. Condition (4.17) implies an existence of at least one element $h^* \in D$ such that $h^* \notin L^\infty(\Omega)$ and h^* is a solution to the homogeneous problem

$$\begin{aligned} -\mathrm{div}\,\big(\nabla y + A\nabla y\big) = 0 \quad \text{in} \quad \Omega, \\ y = 0 \text{ on } \Gamma_D, \qquad \partial y/\partial\nu_A = 0 \text{ on } \Gamma_N. \end{aligned} \tag{4.18}$$

It means that the linear form

$$[h^*,\varphi] = \int_\Omega \big(\nabla\varphi, A(x)\nabla h^*\big)_{\mathbb{R}^N}\,dx, \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^N;\Gamma_D)$$

can have a non-trivial extension onto the entire set D using the rule

$$[h^*,\varphi] = \lim_{\varepsilon\to 0}[h^*,\varphi_\varepsilon], \quad \forall\,\varphi \in D \subset H_0^1(\Omega;\Gamma_D), \tag{4.19}$$

where $\{\varphi_\varepsilon\}_{\varepsilon>0} \subset C_0^\infty(\mathbb{R}^N;\Gamma_D)$ and $\varphi_\varepsilon \to \varphi$ strongly in $H_0^1(\Omega;\Gamma_D)$.

Let L be the following subspace of $H_0^1(\Omega;\Gamma_D)$

$$L = \left\{ h \in D \;:\; \int_\Omega \big(\nabla\varphi, \nabla h + A(x)\nabla h\big)_{\mathbb{R}^N}\,dx = 0 \;\forall\,\varphi \in C_0^\infty(\mathbb{R}^N;\Gamma_D) \right\}, \tag{4.20}$$

i.e., L is the set of all weak solutions of homogeneous problem (4.18).

Proof. Since each of the optimization problems $\big\langle \inf_{(u,v,y)\in\Xi_\varepsilon} I_\varepsilon(u,v,y)\big\rangle$ lives in the corresponding space $L^2(\Gamma_N)\times H^{-\frac{1}{2}}(\Gamma_\varepsilon)\times H_0^1(\Omega_\varepsilon;\Gamma_D)$, we have to show that in this case all conditions of Definition 2.1 hold true. To do so, we divide this proof into three steps.

Step 1. We show that the space $L^2(\Gamma_N)\times H_0^1(\Omega;\Gamma_D)$ possesses the weak approximation property with respect to the weak convergence in scale of spaces (4.14). Indeed, let $\delta = 0$ and let $(u,y) \in L^2(\Gamma_N)\times H_0^1(\Omega;\Gamma_D)$ be an arbitrary pair. Let $h \in C_0^\infty(\Omega;\Gamma_D)$ be such that div $(\nabla h + A(x)\nabla h) \in L^2(\Omega)$. We construct the sequence

$$\Big\{ (u_\varepsilon, v_\varepsilon, y_\varepsilon) \in L^2(\Gamma_N)\times H^{-\frac{1}{2}}(\Gamma_\varepsilon)\times H_0^1(\Omega_\varepsilon;\Gamma_D) \Big\}_{\varepsilon>0}$$

as follows

$$u_\varepsilon = u, \quad v_\varepsilon = \left.\frac{\partial h}{\partial\nu_A}\right|_{\Gamma_\varepsilon}, \quad \text{and} \quad y_\varepsilon = y, \quad \forall\,\varepsilon > 0.$$

In view of (4.9), we have $\chi_{\Omega_\varepsilon} \stackrel{*}{\rightharpoonup} \chi_\Omega$ in $L^\infty(\Omega)$. Hence,

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (\nabla\varphi, \nabla y_\varepsilon)_{\mathbb{R}^N}\,dx &= \lim_{\varepsilon\to 0}\int_\Omega (\nabla\varphi,\nabla y)_{\mathbb{R}^N}\,\chi_{\Omega_\varepsilon}\,dx \\ &= \int_\Omega (\nabla\varphi,\nabla y)_{\mathbb{R}^N}\,dx \quad \forall\,\varphi \in C_0^\infty(\Omega;\Gamma_D), \end{aligned}$$

i.e., $y_\varepsilon \rightharpoonup y$ in $H^1_0(\Omega_\varepsilon;\Gamma_D)$ as $\varepsilon \to 0$.

It remains to show that the sequence $\left\{v_\varepsilon \in H^{-\frac{1}{2}}(\Gamma_\varepsilon)\right\}_{\varepsilon>0}$ is bounded in the sense of Definition 4.3. To do so we make use of the Green formula. As a result, we obtain

$$\left|\left\langle \frac{\partial h}{\partial \nu_A}, \varphi\right\rangle_{H^{-\frac{1}{2}}(\Gamma_\varepsilon);H^{\frac{1}{2}}(\Gamma_\varepsilon)}\right| \le \left|\int_{Q_\varepsilon} \operatorname{div}\,(\nabla h + A(x)\nabla h)\,\varphi\,dx\right| + \left|\int_{Q_\varepsilon} (\nabla\varphi, \nabla h + A(x)\nabla h)_{\mathbb{R}^N}\,dx\right|$$

$$\le \left(\int_{Q_\varepsilon} |\operatorname{div}\,(\nabla h + A(x)\nabla h)\,|^2\,dx\right)^{1/2} \|\varphi\|_{L^2(Q_\varepsilon)} + \|\nabla h\|_{L^2(Q_\varepsilon;\mathbb{R}^N)}\|\nabla\varphi\|_{L^2(Q_\varepsilon;\mathbb{R}^N)}$$

$$\overset{\text{by (4.6)}}{+} c(h)\sqrt{\frac{|\Omega\setminus\Omega_\varepsilon|}{\varepsilon}}\left(\int_{\Omega\setminus\Omega_\varepsilon} |\nabla\varphi|^2_{\mathbb{R}^N}\,dx\right)^{1/2} \le (I_1 + I_2 + I_3)\,\|\varphi\|_{H^1(\Omega\setminus\Omega_\varepsilon)}.$$

Since $|\Omega\setminus\Omega_\varepsilon| = o(\varepsilon^2)$ by the funnel type properties of A, it follows that there exists a suitable change of variables and a constant $C > 0$ independent of ε such that

$$I_2 = \|\nabla h\|_{L^2(Q_\varepsilon;\mathbb{R}^N)} = \left(C\frac{|\Omega\setminus\Omega_\varepsilon|}{\varepsilon}\int_{\Omega\setminus\Omega_1} \|\nabla h(y)\|^2_{\mathbb{R}^N}\,dy\right)^{1/2} \overset{\text{by (4.7)}}{\le} C_1\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}\|h\|_{H^1(\Omega;\Gamma_D)}$$

Following the similar arguments, we get:

$$I_1 = \left\|\operatorname{div}\,(\nabla h + A(x)\nabla h)\,\right\|_{L^2(Q_\varepsilon)} \le C_2(h)\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}.$$

As a result, summing up the previous inequalities, we come to the following conclusion: there exists a constant $C = C(h)$ independent of ε such that

$$\frac{1}{\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}}\left\langle \frac{\partial h}{\partial \nu_A}, \varphi\right\rangle_{H^{-\frac{1}{2}}(\Gamma_\varepsilon);H^{\frac{1}{2}}(\Gamma_\varepsilon)} \le C(h)\|\varphi\|_{H^1(\Omega\setminus\Omega_\varepsilon)} \quad \forall\,\varphi \in C_0^\infty(\Omega;\Gamma_D).$$

Hence,

$$\sup_{\varepsilon>0}\left(\frac{1}{\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}}\left\|\frac{\partial h}{\partial \nu_A}\right\|_{H^{-\frac{1}{2}}(\Gamma_\varepsilon)}\right) \le C. \tag{4.21}$$

Thus, the week approximation property is proved.

Remark 4.5. Note that the funnel-type properties of the stream matrix A together with Sobolev Trace Theorem [1] imply the following estimate

$$\|\varphi\|_{L^2(\Gamma_\varepsilon)} \le \frac{C}{\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}}\|\varphi\|_{H^1_0(\Omega_\varepsilon;\Gamma_D)}, \quad \forall\varphi \in C_0^\infty(\Omega;\Gamma_D) \tag{4.22}$$

with some constant C independent of ε.

Step 2.We show on this step that condition (ddd) of Definition 2.1 holds true with $\delta = 0$. Let $(u^*, y^*) \in \Xi$ be any admissible pair to original OCP (3.3)–(3.4). Note that the set L, defined in (4.20), is not a singleton in this case. Indeed, if L is a singleton in D then $h \equiv 0$ is its unique element and we come into conflict with (4.17). So, we suppose that the set L contains at least one non-trivial element of $D \subset H^1_0(\Omega; \Gamma_D)$. Then, obviously, L is a linear subspace of $H^1_0(\Omega; \Gamma_D)$. Let $h \in D$ be any element of the set L such that h is a non-trivial solution of homogeneous problem (4.18). We construct a $(\Gamma, 0)$-realizing sequence $\{(u_\varepsilon, v_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ as follows:

$$u_\varepsilon := u^* \quad \text{and} \quad v_\varepsilon := w_\varepsilon + \frac{\partial h}{\partial \nu_A} \quad \forall\, \varepsilon > 0, \tag{4.23}$$

where functions w_ε are such that

$$\sup_{\varepsilon>0} \left(\frac{1}{\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}} \, \|w_\varepsilon\|_{H^{-\frac{1}{2}}(\Gamma_\varepsilon)} \right) \le C \tag{4.24}$$

with some C independent of ε.

Let $\left\{y_\varepsilon \in H^1_0(\Omega_\varepsilon; \Gamma_D)\right\}_{\varepsilon>0}$ be a sequence of weak solutions of boundary value problems (4.12) under the corresponding controls $u = u_\varepsilon$ and $v = v_\varepsilon$. In view of the Lax-Milgram lemma and the superposition principle this sequence is defined in a unique way and for every $\varepsilon > 0$ we have the following decomposition $y_\varepsilon = y_{\varepsilon,1} + y_{\varepsilon,2}$, where $y_{\varepsilon,1}$ and $y_{\varepsilon,2}$ are elements of $H^1_0(\Omega_\varepsilon; \Gamma_D)$ such that

$$\begin{aligned} \int_\Omega \left(\nabla\varphi, \nabla y_{\varepsilon,1} + A(x)\nabla y_{\varepsilon,1}\right)_{\mathbb{R}^N} \chi_{\Omega_\varepsilon}\, dx &= \int_\Omega f \chi_{\Omega_\varepsilon} \varphi\, dx + \int_{\Gamma_N} u_\varepsilon \varphi\, d\mathcal{H}^{N-1} \\ &+ \langle w_\varepsilon, \varphi\rangle_{H^{-\frac{1}{2}}(\Gamma_\varepsilon); H^{\frac{1}{2}}(\Gamma_\varepsilon)}, \quad \forall\, \varphi \in C^\infty_0(\Omega; \Gamma_D), \end{aligned} \tag{4.25}$$

$$\begin{aligned} \int_\Omega \left(\nabla\varphi, \nabla y_{\varepsilon,2} + A(x)\nabla y_{\varepsilon,2}\right)_{\mathbb{R}^N} \chi_{\Omega_\varepsilon}\, dx& \\ = \left\langle \frac{\partial h}{\partial \nu_A}, \varphi \right\rangle_{H^{-\frac{1}{2}}(\Gamma_\varepsilon); H^{\frac{1}{2}}(\Gamma_\varepsilon)}&, \ \forall\, \varphi \in C^\infty_0(\Omega; \Gamma_D). \end{aligned} \tag{4.26}$$

Remark 4.6. Hereinafter, we suppose that the functions y_ε of $H^1_0(\Omega_\varepsilon, \Gamma_D)$ are extended by operators P_ε outside of Ω_ε and, therefore, considered as defined in the whole of Ω.

Since $A(x) = \mathbf{T}_\varepsilon(A(x))$ whenever $x \in \Omega_\varepsilon$ for every $\varepsilon > 0$, it means that $A \in L^\infty(\Omega_\varepsilon; \mathbb{S}^N)$. Hence, by the skew-symmetry property of A, we have

$$\int_\Omega \left(\nabla y_{\varepsilon,i}, A(x)\nabla y_{\varepsilon,i}\right)_{\mathbb{R}^N} \chi_{\Omega_\varepsilon}\, dx = 0, \quad i = 1, 2.$$

Then (4.25)–(4.26) lead us to the energy equalities

$$\int_\Omega \|\nabla y_{\varepsilon,1}\|^2_{\mathbb{R}^N} \chi_{\Omega_\varepsilon}\, dx = \int_\Omega f \chi_{\Omega_\varepsilon} y_{\varepsilon,1}\, dx + \int_{\Gamma_N} u_\varepsilon y_{\varepsilon,1}\, d\mathcal{H}^{N-1} + \langle w_\varepsilon, y_{\varepsilon,1}\rangle_{H^{-\frac12}(\Gamma_\varepsilon);H^{\frac12}(\Gamma_\varepsilon)}, \tag{4.27}$$

$$\int_\Omega \|\nabla y_{\varepsilon,2}\|^2_{\mathbb{R}^N} \chi_{\Omega_\varepsilon}\, dx = \left\langle \frac{\partial h}{\partial \nu_A}, y_{\varepsilon,2}\right\rangle_{H^{-\frac12}(\Gamma_\varepsilon);H^{\frac12}(\Gamma_\varepsilon)}. \tag{4.28}$$

Since $h \in L$, condition (iii) of Definition 4.1 implies that

$$\begin{aligned}\left|\left\langle \frac{\partial h}{\partial \nu_A}, \varphi\right\rangle_{H^{-\frac12}(\Gamma_\varepsilon);H^{\frac12}(\Gamma_\varepsilon)}\right| &= \left|\int_{\Omega\setminus\Omega_\varepsilon} \big(\nabla\varphi, \nabla h + A(x)\nabla h\big)_{\mathbb{R}^N}\, dx\right| \\ &\le \sqrt{\frac{|\Omega\setminus\Omega_\varepsilon|}{\varepsilon}}\,(C_1(h)+C_2(h))\,\|\varphi\|_{H^1(\Omega\setminus\Omega_\varepsilon)} \\ &\overset{\text{by (4.5)}}{\le} C(h)\sqrt{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}\|\varphi\|_{H^1(\Omega\setminus\Omega_\varepsilon)}, \quad \forall\,\varphi\in H^1_0(\Omega;\Gamma_D)\end{aligned}$$

with some constant $C(h)$ independent of ε. Hence,

$$\sup_{\varepsilon>0}\big(\mathcal{H}^{N-1}(\Gamma_\varepsilon)\big)^{-1}\left\|\frac{\partial h}{\partial\nu_A}\right\|^2_{H^{-\frac12}(\Gamma_\varepsilon)} < +\infty. \tag{4.29}$$

Thus,

$$\begin{aligned}\left|\langle w_\varepsilon, y_{\varepsilon,1}\rangle_{H^{-\frac12}(\Gamma_\varepsilon);H^{\frac12}(\Gamma_\varepsilon)}\right| &\le C\,\|y_{\varepsilon,1}\|_{L^2(\Gamma_\varepsilon)}\big(\mathcal{H}^{N-1}(\Gamma_\varepsilon)\big)^{\frac12} \\ &\overset{\text{by (4.22)}}{\le} C_1\,\|y_{\varepsilon,1}\|_{H^1_0(\Omega_\varepsilon;\Gamma_D)},\end{aligned} \tag{4.30}$$

$$\begin{aligned}\left|\left\langle \frac{\partial h}{\partial\nu_A}, y_{\varepsilon,2}\right\rangle_{H^{-\frac12}(\Gamma_\varepsilon);H^{\frac12}(\Gamma_\varepsilon)}\right| &\le C\,\|y_{\varepsilon,2}\|_{L^2(\Gamma_\varepsilon)}\big(\mathcal{H}^{N-1}(\Gamma_\varepsilon)\big)^{\frac12} \\ &\overset{\text{by (4.22)}}{\le} C_1\|y_{\varepsilon,2}\|_{H^1_0(\Omega_\varepsilon;\Gamma_D)}.\end{aligned} \tag{4.31}$$

As a result, we come to the a priori estimates

$$\left(\int_\Omega \|\nabla y_{\varepsilon,1}\|^2_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\, dx\right)^{1/2} \le \|f\|_{L^2(\Omega)} + C\left(\|u_\varepsilon\|_{L^2(\Gamma_N)} + 1\right), \tag{4.32}$$

$$\left(\int_\Omega \|\nabla y_{\varepsilon,2}\|^2_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\, dx\right)^{1/2} \le C. \tag{4.33}$$

Hence, the sequences

$$\big\{y_{\varepsilon,1}\in H^1_0(\Omega_\varepsilon;\Gamma_D)\big\}_{\varepsilon>0} \quad\text{and}\quad \big\{y_{\varepsilon,2}\in H^1_0(\Omega_\varepsilon;\Gamma_D)\big\}_{\varepsilon>0}$$

are weakly compact with respect to the weak convergence in variable spaces [10], i.e., we may assume that there exists a couple of functions $\widehat{y}_1$ and $\widehat{y}_2$ in $H_0^1(\Omega;\Gamma_D)$ such that

$$\lim_{\varepsilon\to 0}\int_\Omega \left(\nabla\varphi,\nabla y_{\varepsilon,i}\right)_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx=\int_\Omega \left(\nabla\varphi,\nabla\widehat{y}_i\right)_{\mathbb{R}^N},dx,\ \forall\,\varphi\in C_0^\infty(\Omega;\Gamma_D) \tag{4.34}$$

for $i=1,2$.

Now we can pass to the limit in integral identities (4.25)–(4.26) as $\varepsilon\to 0$. Using $(4.23)_1$, (4.34), (4.29), and L^2-property of $A(x)$, we finally obtain

$$\int_\Omega \left(\nabla\varphi,\nabla\widehat{y}_1+A(x)\nabla\widehat{y}_1\right)_{\mathbb{R}^N}dx=\int_\Omega f\varphi\,dx+\int_{\Gamma_N}u^*\varphi\,d\mathcal{H}^{N-1}, \tag{4.35}$$

$$\int_\Omega \left(\nabla\varphi,\nabla\widehat{y}_2+A(x)\nabla\widehat{y}_2\right)_{\mathbb{R}^N}dx=0 \tag{4.36}$$

for every $\varphi\in C_0^\infty(\Omega;\Gamma_D)$. Hence, $\widehat{y}_1$ and $\widehat{y}_2$ are weak solutions to the boundary value problem (3.1)–(3.2) and (4.18), respectively. Hence, $\widehat{y}_2\in L$ and $\widehat{y}_1\in D$ by Proposition 3.1. As a result, we arrive at the conclusion: the pair $(u^*,\widehat{y}_1+h)$ belongs to the set Ξ, for some $h\in L$. Since by the initial assumptions $(u^*,y^*)\in\Xi$, it follows that having put in (4.23)

$$h=y^*-\widehat{y}_1, \tag{4.37}$$

we obtain

$$h\in L\ \text{ and }\ y_\varepsilon=y_{\varepsilon,1}+y_{\varepsilon,2}\rightharpoonup y^*\quad\text{in }\ H_0^1(\Omega_\varepsilon;\Gamma_D)\ \text{ as }\ \varepsilon\to 0. \tag{4.38}$$

Therefore, properties (2.6)–(2.7) hold true.

It remains to prove inequality (2.8). To do so, it is enough to show that

$$\begin{aligned}I(u^*,y^*)&:=\|y^*-y_d\|^2_{H_0^1(\Omega;\Gamma_D)}+\|u^*\|^2_{L^2(\Gamma_N)}=\lim_{\varepsilon\to 0}I_\varepsilon(u_\varepsilon,v_\varepsilon,y_\varepsilon)\\&=\lim_{\varepsilon\to 0}\left[\|y_\varepsilon-y_d\|^2_{H_0^1(\Omega_\varepsilon;\Gamma_D)}+\|u_\varepsilon\|^2_{L^2(\Gamma_N)}+\frac{1}{\varepsilon^\alpha}\|v_\varepsilon\|^2_{H^{-\frac12}(\Gamma_\varepsilon)}\right],\end{aligned} \tag{4.39}$$

where the sequence $\left\{(u_\varepsilon,v_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon\right\}_{\varepsilon>0}$ is defined by (4.23) and (4.37).

In view of this, we make use the following relations

$$\left.\begin{array}{c}\|v_\varepsilon\|^2_{H^{-\frac12}(\Gamma_\varepsilon)}\le 2\|w_\varepsilon\|^2_{H^{-\frac12}(\Gamma_\varepsilon)}+2\left\|\dfrac{\partial h}{\partial\nu_A}\right\|^2_{H^{-\frac12}(\Gamma_\varepsilon)}<+\infty,\\ \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^\alpha}\|w_\varepsilon\|^2_{H^{-\frac12}(\Gamma_\varepsilon)}\overset{\text{by (4.24)}}{\le}C\lim_{\varepsilon\to 0}\frac{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}{\varepsilon^\alpha}=0,\\ \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^\alpha}\|\frac{\partial h}{\partial\nu_A}\|^2_{H^{-\frac12}(\Gamma_\varepsilon)}\overset{\text{by (4.29)}}{\le}C\lim_{\varepsilon\to 0}\frac{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}{\varepsilon^\alpha}=0,\\ \|u_\varepsilon\|^2_{L^2(\Gamma_N)}=\|u^*\|^2_{L^2(\Gamma_N)}\quad\forall\,\varepsilon>0,\\ \lim_{\varepsilon\to 0}\displaystyle\int_\Omega\left(\nabla y_d,\nabla y_\varepsilon\right)_{\mathbb{R}^N}\chi_{\Omega_\varepsilon}\,dx\overset{\text{by (4.38)}}{=}\int_\Omega\left(\nabla y_d,\nabla y^*\right)_{\mathbb{R}^N}dx.\end{array}\right\} \tag{4.40}$$

In order to obtain the convergence

$$\limsup_{\varepsilon\to 0} \|y_\varepsilon\|^2_{H^1_0(\Omega_\varepsilon;\Gamma_D)} = \|y^*\|^2_{H^1_0(\Omega;\Gamma_D)} \tag{4.41}$$

we make use of the following energy equality which comes from the condition $(u^*, y^*) \in \Xi$

$$\begin{aligned} \|y^*\|^2_{H^1_0(\Omega;\Gamma_D)} &:= \int_\Omega \left\|\nabla y^*\right\|^2_{\mathbb{R}^N} dx \\ &= -[y^*, y^*] + \int_\Omega f y^*\, dx + \int_{\Gamma_N} u^* y^*\, d\mathcal{H}^{N-1}. \end{aligned} \tag{4.42}$$

As for the integral identity for the triplet $(u_\varepsilon, v_\varepsilon, y_\varepsilon)$, we use the following trick.

It is easy to see that the integral identity for the weak solutions y_ε to boundary value problems (4.12) can be represented in the so-called extended form

$$\begin{aligned} \int_\Omega \big(\nabla\varphi, \nabla y_\varepsilon + A(x)\nabla y_\varepsilon\big)_{\mathbb{R}^N} \chi_{\Omega_\varepsilon}\, dx &= \int_\Omega f\chi_{\Omega_\varepsilon}\varphi\, dx + \int_{\Gamma_N} u_\varepsilon \varphi\, d\mathcal{H}^{N-1} \\ &+ \langle w_\varepsilon, \varphi\rangle_{H^{-\frac12}(\Gamma_\varepsilon);H^{\frac12}(\Gamma_\varepsilon)} + \left\langle \frac{\partial h}{\partial \nu_A}, \varphi\right\rangle_{H^{-\frac12}(\Gamma_\varepsilon);H^{\frac12}(\Gamma_\varepsilon)} \\ &- \int_\Omega \big(\nabla\psi, \nabla h^*\big)_{\mathbb{R}^N} dx - [h^*, \psi], \quad \forall\, \varphi, \psi \in C^\infty_0(\Omega;\Gamma_D), \end{aligned} \tag{4.43}$$

where h^* is an arbitrary element of L. Indeed, because of the equality

$$\int_\Omega \big(\nabla\psi, \nabla h^*\big)_{\mathbb{R}^N} dx + [h^*, \psi] \overset{\text{by (4.20)}}{=} 0, \quad \forall\, \psi \in C^\infty_0(\Omega;\Gamma_D),$$

we have an equivalent identity to the classical definition of the weak solutions of boundary value problem (4.12).

As follows from property $(4.38)_2$ and Sobolev Trace Theorem, the sequences

$$\left\{\langle w_\varepsilon, y_\varepsilon\rangle_{H^{-\frac12}(\Gamma_\varepsilon);H^{\frac12}(\Gamma_\varepsilon)}\right\}_{\varepsilon>0} \quad \text{and} \quad \left\{\left\langle \frac{\partial h}{\partial \nu_A}, y_\varepsilon\right\rangle_{H^{-\frac12}(\Gamma_\varepsilon);H^{\frac12}(\Gamma_\varepsilon)}\right\}_{\varepsilon>0}$$

are bounded. Therefore, we can assume, passing to a subsequence if necessary, that there exists a value $\xi_1 \in \mathbb{R}$ such that

$$\langle w_\varepsilon, y_\varepsilon\rangle_{H^{-\frac12}(\Gamma_\varepsilon);H^{\frac12}(\Gamma_\varepsilon)} + \left\langle \frac{\partial h}{\partial \nu_A}, y_\varepsilon\right\rangle_{H^{-\frac12}(\Gamma_\varepsilon);H^{\frac12}(\Gamma_\varepsilon)} \longrightarrow \xi_1 \quad \text{as} \quad \varepsilon \to 0. \tag{4.44}$$

Since $y_\varepsilon \rightharpoonup y^*$ weakly in $H^1_0(\Omega_\varepsilon;\Gamma_D)$ and $y^* \in D$, it follows that there exists a sequence of smooth functions $\{\psi_\varepsilon \in C^\infty_0(\Omega;\Gamma_D)\}_{\varepsilon>0}$ such that $\psi_\varepsilon \to y^*$ strongly in $H^1_0(\Omega_\varepsilon;\Gamma_D)$. Therefore, following extension rule (4.19), we have

$$\lim_{\varepsilon\to 0} \int_\Omega \big(\nabla\psi_\varepsilon, \nabla h^*\big)_{\mathbb{R}^N} dx = \int_\Omega \big(\nabla y^*, \nabla h^*\big)_{\mathbb{R}^N} dx, \quad \lim_{\varepsilon\to 0} [h^*, \psi_\varepsilon] = [h^*, y^*]. \tag{4.45}$$

Because of initial supposition (4.17) (see Remark 4.4), we can assume that the element $h^* \in L$ is such that

$$[h^*, y^*] + \int_\Omega \left(\nabla y^*, \nabla h^*\right)_{\mathbb{R}^N} dx \neq 0.$$

Otherwise, we come into conflict with (4.17). So, due to this observation, we specify the choice of element $h^* \in L$ as follows

$$\widehat{h}^* = \frac{\xi_1 + [y^*, y^*]}{\xi_2 + \xi_3}\, h^*, \quad \forall\, \varepsilon > 0,$$

where

$$\xi_3 := \int_\Omega \left(\nabla y^*, \nabla h^*\right)_{\mathbb{R}^N} dx,\ \ \xi_2 := [h^*, y^*],$$

or, in other words, we aim to ensure the condition

$$\xi_1 - \xi_2 - \xi_3 + [y^*, y^*] = 0.$$

As a result, we have: $\widehat{h}^*$ is an element of L such that

$$\left.\begin{aligned} \lim_{\varepsilon\to 0} \int_\Omega \left(\nabla \psi_\varepsilon, \nabla \widehat{h}^*\right)_{\mathbb{R}^N} dx = \xi_2 \frac{\xi_1 + [y^*, y^*]}{\xi_2 + \xi_3}, \\ \lim_{\varepsilon\to 0} [\widehat{h}^*, \psi_\varepsilon] = \xi_3 \frac{\xi_1 + [y^*, y^*]}{\xi_2 + \xi_3}. \end{aligned}\right\} \tag{4.46}$$

Having put $\varphi_\varepsilon = y_\varepsilon$ and $h^* = \widehat{h}^*$ in (4.43) and used the skew-symmetry property $\int_\Omega \left(\nabla y_\varepsilon, A(x)\nabla y_\varepsilon\right)_{\mathbb{R}^N} \chi_{\Omega_\varepsilon}\, dx = 0$, we arrive at the following energy equality for boundary value problem (4.12)

$$\begin{aligned} \int_\Omega \left(\nabla y_\varepsilon, \nabla y_\varepsilon\right)_{\mathbb{R}^N} \chi_{\Omega_\varepsilon}\, dx &= \int_\Omega f \chi_{\Omega_\varepsilon} y_\varepsilon\, dx + \int_{\Gamma_N} u_\varepsilon y_\varepsilon\, d\mathcal{H}^{N-1} \\ &\quad + \left\langle w_\varepsilon, y_\varepsilon \right\rangle_{H^{-\frac{1}{2}}(\Gamma_\varepsilon); H^{\frac{1}{2}}(\Gamma_\varepsilon)} + \left\langle \frac{\partial h}{\partial \nu_A}, y_\varepsilon \right\rangle_{H^{-\frac{1}{2}}(\Gamma_\varepsilon); H^{\frac{1}{2}}(\Gamma_\varepsilon)} \\ &\quad - \int_\Omega \left(\nabla \psi_\varepsilon, \nabla \widehat{h}^*\right)_{\mathbb{R}^N} dx - [\widehat{h}^*, \psi_\varepsilon]. \end{aligned} \tag{4.47}$$

Further we note that

$$\lim_{\varepsilon\to 0} \int_{\Gamma_N} u_\varepsilon y_\varepsilon\, d\mathcal{H}^{N-1} = \lim_{\varepsilon\to 0} \int_{\Gamma_N} u^* y_\varepsilon\, d\mathcal{H}^{N-1} \overset{\text{by (4.38)}}{=} \int_{\Gamma_N} u^* y^*\, d\mathcal{H}^{N-1}. \tag{4.48}$$

As a result, making use of properties (4.9), (4.38), (4.46), (4.48), we can pass to the limit as $\varepsilon \to 0$ in (4.47). This yields

$$\begin{aligned}
\lim_{\varepsilon\to 0} \|y_\varepsilon\|^2_{H^1_0(\Omega_\varepsilon;\Gamma_D)} &= \lim_{\varepsilon\to 0} \int_\Omega f\chi_{\Omega_\varepsilon} y_\varepsilon\, dx + \lim_{\varepsilon\to 0} \int_{\Gamma_N} u_\varepsilon y_\varepsilon \, d\mathcal{H}^{N-1} \\
&\quad + \lim_{\varepsilon\to 0} \langle w_\varepsilon, y_\varepsilon\rangle_{H^{-\frac{1}{2}}(\Gamma_\varepsilon);H^{\frac{1}{2}}(\Gamma_\varepsilon)} + \lim_{\varepsilon\to 0} \left\langle \frac{\partial h}{\partial \nu_A}, y_\varepsilon \right\rangle_{H^{-\frac{1}{2}}(\Gamma_\varepsilon);H^{\frac{1}{2}}(\Gamma_\varepsilon)} \\
&\quad - \lim_{\varepsilon\to 0} \int_\Omega \left(\nabla\psi_\varepsilon, \nabla\widehat{h}^*\right)_{\mathbb{R}^N} dx - \lim_{\varepsilon\to 0} [\widehat{h}^*, \psi_\varepsilon] \\
&\overset{\text{by } (4.46)}{=} \int_\Omega f y^*\, dx + \int_{\Gamma_N} u^* y^* \, d\mathcal{H}^{N-1} - [y^*, y^*] \overset{\text{by } (4.42)}{=} \|y^*\|^2_{H^1_0(\Omega;\Gamma_D)}. \quad (4.49)
\end{aligned}$$

Hence, turning back to (4.39), we see that this relation is a direct consequence of (4.40) and (4.49). Thus, the sequence $\{(u_\varepsilon, v_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$, which is defined by (4.23) and (4.37), is $(\Gamma, 0)$-realizing. The property (ddd) is established.

Step 3. We prove the property (dd) of Definition 2.1. Let $\{(u_k, v_k, y_k)\}_{k\in\mathbb{N}}$ be a sequence such that $(u_k, v_k, y_k) \in \Xi_{\varepsilon_k}$ for some $\varepsilon \to 0$ as $k \to \infty$,

$$u_k \rightharpoonup u^* \text{ in } L^2(\Gamma_N), \quad y_k \rightharpoonup y^* \text{ in } H^1_0(\Omega_{\varepsilon_k};\Gamma_D),$$

and the sequence of fictitious controls $\left\{v_k \in H^{-\frac{1}{2}}(\Gamma_{\varepsilon_k})\right\}_{k\in\mathbb{N}}$ satisfies inequality (4.16). Our aim is to show that

$$(u^*, y^*) \in \Xi; \quad I(u^*, y^*) \le \liminf_{k\to\infty} I_{\varepsilon_k}(u_k, v_k, y_k). \quad (4.50)$$

Since the integral identity

$$\begin{aligned}
\int_\Omega \left(\nabla\varphi, \nabla y_k + A(x)\nabla y_k\right)_{\mathbb{R}^N} \chi_{\Omega_{\varepsilon_k}}\, dx &= \int_\Omega f\chi_{\Omega_{\varepsilon_k}}\varphi\, dx \\
&+ \int_{\Gamma_N} u_k\varphi\, d\mathcal{H}^{N-1} + \langle v_k, \varphi\rangle_{H^{-\frac{1}{2}}(\Gamma_{\varepsilon_k});H^{\frac{1}{2}}(\Gamma_{\varepsilon_k})}, \quad \forall\, \varphi \in C^\infty_0(\Omega;\Gamma_D) \quad (4.51)
\end{aligned}$$

holds true for every $k \in \mathbb{N}$, we can pass to the limit in (4.51) as $k \to \infty$ using Definition 4.3 and estimate

$$\left| \langle v_\varepsilon, \varphi\rangle_{H^{-\frac{1}{2}}(\Gamma_\varepsilon);H^{\frac{1}{2}}(\Gamma_\varepsilon)} \right| \le C(\Omega)\, \|\varphi\|_{H^1_0(\Omega;\Gamma_D)} \left(\mathcal{H}^{N-1}(\Gamma_\varepsilon)\right)^{\frac{1}{2}}, \quad \forall\, \varphi \in C^\infty_0(\Omega;\Gamma_D)$$

coming from inequality (4.16). Then proceeding as on the Step 2, it can easily be shown that the limit pair (u^*, y^*) is admissible to OCP (3.3)–(3.4). Hence, condition $(4.50)_1$ is valid. As for the inequality $(4.50)_2$, it immediately follows from the lower semicontinuity of the norms $\|\cdot\|_{H^1(\Omega;\Gamma_D)}$ and $\|\cdot\|_{L^2(\Gamma_N)}$ with respect to the weak convergence and the following estimate

$$\frac{1}{{\varepsilon_k}^\alpha} \|v_k\|^2_{H^{-\frac{1}{2}}(\Gamma_{\varepsilon_k})} \le C \frac{\mathcal{H}^{N-1}(\Gamma_{\varepsilon_k})}{{\varepsilon_k}^\alpha} \to 0 \quad \text{as} \quad k \to \infty.$$

The proof is complete. □

Taking into account this result and the main properties of variational convergence of constrained minimization problems (see Theorem 2.1), we arrive at the following variational properties of OCPs (4.10)–(4.12).

Theorem 4.3. *Let $A \in L^2\big(\Omega; \mathbb{S}^N\big)$ be a funnel-type stream matrix such that*

$$\text{the equality} \quad [y,y]=0 \quad \text{does not hold in } D. \tag{4.52}$$

Let $y_d \in H_0^1(\Omega, \Gamma_D)$ and $f \in L^2(\Omega)$ be given functions. Let $\big\{(u_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon\big\}_{\varepsilon>0}$ be a sequence of optimal solutions to regularized problems (4.10)–(4.12). Then a unique optimal solution to OCP (3.3)–(3.4) is attainable in the following sense

$$u_\varepsilon^0 \rightharpoonup u^0 \ \text{ in } \ L^2(\Gamma_N), \quad y_\varepsilon^0 \rightharpoonup y^0 \ \text{ in } \ H_0^1(\Omega_\varepsilon; \Gamma_D), \tag{4.53}$$

$$\inf_{(u,y)\in\Xi} I(u,y) = I\left(u^0, y^0\right) = \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^0, v_\varepsilon^0, y_\varepsilon^0). \tag{4.54}$$

Proof. In order to show that this result is a direct consequence of Theorem 2.1, we have to establish the compactness property for the sequence of optimal solutions $\big\{(u_\varepsilon^0, v_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon\big\}_{\varepsilon>0}$ in the sense of Definition 4.3.

Let $u^* \in L^2(\Gamma_N)$ and $h \in C_0^\infty(\Omega; \Gamma_D)$ be non-trivial functions. We assume that

$$\operatorname{div}\,(\nabla h + A(x)\nabla h) \in L^2(\Omega).$$

We set

$$u_\varepsilon = u^* \quad \text{and} \quad v_\varepsilon = \left.\frac{\partial h}{\partial \nu_A}\right|_{\Gamma_\varepsilon} \in H^{-\frac{1}{2}}(\Gamma_\varepsilon).$$

In view of the initial suppositions and estimate (4.21), there is a constant $C > 0$ independent of ε such that

$$\|u_\varepsilon\|^2_{L^2(\Gamma_N)} \le C, \quad \left\|\frac{\partial h}{\partial \nu_A}\right\|^2_{H^{-\frac{1}{2}}(\Gamma_\varepsilon)} \le C\mathcal{H}^{N-1}(\Gamma_\varepsilon).$$

Let $y_\varepsilon = y_\varepsilon(u_\varepsilon, v_\varepsilon) \in H_0^1(\Omega_\varepsilon; \Gamma_D)$ be a corresponding solution to boundary value problem (4.12). Then following (4.32), we come to the estimate $\|y_\varepsilon\|^2_{H_0^1(\Omega_\varepsilon;\Gamma_D)} \le \widetilde{C}$, where this constant is also independent of ε. As a result, we obtain

$$\begin{aligned} I_\varepsilon(u_\varepsilon^0, v_\varepsilon^0, y_\varepsilon^0) &= \left\|y_\varepsilon^0 - y_d\right\|^2_{H_0^1(\Omega_\varepsilon;\Gamma_D)} + \|u_\varepsilon^0\|^2_{L^2(\Gamma_N)} + \frac{1}{\varepsilon^\alpha}\|v_\varepsilon^0\|^2_{H^{-\frac{1}{2}}(\Gamma_\varepsilon)} \\ &\le I_\varepsilon(u_\varepsilon, v_\varepsilon, y_\varepsilon) \le 2\widetilde{C} + 2\|y_d\|^2_{H_0^1(\Omega;\Gamma_D)} + C + C\frac{\mathcal{H}^{N-1}(\Gamma_\varepsilon)}{\varepsilon^\alpha}. \end{aligned}$$

Since $\varepsilon^{-\alpha}\mathcal{H}^{N-1}(\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$, it follows that the minimal values of cost functional (4.11) bounded above uniformly with respect to ε. Thus, the sequence of optimal solutions $\big\{(u_\varepsilon^0, v_\varepsilon^0, y_\varepsilon^0)\big\}_{\varepsilon>0}$ to problems (4.10)–(4.12) uniformly bounded in

$$L^2(\Gamma_N) \times H^{-\frac{1}{2}}(\Gamma_\varepsilon) \times H_0^1(\Omega_\varepsilon; \Gamma_D)$$

and, hence, it is relatively compact with respect to the weak convergence in the sense of Definition 4.3. For the rest of proof, it remains to apply Theorem 2.1. $\square$

References

1. *R. Adams*, Sobolev spaces, *Academic Press*, New York, 1975.
2. *M. A. Fannjiang, G. C. Papanicolaou*, Diffusion in turbulence, *Probab. Theory and Related Fields*, **105**(1996), 279–334.
3. *D. Cioranescu, P. Donato*, An Introduction to Homogenization, *Oxford University Press*, New York, 1999.
4. *L. C. Evans, R. F. Gariepy*, Measure Theory and Fine Properties of Functions, *CRC Press*, Boca Raton, 1992.
5. *E. Giusti*, Minimal Surfaces and Functions of Bounded Variation, *Birkhäuser*, Boston, 1984.
6. *D. Kinderlehrer, G. Stampacchia*, An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980.
7. *P. I. Kogut, G. Leugering*, Optimal Control Problems for Partial Differential Equations on Reticulated Domains: Approximation and Asymptotic Analysis, Birkhäuser, Boston, 2011.
8. *J.-L. Lions*, Optimal Control of Systems Governed by Partial Differential Equations, *Springer-Verlag*, Berlin 1971.
9. *J.-L. Lions, E. Magenes*, Non-Homogeneous Boundary Value Problems and Applications, *Springer-Verlag*, Berlin, 1972.
10. *V. V. Zhikov*, Weighted Sobolev spaces, *Sbornik: Mathematics*, No. 8, **189**(1998), 27–58.
11. *V. V. Zhikov*, Remarks on the ubiqueness of a solution of the Dirichlet problem for second-order elliptic equations with lower-order terms, *Functional Analysis and Its Applications*, **38**(2004), No. 3, 173-183.

Надійшла до редколегії 25.12.2011