Contemporary Methods in Mathematical Physics and Gravitation

CMMPG, vol. 1, No. 1 (2015), 9-26
http://cmmpg.eu/
ISSN 2367-6760

Step scheme of averaging method for impulsive differential inclusions with fuzzy right-hand side

Natalia Skripnik
Department of Optimal Control and Economic Cybernetics
Odessa I.I. Mechnikov National University, Dvoryanskaya str. 2
Odessa 65082 Ukraine
e-mail: natalia.skripnik@gmail.com

Abstract

In this paper the justification of step scheme of averaging method on a final interval for impulse differential inclusions with the fuzzy right-hand side is considered.

Mathematics Subject Classification 2010: 03E72, 34N27, 34A37, 34A60
Key Words: impulsive differential inclusions with fuzzy right-hand side; averaging method

Received: *February 2, 2015*
Online: *February 15, 2015*

1 Introduction

When a real world problem is transferred into a deterministic initial value problem of ordinary differential equations (ODE), namely

$$\dot{x} = f(t, x),\ x(t_0) = x_0,$$

we cannot usually be sure that the model is perfect. If the underlying structure of the model depends upon subjective choices, one way to incorporate these into the model, is to utilize the aspect of fuzziness, which leads to the consideration of fuzzy differential equations (FDE). The intricacies involved in incorporating fuzziness into the theory of ODE pose a certain disadvantage and other possibilities are being explored to address this problem. One of the approaches is to connect FDE to multivalued differential equations and examine the interconnection between them ([9, 16, 17, 24], etc). The other approach is to transform FDE into differential inclusion with the fuzzy right-hand sides so as to employ the existing theory of differential inclusions ([1, 3, 4, 7, 8, 13, 14], etc).

In this paper the second approach is used.

2 Main Definitions

Let $conv(\mathbb{R}^n)$ be the family of all nonempty compact convex subsets of $\mathbb{R}^n$ with the Hausdorff metric

$$h(F, G) = \inf\{r \geq 0 : F \subset G + S_r(0), G \subset F + S_r(0)\},$$

where $S_r(0) = \{x \in \mathbb{R}^n : ||x|| \leq r\}$, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

Let $\mathbb{E}^n$ be the family of mappings $x : \mathbb{R}^n \to [0, 1]$ satisfying the following conditions:

1) x is normal, i.e. there exists an $y_0 \in \mathbb{R}^n$ such that $x(y_0) = 1$;

2) x is fuzzy convex, i.e. $x(\lambda y + (1-\lambda)z) \geq \min\{x(y), x(z)\}$ whenever $y, z \in \mathbb{R}^n$ and $\lambda \in [0, 1]$;

3) x is upper semicontinuous, i.e. for any $y_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta(y_0, \varepsilon) > 0$ such that $x(y) < x(y_0) + \varepsilon$ whenever $||y - y_0|| < \delta,\ y \in \mathbb{R}^n$;

4) the closure of the set $\{y \in \mathbb{R}^n : x(y) > 0\}$ is compact.

Let $\hat{0}$ be the fuzzy mapping defined by

$$\hat{0}(y) = \begin{cases} 1, & y = 0, \\ 0, & y \in \mathbb{R}^n \backslash 0. \end{cases}$$

Definition 1.1. The set $\{y \in \mathbb{R}^n : x(y) \geq \alpha\}$ is called the α – level $[x]^\alpha$ of a mapping $x \in \mathbb{E}^n$ for $\alpha \in (0,1]$. The closure of the set $\{y \in \mathbb{R}^n : x(y) > 0\}$ is called the 0 - level $[x]^0$ of a mapping $x \in \mathbb{E}^n$.

Theorem 1.1 [16]. *If $x \in \mathbb{E}^n$ then*

1) $[x]^\alpha \in conv(\mathbb{R}^n)$ for all $\alpha \in [0,1]$;

2) $[x]^{\alpha_2} \subset [x]^{\alpha_1}$ for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$;

3) if $\{\alpha_k\} \subset [0,1]$ is a nondecreasing sequence converging to $\alpha > 0$, then $[x]^\alpha = \bigcap\limits_{k \geq 1} [x]^{\alpha_k}$.

Conversely, if $\{A^\alpha : \alpha \in [0,1]\}$ is the family of subsets of $\mathbb{R}^n$ satisfying conditions 1) - 3) then there exists $x \in \mathbb{E}^n$ such that $[x]^\alpha = A^\alpha$ for $\alpha \in (0,1]$ and $[x]^0 = \overline{\bigcup\limits_{\alpha \in (0,1]} A^\alpha} \subset A^0$.

Define the metric $D : \mathbb{E}^n \times \mathbb{E}^n \to [0, +\infty)$ by the equation

$$D(x,y) = \sup_{\alpha \in [0,1]} h([x]^\alpha, [y]^\alpha).$$

Let I be an interval in $\mathbb{R}$.

Definition 1.2. A mapping $F : I \to \mathbb{E}^n$ is called continuous on I if for any $\alpha \in [0,1]$ the multivalued mapping $[F(t)]^\alpha$ is continuous.

Definition 1.3 [16]. An element $G \in \mathbb{E}^n$ is called an integral of $F : I \to \mathbb{E}^n$ over I if $[G]^\alpha = \int\limits_I [F(t)]^\alpha dt$ for any $\alpha \in (0,1]$, where $(A) \int\limits_I [F(t)]^\alpha dt$ is the Aumann integral [2].

Theorem 1.2 [16]. *If the mapping $F : I \to \mathbb{E}^n$ is continuous then it is integrable over I.*

Definition 1.4 [16]. The mapping $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{E}^n$ is said to satisfy the Lipshitz condition in x if there exists such constant $\lambda \geq 0$ that

$$h([F(t,x)]^\alpha, [F(t,\bar{x})]^\alpha)) \leq \lambda ||x - \bar{x}||$$

for all $\alpha \in [0,1]$.

Definition 1.5. The fuzzy mapping $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{E}^n$ is said to be concave in x if

$$\beta[F(t,x)]^\alpha + (1-\beta)[F(t,y)]^\alpha \subset [F(t, \beta x + (1-\beta)y)]^\alpha$$

for any $\beta \in [0,1]$ and $\alpha \in [0,1]$.

In 1990 J. P. Aubin [1] and V. A. Baidosov [3, 4] entered into consideration the differential inclusions with fuzzy right-hand side. Their approach is based on transforming FDE into ordinary differential inclusions.

Consider the differential inclusion with fuzzy right-hand side

$$\dot{x} \in F(t,x),\ x(t_0) = x_0, \tag{1}$$

where $t \in I \subset \mathbb{R}$ is time, $F : I \times \mathbb{R}^n \to \mathbb{E}^n$ is a fuzzy mapping.

Definition 1.6 [7]. The absolutely continuous function $x : I \to \mathbb{R}^n,\ x(t_0) = x_0$ is called the $\alpha-$ solution of inclusion (1) if it satisfies the inclusion $\dot{x(t)} \in [F(t,x(t))]^\alpha$ almost everywhere on I.

Denote by $X_\alpha(t)$ the set of all $\alpha-$ solutions of inclusion (1) at moment t. In case the family $\{X_\alpha(t), \alpha \in [0,1]\}$ satisfies conditions of Theorem 1.1, it defines a fuzzy set $X(t)$ that is called a set of solutions of inclusion (1) at moment t.

The questions of existence of the set $X(t)$ and its properties were considered in [6, 7, 14, 15], etc.

Many processes in biology, management theory, electronics are described by means of impulse differential inclusions with the fuzzy right-hand side [5]:

$$\dot{x} \in F(t,x),\ t \neq \tau_i,\ x(0) = x_0, \tag{2}$$

$$\Delta x|_{t=\tau_i} \in I_i(x),$$

where $t \in I \subset \mathbb{R}$ is time, moments of impulse $\tau_i \in I$, $F : I \times \mathbb{R}^n \to \mathbb{E}^n,\ I_i : \mathbb{R}^n \to \mathbb{E}^n$ are fuzzy mappings.

Definition 1.7 The function $x : I \to \mathbb{R}^n,\ x(t_0) = x_0$ is called the $\alpha-$ solution of inclusion (2) if it is the $\alpha-$ solution of inclusion $\dot{x} \in F(t,x)$ on the intervals between moments of impulse, left - continuous at the moments of impulse and

$$x(\tau_i + 0) - x(\tau_i) \in [I_i(x(\tau_i))]^\alpha$$

for all i.

Denote by $X_\alpha(t)$ the set of all $\alpha-$ solutions of inclusion (2) at moment t. In case the family $\{X_\alpha(t), \alpha \in [0,1]\}$ satisfies conditions of Theorem 1.1, it defines a fuzzy set $X(t)$ that is called a set of solutions of inclusion (2) at moment t.

The questions of existence of the set $X(t)$ were considered in [5].

Many important problems of analytical dynamics are described by nonlinear differential or integro - differential equations. The absence of exact universal research methods for nonlinear systems caused the development of numerous approximate analytic and numerically - analytic methods that can be realized in effective computer algorithms.

All those methods are based on an iterative principle, i.e. either consecutive approximations of phase variables or functional series with members decreasing on size are used. It means that after the initial approximation is chosen then the additives of various order are found using iterations to approach the true solution. This approach is especially effective in investigation of the mathematical models described by nonlinear equations with small parameters. Also there exist various methods of the initial approximation choice: solving of some linear problem (the linearization method) or solving of some nonlinear but significantly easier system (often the averaging method).

Recently, the averaging methods combined with the asymptotic representations (in Poincare sense) began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations. It became possible due to the works of N. M. Krylov, N. N. Bogolyubov, Yu. A. Mitropolskij, A. M. Samojlenko, V. M. Volosov, E. A. Grebennikov, M. A. Krasnoselskiy, S.G. Krein, A. N. Filatov, etc. The application of the averaging method to optimal control problems was considered in the works of N. N. Moiseev, V. N. Lebedev, F. L. Chernousko, L. D. Akulenko, V. A. Plotnikov, etc.

Later in [10, 18, 25] the averaging schemes for differential equations with the set-valued and discontinuous right-hand side, quasidifferential equations, differential equations and inclusions with Hukuhara derivative were considered.

In [11, 19, 20, 21, 22, 23] the possibility of application of averaging method on a final interval for differential inclusions with the fuzzy right-hand side with a small parameter is proved. In [26] the scheme of full averaging for impulsive case is considered.

In this paper we will consider the justification of step scheme of averaging method on a final interval for impulse differential inclusions with the fuzzy right-hand side with a small parameter.

3 Main results

Consider the impulsive differential inclusion with fuzzy right-hand side

$$\dot{x} \in \varepsilon F(t,x),\ t \neq \tau_i,\ x(0) = x_0, \tag{3}$$

$$\Delta x|_{t=\tau_i} \in \varepsilon I_i(x).$$

Along with differential inclusion (3) we will consider the following differential inclusion with fuzzy right-hand side:

$$\dot{y} \in \varepsilon \bar{F}(t,y),\ y(0) = x_0, \tag{4}$$

where the fuzzy mapping

$$\bar{F}(t,x) = \frac{1}{\omega}\int_{j\omega}^{(j+1)\omega} F(t,x)dt + \frac{1}{\omega}\sum_{j\omega \le \tau_i < (j+1)\omega} I_i(x),\ t \in (j\omega,(j+1)\omega],\ j = 0,1,\dots, \tag{5}$$

$\omega > 0$ is the step.

Theorem 2.1. *Let in the domain* $Q = \{t \geq 0,\ x \in G \subset \mathbb{R}^n\}$, *where* G *is convex, the following conditions fulfill:*

1) fuzzy mappings $F : Q \to \mathbb{E}^n$, $I_i : G \to \mathbb{E}^n$ *are continuous, uniformly bounded by constant* M, *satisfy Lipshitz condition in* x *with constant* λ *and are concave in* x*;*

2) the quantity of moments τ_i *on the interval* $(t, t+\tau]$ *does not exceed* $\nu\tau$, *where* $\nu < \infty$;

3) for all $x_0 \in G' \subset G$ *and* $t \geq 0$ *the* α*- solutions of inclusion (4) together with a* ρ*-neighborhood belong to the domain* G *for all* $\alpha \in [0,1]$.

Then for all $L > 0$ *there exist* $\varepsilon^0(L) > 0$ *and* $C(L)$ *such that for all* $\varepsilon \in (0,\varepsilon^0]$ *and* $t \in [0, L\varepsilon^{-1}]$ *the inequality holds:*

$$D(X(t),Y(t)) < C\varepsilon, \tag{6}$$

where $X(t)$ *is the set of solutions of inclusion (3),* $Y(t)$ *is the set of solutions of inclusion (4).*

Proof. From conditions 1), 2) it follows that the fuzzy mapping $\bar{F} : Q \to \mathbb{E}^n$ is uniformly bounded by constant $M_1 = M(1+\nu)$ and satisfies Lipshitz condition in x with constant $\lambda_1 = \lambda(1+\nu)$.

Really, for $t \in (j\omega, (j+1)\omega]$, $j = 0, 1, \ldots$ we have

$$D(\bar{F}(t,x), \{\hat{0}\}) = D\left(\frac{1}{\omega}\int_{j\omega}^{(j+1)\omega} F(t,x)dt + \frac{1}{\omega}\sum_{j\omega \le \tau_i < (j+1)\omega} I_i(x), \{\hat{0}\}\right) \le$$

$$\le \frac{1}{\omega}\int_{j\omega}^{(j+1)\omega} D(F(s,x), \{\hat{0}\})ds + \frac{1}{\omega}\sum_{j\omega \le \tau_i < (j+1)\omega} D(I_i(x), \{\hat{0}\}) \le$$

$$\le M + \nu M = M(1+\nu) = M_1;$$

$$D(\bar{F}(t,x_1), \bar{F}(t,x_2)) =$$

$$= D\biggl(\frac{1}{\omega}\int_{j\omega}^{(j+1)\omega} F(s,x_1)ds + \frac{1}{\omega}\sum_{j\omega \le \tau_i < (j+1)\omega} I_i(x_1), \frac{1}{\omega}\int_{j\omega}^{(j+1)\omega} F(s,x_2)ds +$$

$$+ \frac{1}{\omega}\sum_{j\omega \le \tau_i < (j+1)\omega} I_i(x_2)\biggr) \le$$

$$\le \frac{1}{\omega}\int_{j\omega}^{(j+1)\omega} D(F(s,x_1), F(s,x_2))ds + \frac{1}{\omega}\sum_{j\omega \le \tau_i < (j+1)\omega} D(I_i(x_1), I_i(x_2)) \le$$

$$\le \lambda||x_1 - x_2|| + \lambda\nu||x_1 - x_2|| = \lambda(1+\nu)||x_1 - x_2|| = \lambda_1||x_1 - x_2||.$$

Besides the fuzzy mapping $\bar{F}(t,x)$ is concave in x. Choose any $\alpha \in [0,1]$ and $\beta \in [0,1]$, $x, y \in G$. Then

$$\beta[\bar{F}(t,x)]^\alpha + (1-\beta)[\bar{F}(t,y)]^\alpha =$$

$$= \beta\left[\frac{1}{\omega}\int_{j\omega}^{(j+1)\omega} F(s,x)ds + \frac{1}{\omega}\sum_{j\omega \le \tau_i < (j+1)\omega} I_i(x)\right]^\alpha +$$

$$+(1-\beta)\left[\frac{1}{\omega}\int_{j\omega}^{(j+1)\omega} F(s,y)ds + \frac{1}{\omega}\sum_{j\omega \le \tau_i < (j+1)\omega} I_i(y)\right]^\alpha =$$

$$= \frac{1}{\omega}\int_{j\omega}^{(j+1)\omega} (\beta[F(s,x)]^\alpha + (1-\beta)[F(s,y)]^\alpha)ds +$$

$$+ \frac{1}{T}\sum_{j\omega \le \tau_i < (j+1)\omega} (\beta[I_i(x)]^\alpha + (1-\beta)[I_i(y)]^\alpha) \subset$$

$$\subset \frac{1}{\omega}\int_{j\omega}^{(j+1)\omega}[F(s,\beta x+(1-\beta)y)]^{\alpha}ds+\frac{1}{\omega}\sum_{j\omega\le\tau_i<(j+1)\omega}[I_i(\beta x+(1-\beta)y)]^{\alpha}=$$

$$=[\bar{F}(t,\beta x+(1-\beta)y)]^{\alpha}.$$

Owing to conditions of the theorem the solutions of inclusions (3) and (4) exist [5].

Choose any $\alpha\in[0,1]$. First prove the inclusion

$$[Y(t)]^{\alpha}\subset[X(t)]^{\alpha}+S_{C\varepsilon}(0). \tag{7}$$

Let $y(t)$ be a solution of inclusion

$$\dot{y}(t)\in\varepsilon[\bar{F}(t,y(t))]^{\alpha},\ y(0)=x_0. \tag{8}$$

Divide the interval $[0,L\varepsilon^{-1}]$ with the step ω by the points $t_j=j\omega,\ j=\overline{0,m}$, where $m:\ m\omega\le L\varepsilon^{-1}<(m+1)\omega$. Denote by $t_{m+1}=L\varepsilon^{-1}$ for convenience. Then there exists a measurable selector $v(t)$ of the set-valued mapping $[\bar{F}(t,y(t))]^{\alpha}$ such that

$$y(t)=y(t_j)+\varepsilon\int_{t_j}^{t}v(s)ds,\ t\in[t_j,t_{j+1}],\ j=\overline{0,m},\ y(0)=x_0. \tag{9}$$

Consider the function

$$y^1(t)=y^1(t_j)+\varepsilon\int_{t_j}^{t}v_j(s)ds,\ t\in[t_j,t_{j+1}],\ j=\overline{0,m},\ y^1(0)=x_0, \tag{10}$$

where $v_j(t)$ is a measurable selector of the set-valued mapping $[F(t,y^1(t_j))]^{\alpha}$ such that

$$||v_j(t)-v(t)||=\min_{v\in[\bar{F}(t,y^1(t_j))]^{\alpha}}||v-v(t)||. \tag{11}$$

It is obvious that $v_j(t)$ exists because the function being minimized is continuous in v and the set $[\bar{F}(t,y^1(t_j))]^{\alpha}$ is compact.

Denote by $\delta_j=||y(t_j)-y^1(t_j)||$. For $t\in[t_j,t_{j+1}]$ using (9) and (10) we get

$$||y(t)-y(t_j)||\le M_1\omega\varepsilon,\quad ||y^1(t)-y^1(t_j)||\le M_1\omega\varepsilon. \tag{12}$$

Hence for $t \in (t_j, t_{j+1}]$ the following inequalities hold:

$$||y(t) - y^1(t_j)|| \leq ||y(t_j) - y^1(t_j)|| + ||y(t) - y(t_j)|| \leq \delta_j + \varepsilon M_1(t - t_j) \leq$$
$$\leq \delta_j + M_1\omega\varepsilon,$$

$$h([\bar{F}(t, y(t))]^\alpha, [\bar{F}(t, y^1(t_j))]^\alpha) \leq \lambda_1 ||y(t) - y^1(t_j)|| \leq \lambda_1(\delta_j + M_1\omega\varepsilon). \quad (13)$$

From (11) and (13) it follows that

$$\left|\left|\int_{t_j}^{t_{j+1}} (v(s) - v_j(s))ds\right|\right| \leq \int_{t_j}^{t_{j+1}} h([\bar{F}(s, y(s))]^\alpha, [\bar{F}(s, y^1(t_j))]^\alpha)ds \leq$$
$$\leq \lambda_1(\delta_j + M_1\omega\varepsilon)\omega. \quad (14)$$

Taking into account (9) and (10) we get

$$\delta_{j+1} \leq \delta_j + \varepsilon\lambda_1(\delta_j + \varepsilon M_1\omega)\omega = (1 + \lambda_1\omega\varepsilon)\delta_j + \lambda_1 M_1\omega^2\varepsilon^2. \quad (15)$$

So using inequalities (15) and taking into account that $\delta_0 = 0$ we get

$$\delta_1 \leq \lambda_1 M_1\omega^2\varepsilon^2,$$

$$\delta_2 \leq (1 + \lambda_1\omega\varepsilon)\delta_1 + \lambda_1 M_1\omega^2\varepsilon^2 \leq \lambda_1 M_1\omega^2\varepsilon^2((1 + \lambda_1\omega\varepsilon) + 1).$$

Proceeding by induction we get

$$\delta_{j+1} \leq \lambda_1 M_1\omega^2\varepsilon^2((1 + \lambda_1\omega\varepsilon)^i + (1 + \lambda_1\omega\varepsilon)^{i-1} + \cdots + 1) =$$

$$= M_1\omega\varepsilon\left((1 + \lambda_1\omega\varepsilon)^{i+1} - 1\right) \leq M_1\omega\varepsilon\left((1 + \lambda_1\omega\varepsilon)^{\frac{L}{\omega\varepsilon}} - 1\right) \leq M_1\omega\varepsilon(e^{\lambda_1 L} - 1). \quad (16)$$

From (12) and (16) we have:

$$||y(t) - y^1(t)|| \leq ||y(t) - y(t_j)|| + ||y(t_j) - y^1(t_j)|| + ||y^1(t_j) - y^1(t)|| \leq$$

$$\leq 2M_1\omega\varepsilon + M_1\omega\varepsilon(e^{\lambda_1 L} - 1) \leq M_1\omega\varepsilon(e^{\lambda_1 L} + 1). \quad (17)$$

Using (5) we get

$$[\bar{F}(t, y^1(t_j))]^\alpha = \frac{1}{\omega}\int_{t_j}^{t_{j+1}} [F(s, y^1(t_j))]^\alpha ds + \frac{1}{\omega}\sum_{t_j\le\tau_i<t_{j+1}} [I_i(y^1(t_j))]^\alpha. \tag{18}$$

Therefore there exist vectors $p_{ij} \in [I_i(y^1(t_j))]^\alpha$ and a measurable selector $u_j(t) \in [F(t, y^1(t_j))]^\alpha$ such that

$$\int_{t_j}^{t_{j+1}} v_j(s)ds = \int_{t_j}^{t_{j+1}} u_j(s)ds + \sum_{t_j\le\tau_i<t_{j+1}} p_{ij}. \tag{19}$$

Consider the function

$$x^1(t) = x^1(t_j) + \varepsilon\int_{t_j}^{t} u_j(s)ds + \varepsilon\sum_{t_j\le\tau_i<t} p_{ij},\ t \in (t_j, t_{j+1}],\ j = \overline{0, m},\ x^1(0) = x_0. \tag{20}$$

As $x^1(0) = y^1(0)$, then from (10), (20), (19) and (12) it follows that for $j = \overline{0, m}$, $t \in (t_j, t_{j+1}]$

$$x^1(t_j) = y^1(t_j),\ ||x^1(t) - x^1(t_j)|| \le M_1\omega\varepsilon,\ ||x^1(t) - y^1(t)|| \le 2M_1\omega\varepsilon. \tag{21}$$

Let us show that there exists a solution $x(t)$ of inclusion

$$\dot{x}(t) \in \varepsilon[F(t, x(t))]^\alpha,\ t \ne \tau_i,\ x(0) = x_0, \tag{22}$$

$$\Delta x|_{t=\tau_i} \in \varepsilon[I_i(x)]^\alpha$$

close enough to $x^1(t)$.

Let $\theta_1, \ldots, \theta_p$ be the moments of impulses τ_i, that get in the interval $(t_j, t_{j+1}]$. For convenience denote by $\theta_0 = t_j$, $\theta_{p+1} = t_{j+1}$. Let $\mu_k^+ = ||x^1(\theta_k + 0) - x(\theta_k + 0)||$, $\mu_k^- = ||x^1(\theta_k) - x(\theta_k)||$, $k = \overline{0, p+1}$.

Let $\rho(x, A) = \min_{a\in A} ||x - a||$ be the distance from point $x \in \mathbb{R}^n$ to the set $A \subset \mathbb{R}^n$. Using Lipschitz condition we get

$$\rho\left(x^{1\prime}(t), \varepsilon[F(t, x^1(t))]^\alpha\right) \le h\left(\varepsilon[F(t, y^1(t_j))]^\alpha, \varepsilon[F(t, x^1(t))]^\alpha\right) \le$$

$$\le \varepsilon\lambda||x^1(t) - y^1(t_j)|| \le \lambda M_1\omega\varepsilon^2 = \eta^*,$$

$$\rho\left(\Delta x^1|_{t=\theta_k}, \varepsilon[I_i(x^1(\theta_k))]^\alpha\right) \le h\left(\varepsilon[I_i(y^1(t_j))]^\alpha, \varepsilon[I_i(x^1(\theta_k))]^\alpha\right) \le$$

$$\leq \varepsilon\lambda||y^1(t_j) - x^1(\theta_k)|| \leq \lambda M_1 \omega\varepsilon^2 = \eta^*.$$

From A. F. Filippov's theorem it follows that there exists a solution $x(t)$ of inclusion (22) such that for $t \in (\theta_k, \theta_{k+1}]$ the inequality holds

$$||x(t) - x^1(t)|| \leq \mu_k^+ e^{\varepsilon\lambda(t-\theta_k)} + \int_{\theta_k}^{t} e^{\varepsilon\lambda(t-s)}\eta^* ds.$$

Denote by $\gamma_k = \theta_{k+1} - \theta_k \leq \omega,\ \gamma_0 + \cdots + \gamma_p = \omega$. Then

$$\mu_{k+1}^- \leq \mu_k^+ e^{\varepsilon\lambda\gamma_k} + \frac{\eta^*}{\lambda\varepsilon}\left(e^{\lambda\omega\varepsilon} - 1\right). \tag{23}$$

When getting over the impulse point we have

$$\mu_{k+1}^+ \leq \mu_{k+1}^- + \varepsilon h\left([I_i(y^1(t_j))]^\alpha, [I_i(x(\theta_{k+1}))]^\alpha\right) \leq$$

$$\leq \mu_{k+1}^- + \varepsilon h\left([I_i(x^1(\theta_{k+1}))]^\alpha, [I_i(x(\theta_{k+1}))]^\alpha\right) + \varepsilon h\left([I_i(x^1(t_j))]^\alpha, [I_i(y^1(\theta_{k+1}))]^\alpha\right) \leq$$

$$\leq \mu_{k+1}^- + \varepsilon\lambda\mu_{k+1}^- + \varepsilon h\left([I_i(x^1(t_j))]^\alpha, [I_i(x^1(\theta_{k+1}))]^\alpha\right) \leq$$

$$\leq (1 + \lambda\varepsilon)\mu_{k+1}^- + \eta^*. \tag{24}$$

From (23) and (24) it follows that

$$\mu_{k+1}^+ \leq (1 + \lambda\varepsilon)e^{\varepsilon\lambda\gamma_k}\mu_k^+ + \beta, \quad \beta = \frac{\eta^*}{\lambda\varepsilon}(1 + \lambda\varepsilon)\left(e^{\lambda\omega\varepsilon} - 1\right) + \eta^*.$$

Therefore

$$\mu_1^+ \leq (1 + \lambda\varepsilon)e^{\lambda\varepsilon\gamma_0}\mu_0^+ + \beta \leq (1 + \lambda\varepsilon)e^{\lambda\omega\varepsilon}\mu_0^+ + \beta,$$

$$\mu_2^+ \leq (1 + \lambda\varepsilon)e^{\varepsilon\lambda\gamma_1}\mu_1^+ + \beta \leq (1 + \lambda\varepsilon)^2 e^{\varepsilon\lambda(\gamma_0+\gamma_1)}\mu_0^+ +$$

$$+\beta(1 + \lambda\varepsilon)e^{\varepsilon\lambda\gamma_1} + \beta \leq (1 + \lambda\varepsilon)^2 e^{\lambda\omega\varepsilon}\mu_0^+ + \beta\left((1 + \lambda\varepsilon)e^{\lambda\omega\varepsilon} + 1\right), \text{ etc.,}$$

$$\mu_{k+1}^+ \leq (1 + \lambda\varepsilon)^{k+1}e^{\lambda\omega\varepsilon}\mu_0^+ + \beta\left(e^{\lambda\omega\varepsilon}((1 + \lambda\varepsilon)^k + \ldots + (1 + \lambda\varepsilon)) + 1\right) =$$

$$= (1 + \lambda\varepsilon)^{k+1}e^{\lambda\omega\varepsilon}\mu_0^+ + \beta\left(e^{\lambda\omega\varepsilon}\frac{(1 + \lambda\varepsilon)^k - 1}{\lambda\varepsilon}(1 + \lambda\varepsilon) + 1\right) \leq$$

$$\leq e^{\lambda(1+\nu)\omega\varepsilon}\mu_0^+ + \eta^*\left(\frac{1 + \lambda\varepsilon}{\lambda\varepsilon}(e^{\lambda\omega\varepsilon} - 1) + 1\right)\left(e^{\lambda\omega\varepsilon}\frac{e^{\lambda\nu\omega\varepsilon} - 1}{\lambda\varepsilon}(1 + \lambda\varepsilon) + 1\right) =$$

$$= \kappa\mu_0^+ + \beta_1,$$

where

$$\kappa = e^{\lambda(1+\nu)\omega\varepsilon},$$

$$\beta_1 = M_1\omega\varepsilon\left(\frac{1+\lambda\varepsilon}{\lambda\varepsilon}(e^{\lambda\omega\varepsilon}-1)+1\right)\left(e^{\lambda\omega\varepsilon}\left(e^{\lambda\nu\omega\varepsilon}-1\right)(1+\lambda\varepsilon)+\lambda\varepsilon\right).$$

Therefore,

$$\delta^+_{j+1} = ||x(t_{j+1}) - x^1(t_{j+1})|| \le \kappa\delta^+_j + \beta_1.$$

We obtain the sequence of inequalities

$$\delta^+_0 = 0,\ \delta^+_1 \le \beta_1,\ \delta^+_2 \le \kappa\beta_1 + \beta_1 = (\kappa+1)\beta_1, \dots,$$

$$\delta^+_{j+1} \le (\kappa^j + \dots + 1)\beta_1 = \frac{\kappa^{j+1}-1}{\kappa-1}\beta_1 \le$$

$$\le M_1\omega\frac{e^{\lambda L(1+\nu)}-1}{e^{\lambda(1+\nu)\omega\varepsilon}-1}\left((1+\lambda\varepsilon)\frac{e^{\lambda\omega\varepsilon}-1}{\lambda\varepsilon}+1\right)\left(e^{\lambda\omega\varepsilon}\left(e^{\lambda\nu\omega\varepsilon}-1\right)(1+\lambda\varepsilon)+\lambda\varepsilon\right)\varepsilon.$$

As

$$\lim_{\varepsilon\to 0}\left((1+\lambda\varepsilon)\frac{e^{\lambda\omega\varepsilon}-1}{\lambda\varepsilon}+1\right) = \omega+1$$

and

$$\lim_{\varepsilon\to 0}\frac{e^{\lambda\omega\varepsilon}\left(e^{\lambda\nu\omega\varepsilon}-1\right)(1+\lambda\varepsilon)+\lambda\varepsilon}{e^{\lambda(1+\nu)\omega\varepsilon}-1} = \lim_{\varepsilon\to 0}\frac{e^{\lambda\omega\varepsilon}\frac{e^{\lambda\nu\omega\varepsilon}-1}{\lambda\varepsilon}(1+\lambda\varepsilon)+1}{\frac{e^{\lambda(1+\nu)\omega\varepsilon}-1}{\lambda\varepsilon}} =$$

$$= \frac{\nu\omega+1}{(1+\nu)\omega},$$

then

$$\delta^+_{j+1} \le C_0\varepsilon$$

for $\varepsilon \le \varepsilon_2$.

Therefore, for $t \in (t_j, t_{j+1}]$ the following inequality holds:

$$||x(t) - x^1(t)|| \le ||x(t) - x(t_j)|| + ||x(t_j) - x^1(t_j)|| + ||x^1(t) - x^1(t_j)|| \le$$

$$\le M(1+\nu)\omega\varepsilon + M_1\omega\varepsilon + C_0\varepsilon = (2M_1\omega + C_0)\varepsilon. \quad (25)$$

In view of the inequalities (17), (21) and (25) we get that

$$||x(t) - y(t)|| \le C_1\varepsilon, \quad (26)$$

where $C_1 = M_1\omega(e^{\lambda_1 L} + 5) + C_0$ and the first part of the theorem is proved.

Now let us proof that the following inclusion holds:

$$[X(t)]^\alpha \subset [Y(t)]^\alpha + S_{C\varepsilon}(0). \quad (27)$$

Let $x(t)$ be the solution of inclusion (22). Divide the interval $[0, L\varepsilon^{-1}]$ on the partial intervals with the step ω by the points $t_j = j\omega,\ j = \overline{0, m}$, where $m:\ m\omega \le L\varepsilon^{-1} < (m+1)\omega$. Denote by $t_{m+1} = L\varepsilon^{-1}$ for convenience. Then there exist a measurable selector $u(t)$ of the set-valued mapping $[F(t, x(t))]^\alpha$ and vectors $q_i \in [I_i(x(\tau_i))]^\alpha$ such that

$$x(t) = x(t_j) + \varepsilon \int_{t_j}^{t} u(s)ds + \varepsilon \sum_{t_j \le \tau_i < t} q_i,\ t \in (t_j, t_{j+1}],\ j = \overline{0, m},\ x(0) = x_0. \quad (28)$$

Consider the function

$$x^1(t) = x^1(t_j) + \varepsilon \int_{t_j}^{t} u_j(s)ds + \varepsilon \sum_{t_j \le \tau_i < t} q_{ij},\ t \in (t_j, t_{j+1}],\ j = \overline{0, m},\ x^1(0) = x_0, \quad (29)$$

where the measurable selector $u_j(t)$ of the set-valued mapping $[F(t, x^1(t_j))]^\alpha$ and vectors $q_{ij} \in [I_i(x^1(t_j))]^\alpha$ satisfy the conditions

$$||u_j(t) - u(t)|| = \min_{u \in [F(t, x^1(t_j))]^\alpha} ||u - u(t)||,\ ||q_{ij} - q_i|| = \min_{q \in [I_i(x^1(t_j))]^\alpha} ||q - q_i||. \quad (30)$$

Denote by $\delta_j = ||x(t_j) - x^1(t_j)||$. For $t \in (t_j, t_{j+1}]$ using (28) and (29) we get

$$||x(t) - x(t_j)|| \le \varepsilon M(t - t_j) + \varepsilon M\nu(t - t_j) = \varepsilon M_1(t - t_j) \le M_1\omega\varepsilon,$$

$$||x^1(t) - x^1(t_j)|| \le \varepsilon M(t - t_j) + \varepsilon M\nu(t - t_j) = \varepsilon M_1(t - t_j) \le M_1\omega\varepsilon. \quad (31)$$

Therefore for $t \in (t_j, t_{j+1}]$ the following inequalities hold:

$$||x(t) - x^1(t_j)|| \le ||x(t_j) - x^1(t_j)|| + ||x(t) - x(t_j)|| \le \delta_j + \varepsilon M_1(t - t_j),$$

$$||u(t) - u_j(t)|| \le h\left([F(t, x(t))]^\alpha, [F(y, x^1(t_j))]^\alpha\right) \le$$

$$\le \lambda||x(t) - x^1(t_j)|| \le \lambda(\delta_j + \varepsilon M_1(t - t_j)) \le \lambda(\delta_j + \omega\varepsilon), \tag{32}$$

$$||q_i - q_{ij}|| \le h\left([I_i(x(\tau_i))]^\alpha, [I_i(x^1(t_j))]^\alpha\right) \le \lambda||x(\tau_i) - x^1(t_j)|| \le$$

$$\le \lambda(\delta_j + \varepsilon M_1(\tau_i - t_j)) \le \lambda(\delta_j + \varepsilon M_1(t - t_j)) \le \lambda(\delta_j + \omega\varepsilon).$$

From (28), (29) and (32) we have

$$\delta_{j+1} \le \delta_j + \varepsilon\lambda\left(\delta_j + M_1\omega\varepsilon\right)\omega + \varepsilon\lambda\left(\delta_j + M_1\omega\varepsilon\right)\nu\omega =$$

$$= (1 + \lambda_1\omega\varepsilon)\delta_j + \lambda_1 M_1\omega^2\varepsilon^2. \tag{33}$$

As $\delta_0 = 0$ then from inequality (33) we have

$$\delta_1 \le \lambda_1 M_1\omega^2\varepsilon^2,$$

$$\delta_2 \le (1 + \lambda_1\omega\varepsilon)\delta_1 + \lambda_1 M_1\omega^2\varepsilon^2 \le \lambda_1 M_1\omega^2\varepsilon^2((1 + \lambda_1\omega\varepsilon) + 1), \text{ etc.},$$

$$\delta_{j+1} \le \lambda_1 M_1\omega^2\varepsilon^2((1 + \lambda_1\omega\varepsilon)^i + (1 + \lambda_1\omega\varepsilon)^{i-1} + \cdots + 1) =$$

$$= M_1\omega\varepsilon\left((1 + \lambda_1\omega\varepsilon)^{i+1} - 1\right) \le M_1\omega\varepsilon\left((1 + \lambda_1\omega\varepsilon)^{\frac{L}{\omega\varepsilon}} - 1\right) \le M_1\omega\varepsilon(e^{\lambda_1 L} - 1). \tag{34}$$

So using inequalities (32) we get

$$||x(t) - x^1(t))|| \le ||x(t) - x(t_j)|| + ||x(t_j) - x^1(t_j)|| + ||x^1(t_j) - x^1(t)|| \le$$

$$\le 2M_1\omega\varepsilon + M_1\omega\varepsilon(e^{\lambda_1 L} - 1) = M_1\omega\varepsilon(e^{\lambda_1 L} + 1). \tag{35}$$

From (5) it follows that

$$[\bar{F}(t, x^1(t_j))]^\alpha = \frac{1}{\omega}\int_{j\omega}^{(j+1)\omega} [F(s, x^1(t_j))]^\alpha ds +$$

$$+ \frac{1}{\omega}\sum_{j\omega \le \tau_i < (j+1)\omega} [I_i(x^1(t_j))]^\alpha, \; t \in (j\omega, (j+1)\omega]$$

therefore there exists a measurable selector $v_j(t) \in [\bar{F}(t, x^1(t_j))]^\alpha$ such that

$$\int_{t_j}^{t_{j+1}} v_j(s)ds = \int_{t_j}^{t_{j+1}} u_j(s)ds + \sum_{t_j \le \tau_i < t_{j+1}} q_{ij}. \tag{36}$$

Consider the function

$$y^1(t) = y^1(t_j) + \varepsilon \int_{t_j}^{t} v_j(s)ds, \ j = 0, 1, ..., \ y^1(0) = x_0. \tag{37}$$

As $x^1(0) = y^1(0)$, then from (29), (37) and (36)for $j = \overline{1, m}$ we have

$$x^1(t_j) = y^1(t_j), \ ||y^1(t) - y^1(t_j)|| \le \varepsilon M_1(t - t_j) \le M_1\omega\varepsilon, \tag{38}$$

$$||y^1(t) - x^1(t)|| \le 2M_1\omega\varepsilon.$$

Let us show that there exists a solution $y(t)$ of inclusion (8) close enough to $y^1(t)$.

Using Lipschitz condition we get

$$\rho\left(y^{1\prime}(t), \varepsilon[\bar{F}(t, y^1(t))]^\alpha\right) \le \varepsilon h\left([\bar{F}(t, x^1(t_j))]^\alpha, [\bar{F}(t, y^1(t))]^\alpha\right) \le$$

$$\le \varepsilon\lambda_1||y^1(t) - x^1(t_j)|| \le \lambda_1 M_1\omega\varepsilon^2 = \bar{\eta}^*.$$

From A. F. Filippov's theorem it follows that there exists a solution $x(t)$ of inclusion (22) such that for all t the inequality holds

$$||y(t) - y^1(t)|| \le ||y(0) - y^1(0)||e^{\varepsilon\lambda_1 t} + \int_0^t e^{\varepsilon\lambda_1(t-s)}\bar{\eta}^* ds =$$

$$= \frac{\bar{\eta}^*}{\lambda_1\varepsilon}\left(e^{\varepsilon\lambda_1 t} - 1\right) = M_1\omega\left(e^{\varepsilon\lambda_1 t} - 1\right)\varepsilon \le M_1\omega\left(e^{\lambda_1 L} - 1\right)\varepsilon. \tag{39}$$

In view of the inequalities (35),(38) and (39) we get that

$$||x(t) - y(t)|| \le C_2\varepsilon, \tag{40}$$

where $C_1 = 2M_1\omega(e^{\lambda_1 L} + 1)$ and inclusion (27) is proved. Choosing $C = \max(C_1, C_2)$, we get the conclusion of the theorem.

4 Conclusion

The requirement of a concaveness of the right side of initial inclusion is rather strong and is necessary for ensuring the convexity of $\alpha-$ solutions sets of initial and averaged inclusions for any $\alpha \in [0,1]$. If the solution is considered in the space Σ^n of mappings $x : \mathbb{R}^n \to [0,1]$ that satisfy conditions 1), 3) and 4) from definition of the space $\mathbb{E}^n$ than the requirement of a concaveness is possible to reject, thus the statements of theorem will still hold.

References

[1] J.-P. Aubin, Fuzzy differential inclusions, *Problems of control and information theory,* **19**, 1 (1990), 55 - 67.

[2] R. J. Aumann, Integrals of set - valued functions, *J. Math. Anal. Appl.*, 12 (1965), 1 - 12.

[3] V. A. Baĭdosov, Differential inclusions with fuzzy right-hand side, *Dokl. Akad. Nauk SSSR,* **309** , 4 (1989), 781–783 (in Russian).

[4] V. A. Baĭdosov, Fuzzy differential inclusions, *Prikl. Mat. Mekh.,* **54**, 1 (1990), 12–17 (in Russian).

[5] M. Guo, X. Xue and R. Li, Impulsive functional differential inclusions and fuzzy population models, *Fuzzy Sets and Systems,* **138** (2003), 601 - 615.

[6] E. Hullermeier, Towards modelling of fuzzy functions, *EUFIT'95*, (1995), 150 - 154.

[7] E. Hullermeier, An approach to modelling and simulation of uncertain dynamical system, *Internat. J. Uncertainty, Fuzziness Knowledge - Based Systems* , 7 (1997), 117 - 137.

[8] E. Hullermeier, A fuzzy simulation method, *First International ICSC Symposium on Intelligent Industrial Automation (IIA'96) and Soft Computing (SOCO'96)*, Reading, United Kingdom, 1996.

[9] O. Kaleva, The Cauchy problem for fuzzy differential equations, *Fuzzy Sets and Systems,* **35**(1990), 389 - 396.

[10] M. Kisielewicz, Differential inclusion and optimal control, Warszawa: PWN, 1991.

[11] T. A. Komleva, The full averaging of 2π-periodic fuzzy linear differential inclusions, *Singularitas (MUK Publications), bf 1*, 1 (2015), 22 - 31.

[12] V. Laksmikantham, Set differential equations versus fuzzy differential equations, *Applied Mathematics and Computation,* **164** , (2005), 277 - 294.

[13] V. Lakshmikantham, T. Granna Bhaskar and J. Vasundhara Devi, Theory of set differential equations in metric spaces, Cambridge: Cambridge Scientific Publishers, 2006.

[14] V. Laksmikantham and R. N. Mohapatra, Theory of Fuzzy differential Equations and Inclusions, London: Taylor and Francis Publishers, 2003.

[15] V. Laksmikantham and A. A. Tolstonogov, Existence and interrelation between set and fuzzy differential equations, *Nonlinear Analysis,* **55**(2003), 255 - 268.

[16] J. K. Park and H. K. Han, Existence and uniqueness theorem for a solution of fuzzy differential equations, *Internat. J. Math. and Math. Sci.,* **22**, 2 (1999), 271 - 279.

[17] J. Y. Park and H. K. Han, Fuzzy differential equations, *Fuzzy Sets and Systems,* **110**(2000), 69 - 77.

[18] N. A. Perestyuk, V. A. Plotnikov, A. M. Samoilenko and N. V. Skripnik, Differential equations with impulse effects: multivalued right-hand sides with discontinuities (De Gruyter Studies in Mathematics: 40), Berlin/Boston: Walter De Gruyter GmbH Co., 2011.

[19] A. V. Plotnikov, T. A. Komleva and L. I. Plotnikova, On the averaging of differential inclusions with fuzzy right-hand side when the average of the right-hand side is absent, *Iranian Journal of Optimization,* **2**, 3 (2010), 506 - 517.

[20] A. V. Plotnikov, Averaging of fuzzy integrodifferential inclusions, *Int. J. Control Science and Engineering,* **1**, 1 (2011), 8 - 14.

[21] A. V. Plotnikov, Averaging of fuzzy controlled differential inclusions with terminal criterion of quality, *Nonlin. Oscill.,* **16**, 1 (2013), 105 - 110.

[22] A. V. Plotnikov and T. A. Komleva, Full averaging of control fuzzy integrodifferential inclusions with terminal criterion of quality, *Int. J. Control Science and Engineering,* **3**, 2 (2013), 68 - 72.

[23] A. V. Plotnikov and T. A. Komleva, The partial averaging of fuzzy differential inclusions on finite interval, *Int. J. Diff.Equat., (2014)* doi: 10.1155/2014/307941.

[24] A. V. Plotnikov and N. V. Skripnik, Differential equations with "clear" and fuzzy multivalued right-hand side. Asymptotics methods, Odessa: AstroPrint, 2009 (in Russian).

[25] V. A. Plotnikov, A. V. Plotnikov and A. N. Vityuk, Differential equations with a multivalued right-hand side: Asymptotic methods, Odessa: Astroprint, 1999 (in Russian).

[26] N. Skripnik, Averaging of impulsive differential inclusions with fuzzy right-hand side, *Ukr. Math. J.* (in press) (in Russian).