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Qualitative nature of a discrete predator-prey system

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Abstract

We study the qualitative behavior of a predator-prey model, where the carrying capacity of the predators environment is proportional to the number of prey. We investigate the boundedness character, steady-states, local asymptotic stability of equilibrium points, and global behavior of the unique positive equilibrium point of a discrete predator-prey model given by

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + \gamma x_n}, \ y_{n+1} = \frac{\delta x_n y_n}{x_n + \eta y_n},$$

where parameters $\alpha, \beta, \gamma, \delta, \ \eta$ and initial conditions are positive real numbers. Moreover, the rate of convergence of positive solutions is also discussed. Some numerical examples are given to verify our theoretical results.

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Key Words: predator-prey model; steady-states; local stability; global character; rate of convergence

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1 Introduction

Leslie introduced the following predator-prey model where the "carrying capacity" of the predators environment is proportional to the number of prey:

$$\begin{aligned} \frac{dH}{dt} &= (r_1 - a_1 P - b_1 H) H, \\ \frac{dP}{dt} &= \left(r_2 - a_2 \frac{P}{H} \right) P, \end{aligned} \tag{1}$$

where H, P represent the prey and predator density, respectively and $r_1,\ a_1,\ b_1,\ r_2,\ a_2$ are positive constants. The parameters r_1 and r_2 are the intrinsic growth rates of the prey and the predator, respectively. The value r_1/b_1 denotes the carrying capacity of the prey and r_2H/a_2 takes on the role of a prey-dependent carrying capacity for the predator. There have been many important and interesting results about system (1), such as the global stability, permanence, periodic solutions, almost periodic solutions and so on [1].

It is pointed out in [2] that discrete-time models are more appropriate than continuous ones when the populations have non-overlapping generations. Furthermore, discrete-time systems can also provide efficient computational models of continuous for numerical simulations. In the present work, applying the forward Euler's method followed by a nonstandard difference scheme to system (1), we obtain the discrete-time predator-prey system as follows

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + \gamma x_n},\ \ y_{n+1} = \frac{\delta x_n y_n}{x_n + \eta y_n}, \tag{2}$$

where parameters $\alpha, \beta, \gamma, \delta, \eta$ and initial conditions are positive real numbers. Here, it should be noted that in nonstandard difference scheme, we have used such a transformation that equilibrium points in both cases are conserved. More precisely, our aim is to investigate boundedness character, local asymptotic stability, global asymptotic stability of unique positive equilibrium point, and the rate of convergence of positive solutions of the system (2). For basic theory of difference equations and their applications, we refer to [3, 5, 6, 7, 8, 9]. In [10, 11, 12, 13, 14, 15, 16, 17] some qualitative behavior of difference equations is discussed.

2 Boundedness

The following theorem shows that every positive solution of the system (2) is bounded.

Theorem 1. *Every positive solution $\{(x_n, y_n)\}$ of the system (2) is bounded.*

Proof. Assume that $x_0 \geq 0,\ y_0 \geq 0$, then any arbitrary solution $\{(x_n, y_n)\}$ of (2) is positive if and only if $0 < y_n < \frac{\alpha}{\beta}$ for all $n = 1, 2, \ldots$. Let $\{(x_n, y_n)\}$ be an arbitrary positive solution of (2), then from the system (2) we obtain

$$\begin{aligned} x_{n+1} &= \frac{\alpha x_n - \beta x_n y_n}{1+\gamma x_n} \\ &\leq \frac{\alpha x_n}{1+\gamma x_n} \leq \frac{\alpha}{\gamma},\ n = 0, 1, 2, \ldots, \end{aligned}$$

and

$$\begin{aligned} y_{n+1} &= \frac{\delta x_n y_n}{x_n + \eta y_n} = \frac{\delta y_n}{1+\eta \frac{y_n}{x_n}} \\ &\leq \frac{\delta}{\eta} x_n \leq \frac{\alpha\delta}{\gamma\eta},\ n = 0, 1, 2, \cdots . \end{aligned}$$

Hence, for every positive solution $\{(x_n, y_n)\}$ of (2), one has

$$0 < x_n \leq \frac{\alpha}{\gamma},\ 0 < y_n \leq \min\left\{\frac{\alpha}{\beta}, \frac{\alpha\delta}{\gamma\eta}\right\}$$

for all $n = 1, 2, \ldots$. This completes the proof. □

Theorem 2. *Let $\{(x_n, y_n)\}$ be a positive solution of the system (2). Then, $\left[0, \frac{\alpha}{\gamma}\right] \times [0, B]$ is an invariant set for (2), where $B = \min\left\{\frac{\alpha}{\beta}, \frac{\alpha\delta}{\gamma\eta}\right\}$.*

Proof. For any positive solution $\{(x_n, y_n)\}$ of the system (2) with initial conditions $x_0 \in \left[0, \frac{\alpha}{\gamma}\right]$ and $y_0 \in [0, B]$, we have from the system (2)

$$\begin{aligned} 0 \leq x_1 &= \frac{\alpha x_0 - \beta x_0 y_0}{1+\gamma x_0} \\ &\leq \frac{\alpha x_0}{1+\gamma x_0} \leq \frac{\alpha}{\gamma}, \end{aligned}$$

and

$$\begin{aligned} 0 \leq y_1 &= \frac{\delta x_0 y_0}{x_0 + \eta y_0} \leq \frac{\delta x_0 y_0}{\eta y_0} \\ &= \frac{\delta}{\eta} x_0 \leq \frac{\alpha\delta}{\gamma\eta}. \end{aligned}$$

Hence, $x_1 \in \left[0, \frac{\alpha}{\gamma}\right]$ and $y_1 \in [0, B]$, where $B = \min\left\{\frac{\alpha}{\beta}, \frac{\alpha\delta}{\gamma\eta}\right\}$. Then it follows by induction that

$$0 \leq x_n \leq \frac{\alpha}{\gamma},\ 0 \leq y_n \leq B,$$

for all $n = 1, 2, \ldots$, where $B = \min\left\{\frac{\alpha}{\beta}, \frac{\alpha\delta}{\gamma\eta}\right\}$. □

3 Linearized stability

Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (2), then

$$\bar{x} = \frac{\alpha\bar{x} - \beta\bar{x}\bar{y}}{1 + \gamma\bar{x}},\ \bar{y} = \frac{\delta\bar{x}\bar{y}}{\bar{x} + \eta\bar{y}}.$$

Hence, $P = \left(\frac{(\alpha-1)\eta}{\beta(\delta-1)+\gamma\eta}, \frac{(\alpha-1)(\delta-1)}{\beta(\delta-1)+\gamma\eta}\right)$, $Q = \left(\frac{\alpha-1}{\gamma}, 0\right)$ be only two equilibrium points of the system (2). Then, clearly $P = \left(\frac{(\alpha-1)\eta}{\beta(\delta-1)+\gamma\eta}, \frac{(\alpha-1)(\delta-1)}{\beta(\delta-1)+\gamma\eta}\right)$ be the unique positive equilibrium point of the system (2), if $\alpha > 1$ and $\delta > 1$.

The Jacobian matrix of linearized system of (2) about the fixed point $(\bar{x}, \bar{y})$ is given by

$$F_J(\bar{x}, \bar{y}) = \begin{bmatrix} \frac{\alpha-\bar{y}\beta}{(1+\bar{x}\gamma)^2} & -\frac{\bar{x}\beta}{1+\bar{x}\gamma} \\ \frac{\delta\eta\bar{y}^2}{(\bar{x}+\eta\bar{y})^2} & \frac{\delta\bar{x}^2}{(\bar{x}+\eta\bar{y})^2} \end{bmatrix}.$$

Lemma 1 (Jury condtion)**.** *Consider the second-degree polynomial equation*

$$\lambda^2 + p\lambda + q = 0, \tag{3}$$

where p and q are real numbers. Then, the necessary and sufficient condition for both roots of the Equation (3) to lie inside the open disk $|\lambda| < 1$ is

$$|p| < 1 + q < 2.$$

Theorem 3. *Assume that $\alpha > 1$ and $\delta < 1$, then the equilibrium point $Q = \left(\frac{\alpha-1}{\gamma}, 0\right)$ of the system (2) is locally asymptotically stable.*

Proof. The Jacobian matrix of the linearized system of (2) about the equilibrium point $Q = \left(\frac{\alpha-1}{\gamma}, 0\right)$ is given by

$$F_J\left(\frac{\alpha-1}{\gamma}, 0\right) = \begin{bmatrix} \frac{1}{\alpha} & \frac{(1-\alpha)\beta}{\alpha\gamma} \\ 0 & \delta \end{bmatrix}.$$

Then, characteristic polynomial of the Jacobian matrix $F_J(Q)$ about the equilibrium point $Q = \left(\frac{\alpha-1}{\gamma}, 0\right)$ is given by

$$\Upsilon(\lambda) = \lambda^2 - \left(\frac{1}{\alpha} + \delta\right)\lambda + \frac{\delta}{\alpha}.$$

The roots of characteristic polynomial $\Upsilon(\lambda)$ are

$$\lambda_1 = \frac{1}{\alpha} < 1, \ \lambda_2 = \delta < 1.$$

Hence, equilibrium point $\left(\frac{\alpha-1}{\gamma}, 0\right)$ is locally asymptotically stable, if $\alpha > 1$ and $\delta < 1$. ☐

Theorem 4. *Assume that $\alpha > 2$ and $\delta > 1$, then unique positive equilibrium point $P = \left(\frac{(\alpha-1)\eta}{\beta(\delta-1)+\gamma\eta}, \frac{(\alpha-1)(\delta-1)}{\beta(\delta-1)+\gamma\eta}\right)$ of the system (2) is locally asymptotically stable if and only if*

$$\eta\gamma(\alpha\delta - 1) > (\alpha - 2)\beta(\delta - 1)^2. \tag{4}$$

Proof. Assume that $\alpha > 1$ and $\delta > 1$. Then, characteristic polynomial of the Jacobian matrix $F_J(P)$ about the unique equilibrium point $P = \left(\frac{(\alpha-1)\eta}{\beta(\delta-1)+\gamma\eta}, \frac{(\alpha-1)(\delta-1)}{\beta(\delta-1)+\gamma\eta}\right)$ is given by

$$\Upsilon(\lambda) = \lambda^2 - \left(\frac{\gamma\eta(\alpha+\delta) + \beta\delta^2 - \beta}{\delta(\alpha\gamma\eta + \beta(\delta-1))}\right)\lambda + \frac{\beta(\delta-1)(\alpha(\delta-1) - \delta + 2) + \gamma\eta}{\delta(\alpha\gamma\eta + \beta(\delta-1))}.$$

Let

$$p = \frac{\gamma\eta(\alpha+\delta) + \beta\delta^2 - \beta}{\delta(\alpha\gamma\eta + \beta(\delta-1))}, \ q = \frac{\beta(\delta-1)(\alpha(\delta-1) - \delta + 2) + \gamma\eta}{\delta(\alpha\gamma\eta + \beta(\delta-1))}.$$

Assume that $\eta\gamma(\alpha\delta - 1) > (\alpha - 2)\beta(\delta - 1)^2$. Then, one has

$$\begin{aligned} |p| &= \frac{\gamma\eta(\alpha+\delta) + \beta\delta^2 - \beta}{\delta(\alpha\gamma\eta + \beta(\delta-1))} \\ &= \frac{1}{\delta} + \frac{\gamma\eta + \beta(\delta-1)}{\alpha\gamma\eta + \beta(\delta-1)} \\ &< 1 + \frac{\beta(\delta-1)(\alpha(\delta-1) - \delta + 2) + \gamma\eta}{\delta(\alpha\gamma\eta + \beta(\delta-1))} = 1 + q. \end{aligned}$$

and

$$\begin{aligned} 1 + q &= \frac{\beta(\delta-1)(\alpha(\delta-1) + 2) + \gamma\eta(\alpha\delta + 1)}{\delta(\alpha\gamma\eta + \beta(\delta-1))} \\ &= 2 - \frac{\gamma\eta(\alpha\delta - 1) - (\alpha-2)\beta(\delta-1)^2}{\delta(\alpha\gamma\eta + \beta(\delta-1))} < 2. \end{aligned}$$

Hence, $|p| < 1 + q < 2$. It follows from lemma 1 that unique positive equilibrium point $P = \left(\frac{(\alpha-1)\eta}{\beta(\delta-1)+\gamma\eta}, \frac{(\alpha-1)(\delta-1)}{\beta(\delta-1)+\gamma\eta}\right)$ of the system (2) is locally asymptotically stable if and only if $\eta\gamma(\alpha\delta - 1) > (\alpha - 2)\beta(\delta - 1)^2$. $\square$

4 Global behavior

Theorem 5. *Let $I = [a, b]$ and $J = [c, d]$ be real intervals, and let $f : I \times J \to I$ and $g : I \times J \to J$ be continuous functions. Let us consider two-dimensional discrete dynamical system of the form*

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n),\ n = 0, 1, \ldots, \end{aligned} \tag{5}$$

with initial conditions $(x_0, y_0) \in I \times J$. Suppose that following statements are true:

(i) $f(x, y)$ is non-decreasing in x, and non-increasing in y.

(ii) $g(x, y)$ is non-decreasing in both arguments.

(iii) If $(m_1, M_1, m_2, M_2) \in I^2 \times J^2$ is a solution of the system

$$m_1 = f(m_1, M_2),\ M_1 = f(M_1, m_2)$$
$$m_2 = g(m_1, m_2),\ M_2 = g(M_1, M_2)$$

such that $m_1 = M_1$, and $m_2 = M_2$. Then, there exists exactly one equilibrium point $(\bar{x}, \bar{y})$ of the system (5) such that $\lim_{n\to\infty}(x_n, y_n) = (\bar{x}, \bar{y})$.

Theorem 6. *Assume that $\gamma\eta - \beta\delta \neq 0$. Then, the unique positive equilibrium point $P = \left(\frac{(\alpha-1)\eta}{\beta(\delta-1)+\gamma\eta}, \frac{(\alpha-1)(\delta-1)}{\beta(\delta-1)+\gamma\eta}\right)$ of the system (2) is a global attractor.*

Proof. Let $f(x, y) = \frac{\alpha x - \beta xy}{1+\gamma x}$, and $g(x, y) = \frac{\delta xy}{x+\eta y}$. Then, it is easy to see that $f(x, y)$ is non-decreasing in x for every fixed $y \in [0, \frac{\alpha}{\beta}]$ and non-increasing in y. Moreover, $g(x, y)$ is non-decreasing in both x and y. Let (m_1, M_1, m_2, M_2) be a solution of the system

$$m_1 = f(m_1, M_2),\ M_1 = f(M_1, m_2)$$
$$m_2 = g(m_1, m_2),\ M_2 = g(M_1, M_2)$$

Then, one has

$$m_1 = \frac{\alpha m_1 - \beta m_1 M_2}{1 + \gamma m_1},\ M_1 = \frac{\alpha M_1 - \beta M_1 m_2}{1 + \gamma M_1}, \tag{6}$$

and

$$m_2 = \frac{\delta m_1 m_2}{1 + \eta m_2},\ M_2 = \frac{\delta M_1 M_2}{1 + \eta M_2}. \tag{7}$$

From (6), one has

$$1 + \gamma m_1 = \alpha - \beta M_2,\ 1 + \gamma M_1 = \alpha - \beta m_2. \tag{8}$$

Similarly, from (7) we obtain

$$1 + \eta m_2 = \delta m_1,\ 1 + \eta M_2 = \delta M_1. \tag{9}$$

On subtracting (8), we have $\gamma(M_1 - m_1) = \beta(M_2 - m_2)$, and subtracting (9), one has $\eta(M_2 - m_2) = \delta(M_1 - m_1)$. Then, it follows that $(\gamma\eta - \beta\delta)(M_1 - m_1) = 0$. Hence, $M_1 = m_1$ and similarly one has $M_2 = m_2$. Hence, from theorem 5 the equilibrium point $\left(\frac{(\alpha-1)\eta}{\beta(\delta-1)+\gamma\eta}, \frac{(\alpha-1)(\delta-1)}{\beta(\delta-1)+\gamma\eta}\right)$ of the system (2) is a global attractor. ☐

Lemma 2. *Assume that $\alpha > 2,\ \delta > 1,\ \gamma\eta - \beta\delta \neq 0$ and $\eta\gamma(\alpha\delta - 1) > (\alpha - 2)\beta(\delta - 1)^2$. Then, the unique positive equilibrium point $\left(\frac{(\alpha-1)\eta}{\beta(\delta-1)+\gamma\eta}, \frac{(\alpha-1)(\delta-1)}{\beta(\delta-1)+\gamma\eta}\right)$ of the system (2) is globally asymptotically stable.*

Proof. The proof follows from theorem 4, and theorem 6. ☐

5 Rate of convergence

In this section we will determine the rate of convergence of a solution that converges to the unique positive equilibrium point of the system (2).

The following result gives the rate of convergence of solutions of a system of difference equations

$$X_{n+1} = (A + B(n))\, X_n, \tag{10}$$

where X_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B : \mathbb{Z}^+ \rightarrow$

$C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \tag{11}$$

as $n\to\infty$, where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm

$$\|(x,y)\| = \sqrt{x^2+y^2}$$

Proposition 1. *(Perron's Theorem)[18] Suppose that condition* (11) *holds. If X_n is a solution of* (10)*, then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty}(\|X_n\|)^{1/n} \tag{12}$$

exists and is equal to the modulus of one the eigenvalues of matrix A.

Proposition 2. *[18] Suppose that condition* (11) *holds. If X_n is a solution of* (10)*, then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty}\frac{\|X_{n+1}\|}{\|X_n\|} \tag{13}$$

exists and is equal to the modulus of one the eigenvalues of matrix A.

Let $\{(x_n, y_n)\}$ be any solution of the system (2) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$, where $(\bar{x},\bar{y}) = \left(\frac{(\alpha-1)\eta}{\beta(\delta-1)+\gamma\eta}, \frac{(\alpha-1)(\delta-1)}{\beta(\delta-1)+\gamma\eta}\right)$. To find the error terms, one has from the system (2)

$$\begin{aligned} x_{n+1}-\bar{x} &= \frac{\alpha x_n - \beta x_n y_n}{1+\gamma x_n} - \frac{\alpha\bar{x}-\beta\bar{x}\bar{y}}{1+\gamma\bar{x}} \\ &= \frac{(\alpha-\beta y_n)}{(1+\gamma x_n)(1+\gamma\bar{x})}(x_n-\bar{x}) - \frac{\beta\bar{x}}{1+\gamma\bar{x}}(y_n-\bar{y}), \end{aligned}$$

and

$$\begin{aligned} y_{n+1}-\bar{y} &= \frac{\delta x_n y_n}{x_n+\eta y_n} - \frac{\delta\bar{x}\bar{y}}{\bar{x}+\eta\bar{y}} \\ &= \frac{\delta\eta\bar{y}y_n}{(x_n+\eta y_n)(\bar{x}+\eta\bar{y})}\left(x_n-\bar{x}\right) + \frac{\delta\bar{x}x_n}{(x_n+\eta y_n)(\bar{x}+\eta\bar{y})}\left(y_n-\bar{y}\right). \end{aligned}$$

Let $e_n^1 = x_n - \bar{x}$, and $e_n^2 = y_n - \bar{y}$, then one has

$$e_{n+1}^1 = a_n e_n^1 + b_n e_n^2,$$

and

$$e_{n+1}^2 = c_n e_n^1 + d_n e_n^2,$$

where

$$a_n = \frac{(\alpha - \beta y_n)}{(1+\gamma x_n)(1+\gamma\bar{x})},\ b_n = -\frac{\beta\bar{x}}{1+\gamma\bar{x}},$$
$$c_n = \frac{\delta\eta\bar{y}y_n}{(x_n+\eta y_n)(\bar{x}+\eta\bar{y})},\ d_n = \frac{\delta\bar{x}x_n}{(x_n+\eta y_n)(\bar{x}+\eta\bar{y})}.$$

Moreover,

$$\lim_{n\to\infty} a_n = \frac{\alpha - \bar{y}\beta}{(1+\bar{x}\gamma)^2},\ \lim_{n\to\infty} b_n = -\frac{\bar{x}\beta}{1+\bar{x}\gamma},$$
$$\lim_{n\to\infty} c_n = \frac{\delta\eta\bar{y}^2}{(\bar{x}+\eta\bar{y})^2},\ \lim_{n\to\infty} d_n = \frac{\delta\bar{x}^2}{(\bar{x}+\eta\bar{y})^2}.$$

Now the limiting system of error terms can be written as

$$\begin{bmatrix} e_{n+1}^1 \\ e_{n+1}^2 \end{bmatrix} = \begin{bmatrix} \frac{\alpha-\bar{y}\beta}{(1+\bar{x}\gamma)^2} & -\frac{\bar{x}\beta}{1+\bar{x}\gamma} \\ \frac{\delta\eta\bar{y}^2}{(\bar{x}+\eta\bar{y})^2} & \frac{\delta\bar{x}^2}{(\bar{x}+\eta\bar{y})^2} \end{bmatrix} \begin{bmatrix} e_n^1 \\ e_n^2 \end{bmatrix},$$

which is similar to linearized system of (2) about the equilibrium point $(\bar{x}, \bar{y})$.

Using proposition (1), one has following result.

Theorem 7. *Assume that $\{(x_n, y_n)\}$ be a positive solution of the system (2) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$, where*

$$(\bar{x}, \bar{y}) = \left(\frac{(\alpha-1)\eta}{\beta(\delta-1)+\gamma\eta}, \frac{(\alpha-1)(\delta-1)}{\beta(\delta-1)+\gamma\eta}\right).$$

Then, the error vector $e_n = \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix}$ of every solution of (2) satisfies both of the following asymptotic relations

$$\lim_{n\to\infty} (\|e_n\|)^{\frac{1}{n}} = |\lambda_{1,2} F_J(\bar{x}, \bar{y})|,\ \lim_{n\to\infty} \frac{\|e_{n+1}\|}{\|e_n\|} = |\lambda_{1,2} F_J(\bar{x}, \bar{y})|,$$

where $\lambda_{1,2} F_J(\bar{x}, \bar{y})$ are the characteristic roots of Jacobian matrix $F_J(\bar{x}, \bar{y})$.

6 Examples

In this section, we consider three numerical examples for the system (2). First two examples show that the unique positive equilibrium point of the system (2) is globally asymptotically stable, *i.e.*, the condition (4) of theorem 4 is satisfied, where as from third example it is clear that the unique positive equilibrium point of the system (2) is unstable, *i.e.*, the condition (4) of theorem 4 does not hold.

Example 1. *Let* $\alpha = 12.1,\ \beta = 0.97,\ \gamma = 5,\ \delta = 14.4,\ \eta = 2.1$. *Then, system (2) can be written as*

$$x_{n+1} = \frac{12.1x_n - 0.97x_n y_n}{1 + 5x_n},\ y_{n+1} = \frac{14.4x_n y_n}{x_n + 2.1y_n}, \tag{14}$$

with initial conditions $x_0 = 1,\ y_0 = 6$.

In this case the unique positive equilibrium point $\left(\frac{(\alpha-1)\eta}{\beta(\delta-1)+\gamma\eta}, \frac{(\alpha-1)(\delta-1)}{\beta(\delta-1)+\gamma\eta}\right) = (0.991999, 6.3299)$. *Moreover,* $\gamma\eta(\alpha\delta - 1) = 1819.02$ *and* $(\alpha - 2)\beta(\delta - 1)^2 = 1759.15$. *Hence, stability condition (4) of theorem 4 is satisfied. The plots of* x_n *and* y_n *for the system (14) are shown in Figures (1) and (2), respectively. An attractor of the system (14) is shown in Fig. (3).*

Example 2. *Let* $\alpha = 125,\ \beta = 1.95,\ \gamma = 34,\ \delta = 345,\ \eta = 19.39$. *Then, system (2) can be written as*

$$x_{n+1} = \frac{125x_n - 1.95x_n y_n}{1 + 34x_n},\ y_{n+1} = \frac{345x_n y_n}{x_n + 19.39y_n}, \tag{15}$$

with initial conditions $x_0 = 1.8,\ y_0 = 32$.

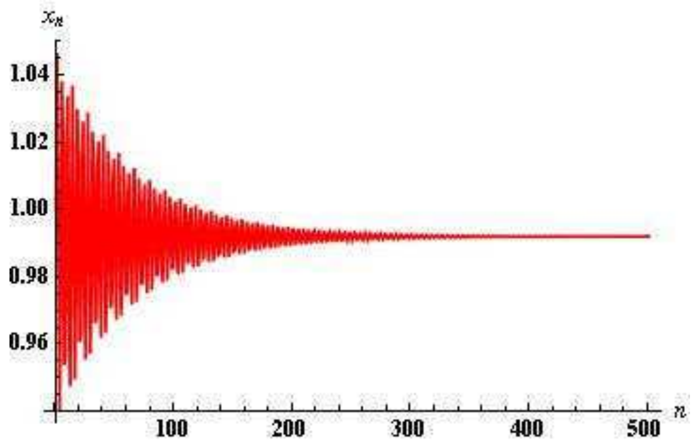

Figure 1: Plot of x_n for the system (14)

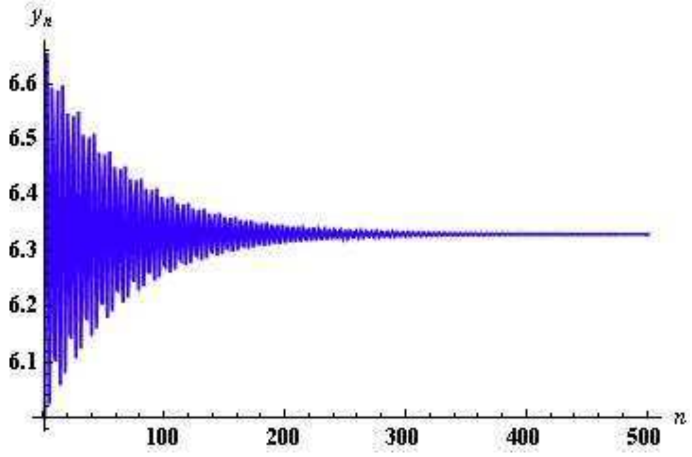

Figure 2: Plot of y_n for the system (14)

In this case the unique positive equilibrium point $\left(\frac{(\alpha-1)\eta}{\beta(\delta-1)+\gamma\eta}, \frac{(\alpha-1)(\delta-1)}{\beta(\delta-1)+\gamma\eta}\right) = (1.80771, 32.0707)$. *Moreover,* $\gamma\eta(\alpha\delta - 1) = 2.84299 \times 10^7$ *and* $(\alpha - 2)\beta(\delta - 1)^2 = 2.83829 \times 10^7$. *Hence, stability condition (4) of theorem 4 is satisfied. The plots of* x_n *and* y_n *for the system (15) are shown in Figures (4) and (5), respectively. An attractor of the system (15) is shown in Fig. (6).*

Example 3. *Let* $\alpha = 279,\ \beta = 6,\ \gamma = 130.1,\ \delta = 657,\ \eta = 29.99$. *Then, system (2) can be written as*

$$x_{n+1} = \frac{279x_n - 6x_ny_n}{1 + 130.1x_n},\ y_{n+1} = \frac{657x_ny_n}{x_n + 29.99y_n}, \tag{16}$$

with initial conditions $x_0 = 1.2,\ y_0 = 25$.

In this case the unique positive equilibrium point $\left(\frac{(\alpha-1)\eta}{\beta(\delta-1)+\gamma\eta}, \frac{(\alpha-1)(\delta-1)}{\beta(\delta-1)+\gamma\eta}\right)$ *is unstable. Moreover,* $\gamma\eta(\alpha\delta - 1) = 7.15189 \times 10^8$ *and* $(\alpha - 2)\beta(\delta - 1)^2 = 715218432$. *Hence, stability condition (4) of theorem 4 does not hold. The plots of* x_n *and* y_n *for the system (16) are shown in Figures (7) and (8), respectively. The parametric plot of the system (16) is shown in Fig. (9).*

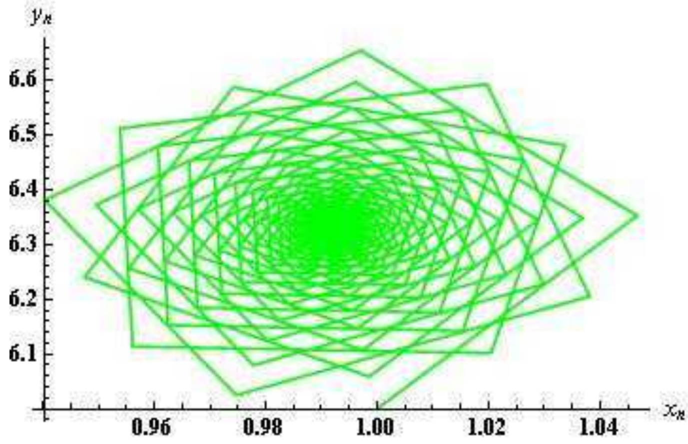

Figure 3: An attractor of the system (14)

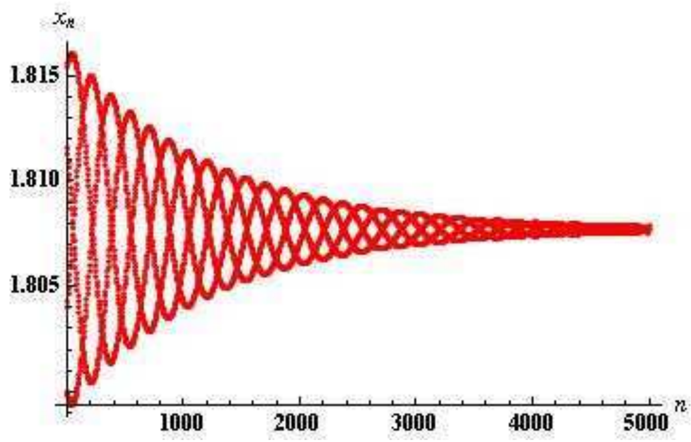

Figure 4: Plot of x_n for the system (15)

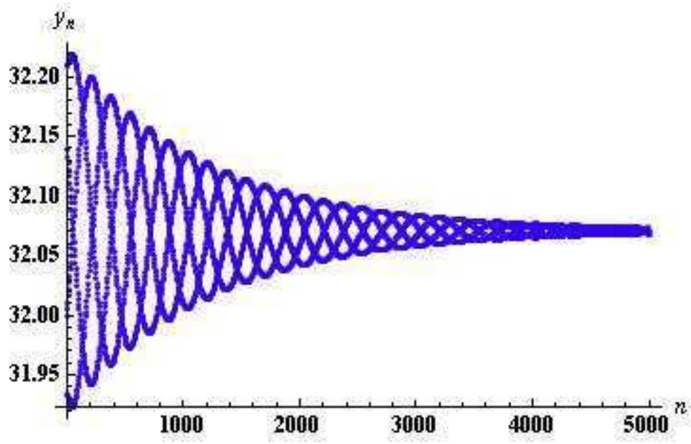

Figure 5: Plot of y_n for the system (15)

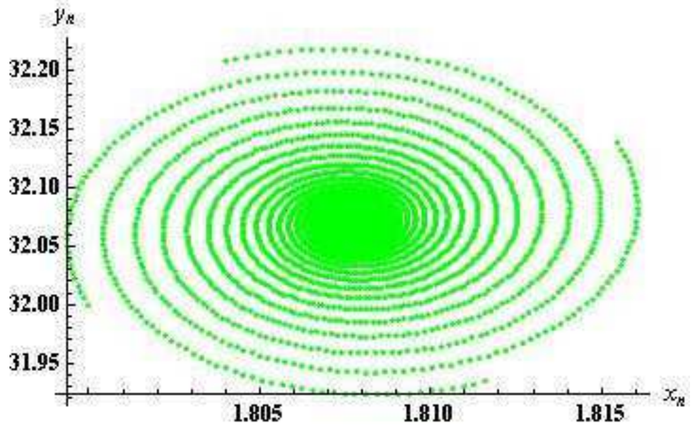

Figure 6: An attractor of the system (15)

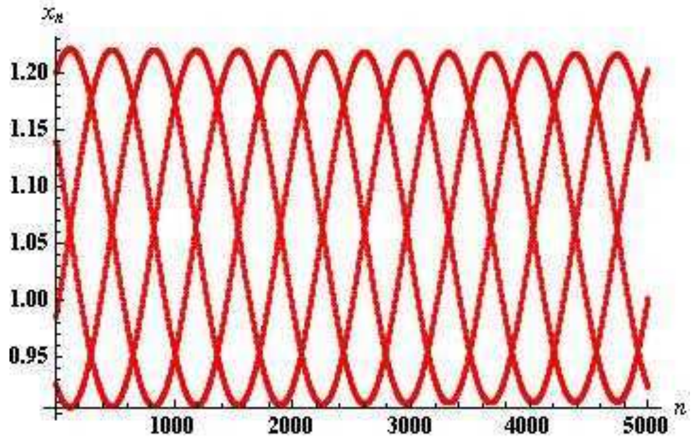

Figure 7: Plot of x_n for the system (16)

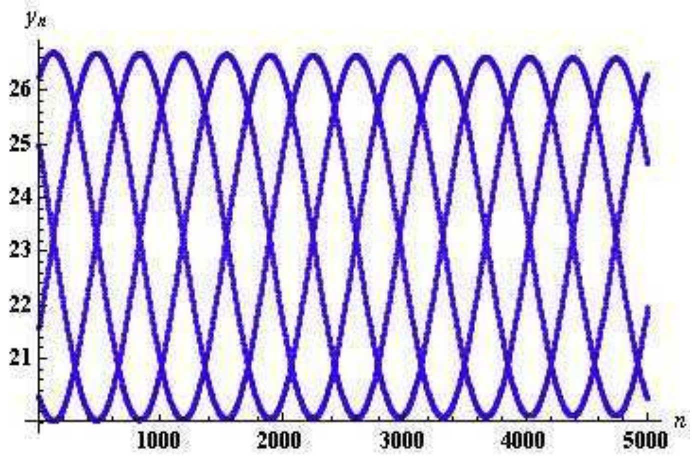

Figure 8: Plot of y_n for the system (16)

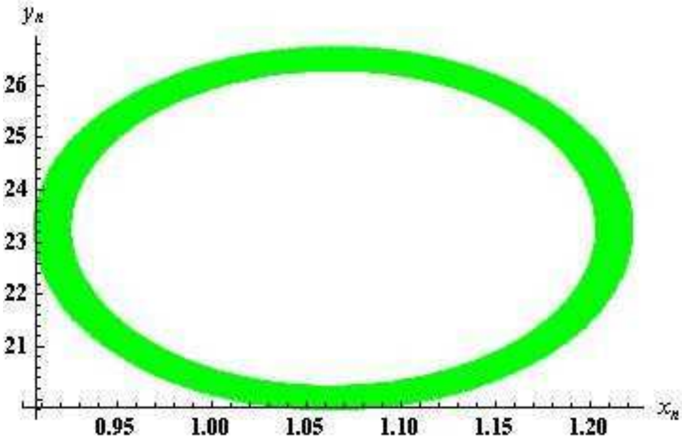

Figure 9: Phase portrait of system (16)

7 Conclusions

This work is related to the qualitative behavior of a discrete-time predator-prey model. We proved that the system (2) has a unique positive equilibrium point, which is locally asymptotically stable. The method of linearization is used to prove the local asymptotic stability of unique equilibrium point. Linear stability analysis shows that the steady states of the system (2) will be stable under the condition $\eta\gamma(\alpha\delta - 1) > (\alpha - 2)\beta(\delta - 1)^2$. The experimental verification of this necessary and sufficient condition is clear from numerical examples. The main objective of dynamical systems theory is to predict the global behavior of a system based on the knowledge of its present state. An approach to this problem consists of determining the possible global behaviors of the system and determining which initial conditions lead to these long-term behaviors. In case of nonlinear dynamical systems, it is very crucial to discuss global behavior of the system. In the paper, we prove the global asymptotic stability of the unique equilibrium point the system (2). Moreover, we investigated the rate of convergence of a solution that converges to the unique positive equilibrium point of the system (2). Some numerical examples are provided to support our theoretical results. These examples are experimental verifications of theoretical discussions. The main result of this paper is to prove the global asymptotic stability of the unique positive equilibrium point of the system (2).

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