Contemporary Methods in Mathematical Physics and Gravitation

CMMPG, vol. 1, No. 2 (2015), 43-54
http://cmmpg.eu/
eISSN: 2367-6760

A new variable shaped radial basis function approach for solving European option pricing model

Ahmad Golbabai
(Correspondence author)
School of Mathematics, Iran University of Science & Technology
Narmak, Tehran 16844-13114, Iran
e-mail: golbabai@iust.ac.ir

Ehsan Mohebianfar
School of Mathematics, Iran University of Science & Technology
Narmak, Tehran 16844-13114, Iran
e-mail: e_mohebianfar@iust.ac.ir

Abstract

In this paper, a new radial basis function (RBF) approach is developed for solving European option pricing model. Without any simplifications, a simple discretization pattern directly leads to a system $Ax = b$, moreover, employing a new variable shape parameter (VSP) strategy named binary shape parameter (BSP) strategy leads to more accurat results rather than constant shape parameter (CSP) strategy where they are compared with exact solution for European put option model.

Mathematics Subject Classification 2010: 35K20, 65D99.
Key Words: option pricing models; radial basis function; variable shape parameter strategies.

Received: *March 9, 2015*
Online: *March 24, 2015*

1 Introduction

The simplest types of options, called European, come in two main brands, Calls and Puts. If one is long a call, then he has the right, at some known point in the future (called expiry, T) to purchase a unit of the underlying asset, whatever that may be, for a pre-determined price (called the exercise, or strike price, K). As well as, if one is long a put, then the same applies, but he instead has the right to sell the underlying. If one is short either of these options, he has received a premium, but may be forced to either buy or sell the underlying in future, according to the terms of the contract. Another one is American option which can be exercised at any moment before the expiry time and this property makes it more flexible than the European option. More kind of options can be seen in [4]. The aim of the mathematics we are about to discuss is to determine what the fair price for this premium should be.

Black and Scholes published their seminal work on option pricing in 1973 [1]. In it, they described a mathematical framework for calculating the fair price of a European option in which they used a no-arbitrage argument to derive a partial differential equation which governs the evolution of the option price with respect to the time to expiry, t, and the price of the underlying asset, S.

We consider the values of the European options which satisfied the following Black-Scholes equation,

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0, \tag{1}$$

where r is the risk-free interest rate, σ is the volatility of the stock price S, and $V(S,t)$ is the option value at time t and stock price S. The initial condition is given by the terminal payoff valuation,

$$V(S,T) = \begin{cases} \max\{K - S,\ 0\} & \text{for } \textit{put} \\ \max\{S - K,\ 0\} & \text{for } \textit{call} \end{cases} \tag{2}$$

and the boundary conditions are as follows,

$$\begin{cases} V(0,t) = Ke^{-r(T-t)} & \text{and} \quad \lim_{S\to\infty} V(S,t) = 0 \quad \text{for } \textit{put} \\ V(0,t) = 0 & \text{and} \quad \lim_{S\to\infty} V(S,t) = S \quad \text{for } \textit{call} \end{cases} \tag{3}$$

Some numerical approaches such as finite element and finite-difference approximations have been presented in [6, 18, 21]. Despite their increasing success in the scientific community, surprisingly, RBF are almost unknown in finance. With the exception of a handful of articles in the financial literature, Hon and Mao in [12] apply radial basis functions (RBFs) for solving option pricing model, they used the transformation $S = e^y$ and converted (1) to an equation with constant coefficients. Goto et al. [10] directly solved (1) but did not apply boundary conditions and this is a theorical defect, in both previous articles, authors use the iterative methods for solving the resulting systems raised from discretization, as well as, their computational cost increase because of existance of matrices with badly condition numbers and their inverses. In this paper, we propose a new radial basis function (RBF) scheme that overcomes these difficulties, as well as, a new variable shape parameter (VSP) strategy is used that has advantages rather than previous ones.

The rest of this paper is as follows. The RBF method and different VSP strategies is introduced briefly in Section 2, as well as, a new VSP is presented in this section. In Section 3, our new approach is provided. The numerical results and their interpretations are given in Section 4. Finally, we conclude this paper in Section 5.

2 A brief review on the RBF method

One of the most popular meshless methods is constructed by radial kernels as basis called radial basis function (RBF) method. It is (conditionally) positive definite, rotationally and translationally invariant. These properties make its application straightforward specially for approximation problems with high dimensions. Some of the well-known RBFs are as follows,

$$\begin{aligned} &Gaussian\ (GA): && exp(-\varepsilon^2 r^2) \\ &Multiquadric\ (MQ): && \sqrt{1+\varepsilon^2 r^2} \\ &Inverse\ \ Multiquadric(IMQ): && (\sqrt{1+\varepsilon^2 r^2})^{-1} \end{aligned}$$

where r is the Euclidean distance between any two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, *i.e.* $r = \|\mathbf{x} - \mathbf{y}\|_2$, [22, 2]. The RBFs include two useful characteristics: a set of scattered centers $X_C = \{\mathbf{x}_1^c, ..., \mathbf{x}_N^c\} \subseteq \mathbb{R}^d$ with possibility of selecting their locations and existence of a free positive parameter, ε, known as the shape parameter.

Assume the ε_j be the shape parameter corresponding to j^{th} center $\mathbf{x}_j^c$, we use following notation for translation of RBFs at j^{th} center,

$$\phi_j(\mathbf{x}, \varepsilon_j) = \phi(\|\mathbf{x} - \mathbf{x}_j^c\|_2, \varepsilon_j), \quad j = 1, \ldots, N.$$

Let data values $f_j^c = f(\mathbf{x}_j^c)$ are given, the function $f(\mathbf{x})$ will be approximated using a linear combination of translates of a single RBF so that,

$$f(\mathbf{x}) \simeq S(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \phi_j(\mathbf{x}, \varepsilon_j), \tag{4}$$

where the unknown coefficients $\{\alpha_j\}_{j=1}^N$ will be determined by collocating (4) at the same set of centers, X_C.

The shape parameter plays an important role in RBFs, the choice of it controls the shape of the basis functions and interchanges the error and stability of interpolation process. This behavior is manifested as a classical trade off between accuracy and stability or *Uncertainty Principle* [20] which refers to the fact that an RBF approximant can not be accurate and well-conditioned at the same time.

Two scenarios are available for choosing shape parameters: constant shape parameter (CSP) strategies that all of shape parameters take the same value and variable shape parameter (VSP) strategies that assign different values to shape parameters corresponding to each center. Many scientists and mathematicians use CSPs in RBF approximations [11, 5, 13] because of their simple analysis as well as solid theoretical background rather than VSPs, but there are numerous results from a large collection of applications [3, 14, 15, 19, 7, 8] indicating the advantages of using VSPs. We have given a good review on the available strategies in the literature and their properties by focusing VSPs then introduce two alternative VSP, named hybrid shape parameter (HSP) and binary shape parameter (BSP) strategies, which leads to better results in RBF approximations rather than previous strategies in [9]. In the following subsection, we give a shortly review on the VSP strategies.

2.1 Variable shape parameter strategies

In general, using variable shape parameters can be interpreted as a superposition of different shaped basis functions because the values of shape parameters control the shape of basis functions. It also in turn, may result in more accurate approximations, moreover, implies that each column of RBF approximation matrix has different shape parameter, so it leads to more distinct entries in RBF matrix which may cause the lower condition number [9, 16, 17]. Some VSP strategies are summarized in Table 1, where ε_{min} and ε_{max} are positive parameters and denote the minimum and maximum of $\varepsilon_j^{'}$s respectively.

In Table 1, the first three strategies proposed by Kansa [14, 15] so that the exponential shape parameter (ESP) strategy assigns the value of shape parameter exponentially to each center, the increasing linear shape parameter (ILSP) and decreasing linear shape paremeter (DLSP) strategies have increasing and decreasing linear trends. In comparison to previous ones, the random shape parameter (RSP) [19] strategy assigns the value of shape parameter randomly which *rand(1,N)* is a MATLAB's function that produces N random values in the interval $[0, 1]$. As well as the trigonometric shape parameter (TSP) startegy suggested by Xiang et al. [23] introduces trigonometrical values as shape parameters with function *sin(j)*. However, applying this strategy resulted in more accurate approximations, it has a theoretical defect because of producing non-positive shape parameters [2, 22]. Golbabai and [8] modified the TSP formula called it sinusoidal shape parameter (SSP) strategy which produces N shape parameters in the interval $[\varepsilon_{min}, \varepsilon_{max}]$. They [8] also combined three strategies and introduced hybrid shape parameter (HSP) strategy as follows,

$$\varepsilon_j = \begin{cases} SSP_j, & j = 3k+1 \\ \\ DLSP_j, & j = 3k+2 \\ \\ ESP_j, & j = 3k+3 \end{cases} \tag{5}$$

where $k = 0, 1, \ldots, \lfloor \frac{N}{3} \rfloor$ and ESP_j, SSP_j and $DLSP_j$ denote the j^{th} shape parameter ε_j genereted by ESP, SSP and DLSP strategies *i.e.* after producing three vector of shape parameters by these strategies, j^{th} element (values of j is specified for each strategy in (5)) of them selected as j^{th} element of HSP's vector of shape parameters. The HSP strategy have some advantages rather than previous ones so that Kansa's strategies have larger errors in regions where the shape parameter is largest [19], although Sarra and Sturgill [19] treat this problem by proposing RSP strategy but it suffers from randomly nature that results in different results for a single problem, the HSP strategy overcomes all of these issues.

In numerical experiments, we used binary shape parameter (BSP) strategy defined in [9] which produces variable shape parameters using a biconditional rule. While its definition causes non-monotonicity in shape parameter sequence, unlike the other strategies, it

Table 1: Some common VSPs.

VSPs	$\varepsilon_j \quad j = 1, ..., N$
ESP	$[\varepsilon_{min}^2 (\frac{\varepsilon_{max}^2}{\varepsilon_{min}^2})^{(j-1)/(N-1)}]^{1/2}$
ILSP	$\varepsilon_{min} + (\frac{\varepsilon_{max}-\varepsilon_{min}}{N-1})j$
DLSP	$\varepsilon_{max} + (\frac{\varepsilon_{min}-\varepsilon_{max}}{N-1})j$
RSP	$\varepsilon_{min} + (\varepsilon_{max} - \varepsilon_{min}) \times rand(1, N)$
TSP	$\varepsilon_{min} + (\varepsilon_{max} - \varepsilon_{min}) sin(j)$
SSP	$\varepsilon_{min} + (\varepsilon_{max} - \varepsilon_{min}) sin((j-1)\frac{\pi}{2(N-1)})$

is constructed based on a simple structure. As shown in [9], the BSP strategy results in more accurate results rather than other strategies. Let ε_{min} and ε_{max} be as before, then the binary shape parameter (BSP) strategy is defined as follows,

$$\varepsilon_j = \begin{cases} \varepsilon_{min}, & \text{j be odd} \\ \\ \varepsilon_{max}, & \text{j be even} \end{cases}$$

the definition remains true by changing ε_{min} with ε_{max} in the above formula.

3 RBF Approach

In this section, we use the RBF method for solving the equation (1) with the initial and boundary conditions (2) and (3). At the first, we discretize the time derivative of V in time variable using the following finite difference approximation with uniform step size Δt,

$$\frac{\partial V^{(n)}}{\partial t} = \frac{V^{(n+1)} - V^{(n)}}{\Delta t}, \tag{6}$$

where $V^{(n)} = V(S, t^{(n)})$ so that $t^{(n)} = T - n\Delta t$ and $n = 0, 1, ..., M$. Substituting (6) in (1), we obtain,

$$\frac{V^{(n+1)} - V^{(n)}}{\Delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V^{(n)}}{\partial S^2} + rS\frac{\partial V^{(n)}}{\partial S} - rV^{(n)} = 0, \tag{7}$$

it leads to,

$$V^{(n+1)} = HV^{(n)}, \tag{8}$$

where

$$HV^{(n)} = V^{(n)} - \frac{1}{2}\sigma^2 S^2 \Delta t \frac{\partial^2 V^{(n)}}{\partial S^2} - rS\Delta t \frac{\partial V^{(n)}}{\partial S} + r\Delta t V^{(n)}. \tag{9}$$

The idea of the proposed scheme is to approximate the unknown function $V^{(n+1)}$ in (8) by (4) using a set of points $\{S_j\}_{j=1}^N$ as centers so that,

$$V^{(n+1)} \simeq \sum_{j=1}^{N} \alpha_j^{(n+1)} \phi_j(S, \varepsilon_j), \tag{10}$$

where $\{\alpha_j^n\}_{j=1}^N$ are unknown coefficients depending on time and the shape parameters $\{\varepsilon_j\}_{j=1}^N$ are variable selected by VSPs. Substituting (10) in (8) and collocating it at the same nodes S_j, we obtain the following system,

$$\sum_{j=1}^{N} \alpha_j^{(n+1)} \phi_j(S_i, \varepsilon_j) = HV_i^{(n)}, \quad i = 1, \ldots, N, \tag{11}$$

which $V_i^{(n)} = V(S_i, t^{(n)})$. The matrix form of (11) is as follows,

$$\mathbf{L}\alpha^{(n+1)} = \mathbf{HV}^{(n)}, \tag{12}$$

such that,

$$\mathbf{L} = (\phi_j(S_i, \varepsilon_j))_{N\times N}, \tag{13}$$

$$\alpha^{(n+1)} = (\alpha_1^{(n+1)}, ..., \alpha_N^{(n+1)})^T, \tag{14}$$

and,

$$\mathbf{HV}^{(n)} = (HV_1^{(n)}, ..., HV_N^{(n)})^T. \tag{15}$$

In the notation $HV_i^{(n)}$, it is understood that H is applied to the first variable S, then evaluated. Determining unknown vector $\alpha^{(M)}$, one can obtain the option price at the given stock price S using,

$$V^{(M)}(S, 0) = \sum_{j=1}^{N} \alpha_j^{(M)} \phi_j(S, \varepsilon_j). \tag{16}$$

It can be done by using the following algorithm,

Algorithm:

1. Let $n = 0$.
2. Compute $\mathbf{HV}^{(n)}$ using initial condition (2) and (9).
3. Solve (12) for $\alpha^{(n+1)}$.
4. Obtain $\mathbf{HV}^{(n+1)}$ using (10) and (9).
5. Substitute the first element of $\mathbf{HV}^{(n+1)}$ from the boundary condition (3).
6. Iterate steps 3 to 5 for $n = 1, ..., M - 2$.
7. Solve (12) for $\alpha^{(M)}$ with the right-hand side $\mathbf{HV}^{(M-1)}$.
8. Substituting $\alpha^{(M)}$ into (16) results the option price $V^{(M)}(S, 0)$.

Notice that in the above algorithm, the RBF matrix L is only evaluated once. As well as for $n = 0, 1, ..., M - 2$, one can obtain $\partial V^{(n+1)} \partial S$ and $\partial^2 V^{(n+1)} \partial S^2$ using (10) such that,

$$\frac{\partial V^{(n+1)}}{\partial S} = \sum_{j=1}^{N} \alpha_j^{(n+1)} \frac{\partial \phi_j(S, \varepsilon_j)}{\partial S}, \tag{17}$$

$$\frac{\partial^2 V^{(n+1)}}{\partial S^2} = \sum_{j=1}^{N} \alpha_j^{(n+1)} \frac{\partial^2 \phi_j(S, \varepsilon_j)}{\partial S^2}. \tag{18}$$

4 Numerical Experiments

In this section, we apply new proposed variable shaped RBF approach to solve European put option. The multiquadric (MQ) functions,

$$\phi_j(\mathbf{x}, \varepsilon_j) = \sqrt{1 + \varepsilon_j^2 \|\mathbf{x} - \mathbf{x}_j^c\|_2}$$

has been selected as basis functions and the BSP strategy defined in Subsection 2.1 has been employed, as well as in all of the reported results $M = 5$ and other related parameters for the problem are listed in Table 2. Since in the real markets the stock price S never tend to infinite, so the S_{max} is choosen sufficently large to satisfy the right end boundary conditions,

$$\begin{cases} \lim_{S\to\infty} V(S,t) = 0 \quad for\ \ put \\ \\ \lim_{S\to\infty} V(S,t) = S \quad for\ \ call \end{cases} \tag{19}$$

The approximation solutions obtained for $N = 22$ centers are compared with the exact soultion in Table 3. Where in the case of constant shape parameter strategy the "brute

Table 2: Parameters for numerical experiments.

Expiry time	$T = 0.5$ (year)
Strike price	$K = 10$
Risk-free interest rate	$r = 0.05$
Volatility	$\sigma = 0.2$
Minimum of stock price	$S_{min} = 1$
Maximum of stock price	$S_{max} = 30$

Table 3: Numerical results for European put option using new approach.

Stock S	Error	
	Constant Strategy	BSP Strategy
2	7.9276×10^{-2}	9.8544×10^{-3}
5	4.4411×10^{-1}	1.0576×10^{-5}
8	2.0128×10^{-1}	1.2214×10^{-3}
12	7.6656×10^{-2}	1.1340×10^{-3}
15	5.5821×10^{-4}	5.0658×10^{-4}
18	2.9003×10^{-6}	2.9003×10^{-6}

force" (BF) method is applied; one is calculating the errors with different shape parameters and choosing the shape parameter whose corresponding error is locally minimum (trial and error procedure). As well as, the comparison is performed for different values of stock price S. It is known that the accuracy increase when the number of centers increased, we also observed this issue in our results. However this issue needed more carefully because of increasing the number of centers increase the condition number of system matrix.

Because of significantly role of shape parameter on accuracy and stability in approximation, the results are compared over a range of average shape parameter,

$$\varepsilon_{avg} = \frac{1}{2}(\varepsilon_{min} + \varepsilon_{max}) \tag{20}$$

so that the distance $K = \varepsilon_{max} - \varepsilon_{min}$ has been specified as $K = 1$ recommended by authors in [19, 9]. However, the minimum error obtained for optimum shape parameter in the constant case and optimum interval for the BSP case, are inserted in Table 3. As shown in Table 3, the results obtained from new scheme are in good agreement with exact solutions, as well as applying BSP strategy is effective for decreasing computational error rather than constant strategy and results in more accurate solutions.

In Figure 1 obtained with MATLAB's function *plot()*, both constant and variable BSP

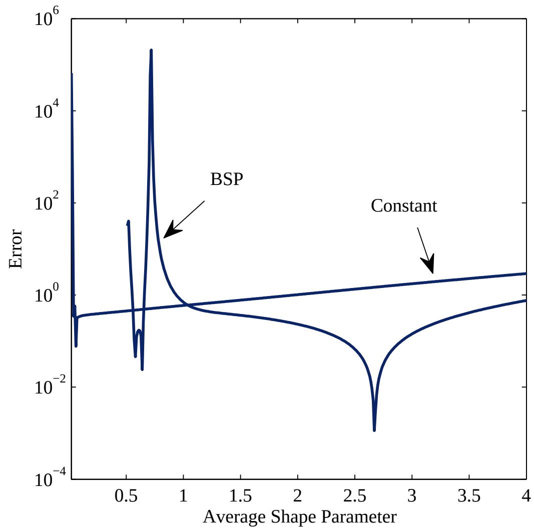

Figure 1: Comparison of constant and BSP strategies.

strategies are compared over a rang of average shape parameter for $S = 12$ and $N = 22$. It is shown that results are completely sensitive to vary the value of shape parameter and apply BSP startegy leads to more accurate results rather than constant one. As well as because of producing a system matrix with more distinct elements using BSP strategy, the error can be locally minimized as shown in Figure 1.

5 Conclusion

European options pricing model is very important in finance, a new discretization approach using variable shaped radial kernels is presented for underlying options. As well as a new variable strategy was named as Binary which assigns different shape parameters to odd and even centers is employed. It has a simple construction, however, it can make reasonable differences between the rows of system matrix of RBF approximation and its advantages rather than previous strategies was approved by authors in [9].

As test example, we applied our new scheme with variable BSP strategy to European put option and compared our results with the exact solutions. This comparison showed the results are in good agreement with exact solutions, moreover, using BSP strategy is competitive with traditional constant ones.

References

[1] F. Black and M. Scholes, The pricing of options and corporate liabilities, *J. Polit. Econ.*, **81** (1973), 637-654.

[2] M. D. Buhmann, *Radial Basis Functions: Theory and Implementation, University of Gissen*, Cambridge University Press, 2004.

[3] R. E. Carlson and T. A. Foley, The parameter R^2 in multiquadric interpolation, *Comput. Math. Appl.*, **21** (1991), 29-42.

[4] D. J. Duffy, *Finite Difference Methods in Financial Engineering*, John Wiley Sons, England, 2006.

[5] R. Franke, Scattered data interpolation: tests of some methods, *Math. Comput.*, **38** (1982), 181-200.

[6] R. Geske and K. Shastri, Valuation by approximation: a comparison of alternative option valuation techniques, *J. Financ. Quant. Anal.*, **20** (1985), 45-71.

[7] A. Golbabai, E. Mohebianfar and H. Rabiei, On the role of shape parameter in approximating the Eigenvalues of fredholm integral equations: an RBF-Simpson approach, *Appl. Math. Inf. Sci.*, 2012, In Press.

[8] A. Golbabai and H. Rabiei, Hybrid shape parameter strategy for the RBF approximation of vibrating systems, *Int J. Comput. Math.*, **89** (2012), 2410-2427.

[9] A. Golbabai, E. Mohebianfar and H. Rabiei, On the new variable shape parameter strategies for radial basis functions, *Comput. Appl. Math.*, 2014, In Press.

[10] Y. Goto, Z. Fei, S. Kan and E. Kita, Options valuation by using radial basis function approximation, *Eng. Anal. Bound. Elem.*, **31** (2007), 836-843.

[11] R. L. Hardy, Multiquadric equations of topography and other irregular surfaces, *J. Geophys. Res.*, **76** (1971), 1905-1915.

[12] Y. C. Hon and X. Z. Mao, A Radial Basis Function Method for Solving Options Pricing Models, *IAFE*, **8** (1999), 31-49.

[13] C. S. Huang, C. F. Lee and A. D. Cheng, Error estimate, optimal shape factor, and high precision computation of multiquadric collocation method, *Eng. Anal. Bound. Elem.*, **31** (2007), 614-623.

[14] E. J. Kansa and R. E. Carlson, Improved accuracy of multiquadric interpolation using variable shape parameters, *Comput. Math. Appl.*, **24** (1992), 99-120.

[15] E. J. Kansa, R. C. Aldredge and L. Ling. Numerical simulation of two-dimensional combustion using mesh-free methods, *Eng. Anal. Bound. Elem.*, **33** (2009), 940-950.

[16] E. J. Kansa, Multiquadrics-a scattered data approximation scheme with applications to computational fluid dynamics-I Surface approximations and partial derivative estimates, *Comput. Math. Appl.*, **19** (1990), 127-145.

[17] E. J. Kansa, Multiquadrics-a scattered data approximation scheme with applications to computational fluid dynamics-II solutions to parabolic, hyperbolic and elliptic partial differential equations, *Comput. Math. Appl.*, **19** (1990), 147-161.

[18] O. Pironneau and F. Hecht, Mesh adaptation for the black and scholes equations, *East-West J. Numer. Math.*, **8** (2000), 25-35.

[19] S. A. Sarra, D. Sturgill, A random variable shape parameter strategy for radial basis function approximation methods, *Eng. Anal. Bound. Elem.*, **33** (2009), 1239-1245.

[20] R. Schaback, Improved error bounds for scattered data interpolation by radial basis functions, *Math. Comput.*, **68** (1999), 201-206.

[21] P. Wilmott, S. Howison and J. Dewynne, *The Mathematics of Financial Derivatives*, Cambridge University Press, New york, 1995.

[22] H. Wendland, *Scattered Data Approximation*, Cambridge University Press, 2005.

[23] S. Xiang, K. Wang, Y. Ai, Y. Sha and H. Shi, Trigonometric variable shape parameter and exponent strategy for generalized multiquadric radial basis function approximation, *Appl. Math. Model.*, **36** (2012), 1931-1938.