SPOUDAI Journal, Vol. 63 (2013), Issue 1-2, pp. 94-99

University of Piraeus

SPOUDAI
Journal of Economics and Business
Σπουδαί
http://spoudai.unipi.gr

On the Relation between the Equilibrium Set and the Demand Functions

Nickolas J. Michelacakis

University of Piraeus, Department of Economics, 80, M. Karaoli & A. Dimitriou Str, 185 34 Piraeus, Greece, Email: njm@unipi.gr

Abstract

It is shown that, in pure exchange economies, the individual demand functions can be recovered from the equilibrium set regardless of the way we evaluate wealth. Following Balasco (2004), the demand functions do not have to be differentiable, not even continuous nor utility maximizing. Thus, the set of equilibria does not necessarily carry the structure of a manifold. Further, it is proved that the inner product structure is redundant in the sense that the result holds true even if wealth is being evaluated by means of a more general function on the price commodity space satisfying minimal requirements.

JEL Classifications: D50, D51.

Keywords: Equilibrium set (manifold), Demand functions.

1. Introduction

In pure exchange economies, the equilibrium manifold stores the memory of individual preferences regardless of the way we measure wealth. In fact, since no assumption of differentiability not even continuity on the demand functions is being made the set of equilibria does not necessarily carry the structure of a manifold and we just speak of the equilibrium set. Following Samuelson's (1948) revealed preference theory a number of authors focused their research on understanding the relationship between observable market data and unobservable consumer preferences.

In Chiappori *et al.* (2004), the demand functions are required to be analytic and utility maximizing. Balasco (2004) provides an elementary proof that the individual demand functions can be recovered from the equilibrium set. His proof works for demand functions that are not even continuous nor utility maximizing.

In this paper, it is shown that the result holds true regardless of the wealth evaluating function eliminating, therefore, the possibility that the inner product structure interferes with the result. Specifically, the demand functions are not continuous nor utility maximizing and a function, $g : S \times X \to R_+$ on the price-commodity space is introduced to replace the classical inner product with sole requirement that $g(p,0) = 0$.

Section 2 provides all necessary notation and definitions. The aggregate demand function, first and subsequently the individual demand functions are being constructed in Section 3.

2. Definitions and notation

Let m denote the number of consumers sharing l commodities. Individual i's consumption bundle $x^i = (x^i{}_1, ..., x^i{}_l) \in X := R^l{}_+$ the non-negative orthant of R^l, and the total consumption space containing consumption bundles as well as initial endowments is given by

$$R_+^{lm} \supseteq X^m =: \left\{ \left(x_1^1, ..., x_l^1 ..., x_1^m, ..., x_l^m \right) : x_j^i \geq 0, j = 1, ..., l, i = 1, ..., m \right\}. \quad (1)$$

Prices are non-negative and adopting the numeraire convention of taking the priceof the last commodity, $p_l = 1$, they can be identified either by the positive part of the real projective space or by the subset R^l.

$$R^l \supset S := \{ p = (p_1, ..., p_{l-1}, 1) \mid p_i j = 1, ..., l - 1 \}. \quad (2)$$

We define, further, a commodity-bundle valuation function

$$g : S \times X \to R_+ \quad (3)$$

satisfying the property $g(p, x) = 0 \Leftrightarrow x = 0, p \in S$.

Individual i's demand function is given by $f^i : S \times R_+ \to X$, where $f^i(p, w^i)$ express the commodity bundle sought by the consumer given the price p and her wealth $w^i > 0$.

Total demand can then be expressed in terms of a functic $f : S \times R_+^m \to S \times X^m$, where,

$$f(p, w^1, ..., w^m) = (p, f^1(p, w^1), ..., f^m(p, w^m)), \quad (4)$$

while, aggregate demand is given by $F : S \times \mathbb{R}^m_+ \rightarrow X = \mathbb{R}^l_+$ with

$$F(p, w^1, ..., w^m) = f^1(p, w^1) + ... + f^m(p, w^m). \quad (5)$$

We assume throughout that the individual demand functions satisfy the generalized Walras law, i.e. $g(p, f^i(p, w^i)) = w^i$, for all price vectors and $w^i > 0$.

Initial endowments represent the initial wealth of consumers and we define a total evaluation map $\Phi_g : S \times X^m \rightarrow S \times \mathbb{R}^m_+$ by letting

$$\Phi_g(p, x^1, ..., x^m) = (p, g(p, x^1), ..., g(p, x^m)). \quad (6)$$

It will be of use to further define the map $\sigma : S \times X^m \rightarrow \mathbb{R}^l_+$ with

$$\sigma(p, x^1, ..., x^m) = x^1 + ... + x^m. \quad (7)$$

In pure exchange economies with endowments in X an equilibrium is a price-endowment vector $(p, x) \in S \times X^m$ such that equation

$$F(p, g(p, x^1), ..., g(p, x^m)) = x^1 + ... + x^m = \sigma(p, x^1, ..., x^m) \quad (8)$$

be satisfied and the equilibrium set, E, is composed of all such equilibria.

A no-trade equilibrium is by definition an element (p, x) of $S \times X^m$ such that

$$f^i(p, g(p, x^i)) = x^i, \forall i = 1, ..., m. \quad (9)$$

Notice that a no-trade equilibrium trivially satisfies the generalized Walras law. No-trade equilibria are, in fact, equilibria as they satisfy equation 8.

3. Aggregate demand and individual demand functions

The next proposition shows that the aggregate demand function remains invariant on the fibres of φ_g.

Proposition 3.1 *On the equilibrium set E,*

$$F \circ \varphi_g \equiv \sigma. \quad (10)$$

Proof. Let $(p, x^1, ..., x^m)$ and $(p^l, y^1, ..., y^m)$ two equilibria belonging into the same fibre of φ_g, i.e.

$$(p,g(p,x^1), ..., g(p,x^m)) = \varphi_g((p,x^1,...,x^m))$$
$$= \varphi_g(p^l,y^1,...,y^m) = (p^l,g(p^l, y^1), ...g(p^l,y^m)) \quad (11)$$

which implies $p=p'$ and $g(p,x^i)=g(p,y^i)=w^i$ for all $i=1,...,m$. But then, since $(p,x^1,...,x^m)$ and $(p,y^1, ..., y^m)$ are both equilibria, they both satisfy equation 8

$$\sigma(p,x^1,...,x^m) = x^1 + ... + x^m = F(p,g(p,x^1),..., g(p,x^m))$$
$$= F(p^l,g(p,y^1),...,g(p,y^m)) = y^1 + ... + y^m = \sigma(p^l,y^1,...,y^m \quad (12)$$

and the result follows readily.

Because of Proposition 3.1. one can define well an "aggregate" demand function, F, on the fibres of φ_g. Subsequently, a choice for a corresponding vector of m individual demand functions could be created by taking the natural embedding $e^i : S\,x\,\mathrm{R}_+ \to S\,x\,\mathrm{R}_+^m$ followed by F,

$$(p,w) \xrightarrow{e^i} (p,0,...,0,w^i,0,...0,) \xrightarrow{F} F(p,0,...,0,w^i,0,...0), \quad (13)$$

where $w^i=w$. It is part of Theorem 3.4 that this choice works. The following proposition shows that the set of no-trade equilibria is a section of φ_g.

Proposition 3.2. *The set of no-trade equilibria can be identified with the image of* $S\,x\,\mathrm{R}_+^m$ *under f and intersects each fibre of* φ_g at a single element.

Proof. Let $(p,x^1, ..., x^m)$ be a no-trade equilibrium. Consider the element of $S\,x\,\mathrm{R}_+^m$ given by $(p,w^1,...,w^m) := \varphi_g(p,x^1,..x^m)$. Then,

$$f(p,w^1,..., w^m) = (p,f^1(p, w^1), ..., f^m(p,w^m)) = (p,x^1, ..., x^m) \quad (14)$$

because by definition at a no-trade equilibrium, $f^i(p,g(p,x^i)) = x^i$ for all $i=1,...,m$.

Conversely $(p,x^1,...,x^m) = f(p,w^1,..., w^m)$ for some

$(p,w^1,...,w^m) \in S\,x\,\mathrm{R}_+^m$ Then,

$$(p,x^1,...,x^m) = f(p,w^1,...,w^m) = (p,f^1(p,w^1),..., f^m(p,w^m)) \quad (15)$$

i.e $x^i = f^i(p,w^i)$ for all $i=1,...,m$ and $(p,x^1, ..., x^m)$ is a no-trade equilibrium.

For the second part of the proposition the stronger statement

$$f(S\,x\,\mathrm{R}_+^m) \cap \phi_g^{-1}(p,w^1,...,w^m) = \{f(p,w^1,...,w^m)\}. \quad (16)$$

can be proved by observing

$$
\begin{aligned}
\varphi_g(f(p,w^1,\ldots,w^m)) &= \varphi_g(p,f^1(p,w^1),\ldots,f^m(p,w^m)) \\
&= (p,g(p,f^1(p,w^1)),\ldots,g(f^m(p,w^m))) \\
&= (p,w^1,\ldots,w^m) \qquad (17)
\end{aligned}
$$

which shows at the same time that f is one-to-one.

The extra condition $g(p,x)=0 \Leftrightarrow x=0,\ p\in S$ imposed on $g: S\,x\,X \rightarrow_+$ is needed in

Lemma 3.3. *The individual demand is zero on zero wealth, for all price vectors p.*

Proof. We need to prove that for the individual demand functions $f^i : S\,x\,\mathrm{R}_+ \rightarrow X$, for all p and $i=1,\ldots m$. But this is straightforward for $f^i(p,0)$ satisfies the generalized Walras law, i.e. $g(p,f^i(p,0)) = 0$ which implies that $f^i(p,0) = 0$.

Following Proposition 3.2, the vector of m values of the m individual preferences associated with price-income vector $(p,w^1,\ldots,w^m)$ can be identified with the corresponding unique equilibrium in the fibre $\varphi_g(p,w^1,\ldots,w^m)$. It is this property together with Lemma 3.3 that allows us to recover the individual preferences from the equilibrium set via the construction 13.

Theorem 3.4 *$F : S\,x\,(\mathrm{R}_+)^m \rightarrow X$ is the aggregate demand function then*

$$f^i(p,w^i) = F(p,0,\ldots,0,w^i,0,\ldots 0) = (\sigma \circ f \circ e^i)(p,w^i), \qquad (18)$$

where $e^i : S\,x\,\mathrm{R}_+ \rightarrow S\,x\,\mathrm{R}^m_+$ is the natural embedding defined in eq. 13 and f^i : $S\,x\,\mathrm{R}_+ \rightarrow X$ is the consumer's demand function.

Proof. *Propositions 3.2 and 3.1 prove that $F=\sigma \circ f$. Lemma 3.2 assures that*

$$
\begin{aligned}
F(p,0,\ldots,0,w^i,0,\ldots 0) &= f^1(p,0)+\ldots+f^{i-1}(p,0)+f^i(p,w^i) \\
&\quad +f(i+1)(p,0)+\ldots+f^m(p,0)=f^i(p,w^i) \qquad (19)
\end{aligned}
$$

and the proof is complete.

Remark 3.5 It is worth observing that g is not linear not even continuous. If one takes as $g : S\,x\,X \rightarrow \mathrm{R}_+$ the inner product $g(p,x) = p.x$ the results in (Balasco, 2004) are immediately obtained. In particular, no inner product structure is required for the recoverability property to hold.

Appendix

In mathematics, a bundle is a triple $\xi := (E,p,B)$, where $p:E \to B$ is a map from the total space E to the base space B. The inverse image, $p^{-1}(\{b\}) = p^{-1}(b)$, of a singleton $\{b\}$, with $b \in B$, is the fibre of p over b. Intuitively, the total space is composed of the distinct union of the fibres $p\text{-}1(b),\ b \in B$ "glued together" by the topology of E.

A function $f:E \to R$ remains invariant on the fibres of ξ if and only if it is constant on the sets $p^{-1}(b)$ for all $b \in B$.

A subbundle of (E,p,B) is a triple (E',p',B') with $E' \subseteq E, B' \subseteq B$ and $p' = p|: E' \to B'$ the restricton of p to E'.

A section of a bundle $\xi := (E,p,B)$ is a map $s:B \to E$ with the property that $p^{oS} \equiv {}^{id}B$. A complete account on fibre bundles can be found in Husemoller (1994).

References

Balasco, Y., 2004. The equilibrium manifold keeps the memory of individual demand functions, Economic Theory 24, 493–501.

Chiappori, P. A., Ekeland, I., Kübler, H., and Polemarchakis, H., 2004. Testable implications of general equilibrium theory: a differentiable approach, Journal of Mathematical Economics 40, 105–119.

Husemoller, D., 1994. Fibre bundles, Springer-Verlag.

Samuelson, P. A., 1948. Consumption theory in terms of revealed preferences, Economica 15, 243–253.