BULLETIN OF INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN 1840-4367
Vol. 2(2012), 173-184

Former
BULLETIN OF SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

EXTERNALLY EQUITABLE COLORING IN GRAPHS

D. Lakshmanaraj and V. Swaminathan

ABSTRACT. Let $G = (V, E)$ be a simple graph. A partition of $V(G)$ into independent,externally equitable sets is called externally equitable proper color partition of G or externally equitable proper coloring of G. The minimum cardinality of an externally equitable proper coloring of G is called externally equitable chromatic number of G and is denoted by $\chi_{ee}(G)$. Since $\Pi = \{\{u_1\}, \{u_2\}, \cdots, \{u_n\}\}$ where $V(G) = \{u_1, u_2, \cdots, u_n\}$ is an externally equitable proper coloring of G, externally equitable proper color partition exists in any graph G . In this paper, this new parameter is introduced and studied.

1. Introduction

The concept of equitability has been widely studied in coloring. A proper color partition is said to be equitable if the cardinalities of the color classes differ by at most one. E. Sampathkumar introduced degree equitability in graphs. A subset S of the vertex set of a graph is said to be degree equitable if the degrees of any two vertices of S differ by at most one. Arumugam et. al. [**2**] studied degree equitable sets and degree equtable proper coloring of vertices of a graph. K.M.Dharmalingam defined degree equitability and out degree equitability studied dominating sets which are (i) degree equitable and (ii) out degree equitable. A subset S of the vertex set V of a graph G is said to be externally equitable, if for any $x, y \in V - S$, $||N(x) \cap S| - |N(y) \cap S|| \leqslant 1$.

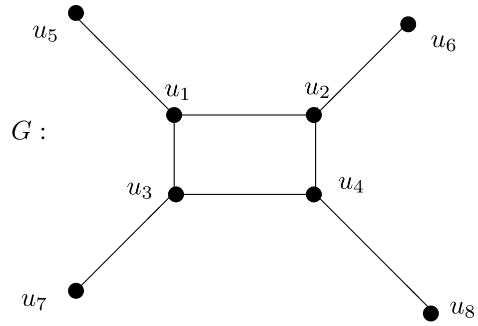
2. Main Results

DEFINITION 2.1. A partition of $V(G)$ into independent,externally equitable sets is called externally equitable proper color partition of G or externally equitable proper coloring of G. The minimum cardinality of an externally equitable proper coloring of G is called externally equitable chromatic number of G and is denoted

2010 *Mathematics Subject Classification.* Primary 05C.
Key words and phrases. externally equitable sets, externally equitable proper coloring, externally equitable chromatic number.

by $\chi_{ee}(G)$. Since $\Pi = \{\{u_1\}, \{u_2\}, \cdots, \{u_n\}\}$ where $V(G) = \{u_1, u_2, \cdots, u_n\}$ is an externally equitable proper coloring of G, externally equitable proper color partition exists in any graph G .

ILLUSTRATION 2.1.



$\{\{u_5, u_6, u_7, u_8\}, \{u_1\}, \{u_2\}, \{u_3\}, \{u_4\}\}$ is an externally equitable independent partition of $V(G)$.

REMARK 2.1. Since any $\chi_{ee}-$ partition of G is a proper color partition of G, $\chi(G) \leqslant \chi_{ee}(G)$ for any graph G.

2.1. $\chi_{ee}-$ proper color partition for standard graphs.

OBSERVATION 2.1. (1) $\chi_{ee}(K_n) = n$

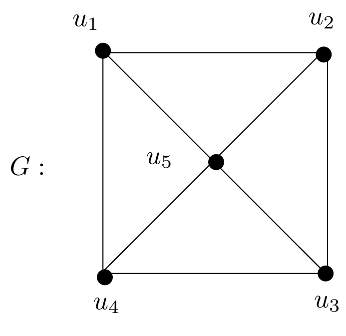
(2) $\chi_{ee}(P_n) = \chi(P_n) = 2.$

(3) $\chi_{ee}(C_n) = \chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$

(4) $\chi_{ee}(K_{1,n}) = \chi(K_{1,n}) = 2.$

THEOREM 2.1. $\chi_{ee}(W_{n+1}) = \begin{cases} n+1 & \textit{when } n \geqslant 7 \\ 4 & \textit{when } n = 3 \textit{ or } 5 \\ 3 & \textit{when } n = 4 \textit{ or } 6 \end{cases}$

PROOF. When $n = 3$, $W_4 = K_4$ and hence $\chi_{ee}(W_4) = 4$.
When $n = 4$, W_5 is



It is easily seen that $\{\{u_1, u_3\}, \{u_2, u_4\}, \{u_5\}$ is a χ_{ee}-partition of W_5. Therefore $\chi_{ee}(W_5) = 3$.

When $n = 5$, W_6 is

It is easily seen that $\{\{u_1, u_3\}, \{u_2, u_4\}, \{u_5\}, \{u_6\}\}$ is a χ_{ee}-partition of W_6. Therefore $\chi_{ee}(W_6) = 4$.

When $n = 6$, W_7 is

It is easily seen that $\{\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}, \{u_7\}\}$ is a χ_{ee}-partition of W_7. Therefore $\chi_{ee}(W_6) = 3$. Let $n \geqslant 7$. Consider W_{n+1}. let $\{u_1, u_2, \ldots, u_n\}$ be the vertices on the cycle of W_{n+1} and v be the central vertex. In any χ_{ee}-partition of W_{n+1} , $\{v\}$ is an element of the partition. Let V_1 be any other element of the partition. Then $|N(v) \cap V_1| = |V_1|$. Since for any $x \in V(W_{n+1}) - V_1$, $x \neq v$, $|N(x) \cap V_1| \leqslant 2$, we get that $|V_1| \leqslant 3$. If V_1 contains $u_{i+1}, u_{i+3}, u_{i+5}$, then $|N(u_{i+7} \cap V_1| = 0$. A similar argument shows that $|V_1| \neq 2$. Therefore $|V_1| = 1$. Therefore $\chi_{ee}(W_{n+1}) = n + 1$. □

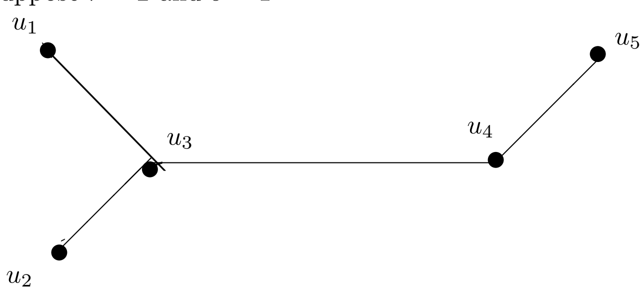
Theorem 2.2. $\chi_{ee}(D_{r,s}) = \begin{cases} \max\{r,s\} + 2 & when\ |r-s| \geqslant 2 \\ 3 & when\ |r-s| \leqslant 1 and\ r,s \geqslant 3 \\ 2 & when\ r = 2, s = 1\ or\ 2 \\ & \ or\ r = 1, s = 1\ or\ s = 2 \end{cases}$

Proof. **Case(i):** $|r-s| \geqslant 2$. Let $r = max\{r, s\}$. Let u, v be the centers of $D_{r,s}$ and let $u_1, u_2, \ldots, u_r$ be the pendant vertices at u and $v_1, v_2, \ldots, v_s$ be the pendant vertices at v. Let $\pi = \{V_1, V_2, \ldots, V_t\}$ be a χ_{ee}-partition of $D_{r,s}$. If V_1 contains two pendants at u, then any pendant at v not in V_1 will have no neighbours in V_1 and $u \notin V_1$ has two neighbours in V_1, a contradiction. Therefore V_1 contains all pendant vertices at v. Suppose V_1 does not contain a pendant at u. Then that pendant at u will have no neighbour in V_1, a contradiction since u has two neighbours in V_1. Therefore V_1 contains all pendants at u. Then $|N(u) \cap V_1| = r$ and $|N(v) \cap V_1| = s$ and $|r - s| \geqslant 2$, a contradiction. Therefore, V_1 cannot contain two pendants at u. Similarly, V_1 cannot contain two pendant at v. Suppose $V_1 = \{u_i, v_j\}$ where $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$. Then V_1 is externally equitable independent.Thus, $V_1 = \{u_1, v_1\}$, $V_2 = \{u_2, v_2\}$, $V_3 = \{u_3, v_3\}, \cdots, V_s = \{u_s, v_s\}$ are elements of any χ_{ee}-partition of $D_{r,s}$. The remaining pendants at u and the two centers must appear as singletons in π. Therefore $\chi_{ee}(D_{r,s}) = max\{r, s\} + 2$.

Case (ii): $|r - s| \leqslant 1$ and $r, s \geqslant 3$. Let $\pi = \{\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s\}, \{u\}, \{v\}\}$. Then π is an externally equitable independent partition of $D_{r,s}$. Therefore $\chi_{ee}(D_{r,s}) \leqslant 3$. Suppose $\pi = \{V_1, V_2\}$ be a χ_{ee}-partition of $D_{r,s}$. Clearly $u \in V_1$ and $v \in V_2$. Therefore V_2 contains all the pendants at u and V_1 contains all the pendants at v. Then $|N(u_1) \cap V_1| = 1$ and $|N(v) \cap V_1| = r$, a contradiction, since $r \geqslant 3$. Therefore $\chi_{ee}(D_{r,s}) \geqslant 3$. Therefore $\chi_{ee}(D_{r,s}) = 3$.

Case(iii): Suppose $r = 2$ and $s = 1$

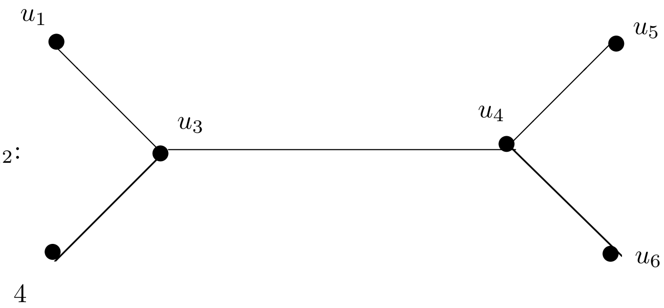
$D_{2,1}$:



Here $\pi = \{\{u, v_1\}, \{v, u_1, u_2\}\}$ is a χ_{ee}-partition of $D_{2,1}$.Therefore $\chi_{ee}(D_{2,1}) = 2$.

Suppose $r = 2$ and $s = 2$

$D_{2,2}$:

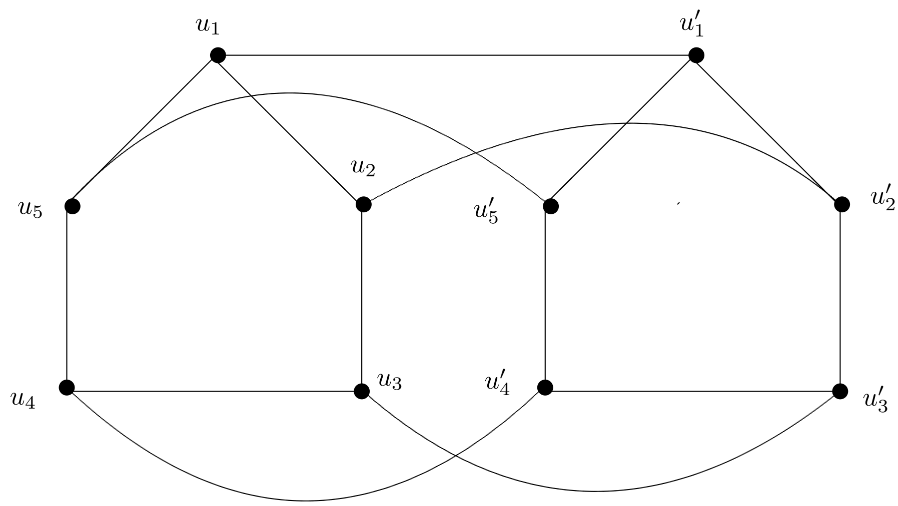


Then $\pi = \{\{u, v_1, v_2\}, \{v, u_1, u_2\}\}$ is a χ_{ee}-partition of $D_{2,2}$.
Therefore $\chi_{ee}(D_{2,2}) = 2$. When $r = 1,\ s = 1$ then $\chi_{ee}(D_{1,1}) = P_4$ and $\chi_{ee}(P_4) = 2$. □

Remark 2.2. There exists regular graph G such that $\chi_{ee}(G) > \chi(G)$.

For: let

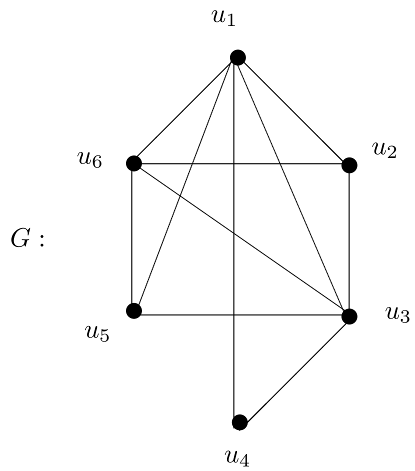
G :



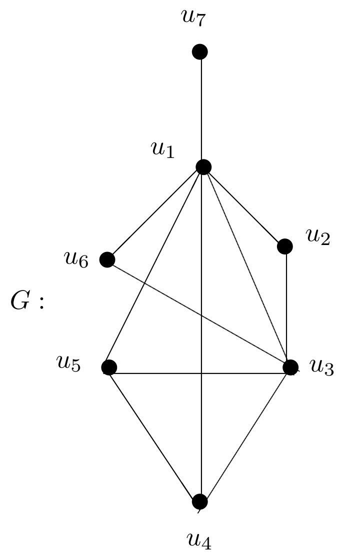
$\pi = \{\{u_1, v_3\}, \{u_2, v_4\}, \{u_3, v_5\}, \{u_4, v_1\}, \{u_5, v_2\}\}$ is an externally equitable independent partition of $V(G)$. $\chi(G) = 3$, $\chi_{ee}(G) = 5$, since
$\beta_0^{ee}(G) = 2$.

PROPOSITION 2.1. *Given any positive integer k there exists a graph G such that $\chi_{ee}(G) - \chi(G) = k$.*

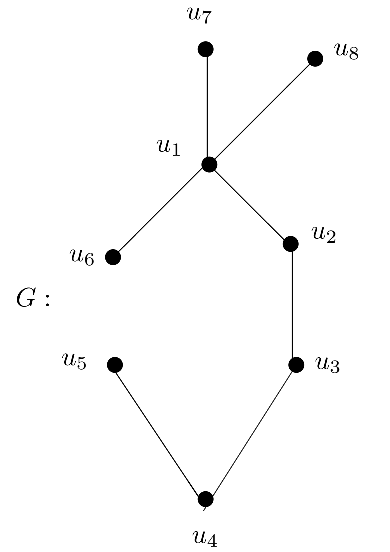
PROOF. Let $k \geqslant 6$. Let $G = D_{2,k}$. Then $\chi_{ee}(G) = max\{2, k\} + 2 = k + 2$, $\chi(G) = 2$. Therefore $\chi_{ee}(G) - \chi(G) = k$. $\chi(D_{3,4}) = 2$ *and* $\chi_{ee}(D_{3,4}) = 3$. Therefore $\chi_{ee}(G) - \chi(G) = 1$. $\square$



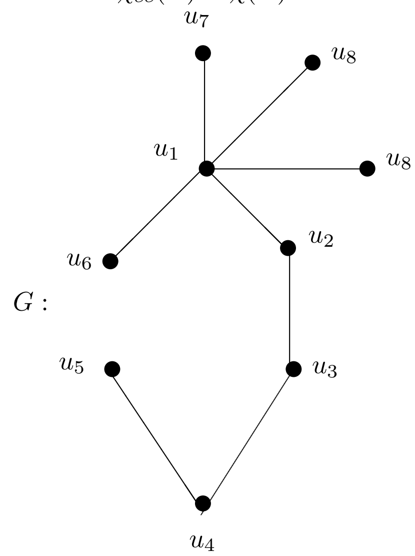
$\chi(G) = 4$, $\chi_{ee}(G) = 6$. Therefore $\chi_{ee}(G) - \chi(G) = 2$.



$\chi(G) = 4$, $\chi_{ee}(G) = 7$. Therefore $\chi_{ee}(G) - \chi(G) = 3$.



$\chi(G) = 4$, $\chi_{ee}(G) = 8$. Therefore $\chi_{ee}(G) - \chi(G) = 4$.



$\chi(G) = 4$, $\chi_{ee}(G) = 9$. Therefore $\chi_{ee}(G) - \chi(G) = 5$.

THEOREM 2.3. *$\chi_{ee}(G) = 1$ if and only if G is $\overline{K_n}$.*

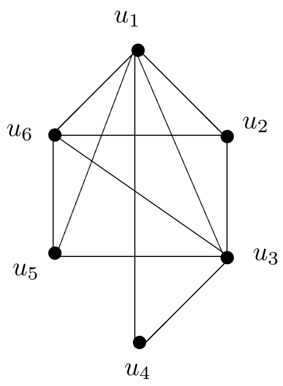
PROOF. Suppose $\chi_{ee}(G) = 1$. Then no two vertices of G are adjacent. Therefore $G = \overline{K_n}$. Converse is obvious. $\square$

THEOREM 2.4. *$\chi_{ee}(G) = 2$ if and only if G is bipartite and the degrees of any two elements of the same partition differ by at most one.*

PROOF. Suppose $\chi_{ee}(G) = 2$. Then $V(G) = X \cup Y$ where X and Y are independent and any two vertices of X have almost equal number of neighbors in Y and vice versa .(that is, for every $u, v \in X$, $||N(u) \cap Y| - |N(v) \cap Y|| \leqslant 1$ and for every $u, v \in Y$, $||N(u) \cap X| - |N(v) \cap X|| \leqslant 1$). The converse is obvious. $\square$

EXAMPLE 2.1.

$G :$



G is a bipartite Graph. Therefore $\chi(G) = 2$. But $\chi_{ee}(G) = 3$ and the color classes are $\{\{u_1, u_2\}, \{v_1, v_3, v_4, v_5\}, \{v_2, v_3\}\}$.

THEOREM 2.5. *$\frac{n}{\beta_0^{ee}(G)} \leqslant \chi_{ee}(G) \leqslant n - \beta_0^{ee}(G) + 1$.*

PROOF. Let $\chi_{ee}(G) = k$. Let π be a partition of $V(G)$ into k externally equitable independent sets $V_1, V_2, \cdots, V_k$. Then $|V_i| \leqslant \beta_0^{ee}(G) \ \vee i, 1 \leqslant i \leqslant k$.

$n = |V_1| + |V_2| + |V_3| + \cdots + |V_k| \leqslant k\beta_0^{ee}(G)$. Therefore $\frac{n}{\beta_0^{ee}(G)} \leqslant k$. Let S be a β_0^{ee} - set of G. Let $\pi = \{S, \{v_1\}, \{v_2\}, \ldots, \{v_t\}\}$ where $V - S = \{v_1, v_2, \ldots, v_t\}$ and $t = n - \beta_0^{ee}(G)$. Then π is an externally equitable independent partition of G. Therefore $\chi_{ee}(G) \leqslant n - \beta_0^{ee}(G) + 1$. $\square$

PROPOSITION 2.2. *Let S be any externally equitable independent set of C_n^+. Let $V(C_n) = \{u_1, u_2, u_3, \ldots, u_n\}$ and $V(C_n^+) = \{u_1, u_2, u_3, \ldots, u_n, u_1', u_2', u_3', \ldots, u_n'\}$ where u_i' is the pendant vertex of C_n^+ adjacent with u_i, $1 \leqslant i \leqslant n$.Then S cannot contain any pair of vertices of the form u_i, u_{i+2}, $1 \leqslant i \leqslant n$ ($i + 2$ taken mod n).*

PROOF. Suppose $u_i, u_{i+2} \in S$. Then $|N(u_{i+1}) \cap S| \geqslant 2$. Therefore, $|N(x) \cap S| \geqslant 1$, for every $x \in V - S$. Therefore S is a dominating set of C_n^+. Since S is independent, $u_{i+1} \notin S$ and as S is a dominating set of C_n^+ we get that $u_{i+1}' \in S$. Therefore $|N(u_{i+1}) \cap S| = 3$. Hence $|N(x) \cap S| \geqslant 2$, for every $x \in V - S$. Since $V - S$ contains pendant vertices, this is not possible. $\square$

PROPOSITION 2.3. *When $n \cong 1, 2$ (mod 3), $\chi_{ee}(C_n^+) \geqslant 4$.*

Proof. Suppose $n \cong\ 1\ (mod\ 3)$. Let $\pi = \{V_1, V_2, V_3\}$ be a χ_{ee}- partition of C_n^+. Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$. By Proposition 2.2, $|V_i \cap V(C_n)| \leqslant \frac{n-1}{3}$, $1 \leqslant i \leqslant 3$. Therefore $\sum\limits_{i=1}^{i=3} |V_i \cap V(C_n)| \leqslant n - 1$ a contradiction,

since $V_1 \cup V_2 \cup V_3 = V(C_n^+)$. Suppose $n \cong 2(mod\ 3)$. Arguing as before $\sum\limits_{i=1}^{i=3} |V_i \cap V(C_n)| \leqslant n - 2$ a contradiction. Therefore $\chi_{ee}(C_n^+) \geqslant 4$, when $n \cong 1, 2(mod\, 3)$. $\square$

Proposition 2.4. *Let $n \cong\ 0\ (mod\ 3)$. Let V_1 be an externally equitable independent set of C_n^+. Let $V(C_n) = \{u_1, u_2, u_3, \ldots, u_n\}$. Suppose $|V_1 \cap V(C_n)| = \frac{n}{3}$ and suppose $u_j^{'} \in V_1$ for some j such that $u_j \notin V_1$. Then $|V_1| = n$.*

Proof. Let $n \cong 0(mod\, 3)$. Let without loss of generality $u_1, u_4, \ldots, u_{n-2} \in V_1$ and $u_2^{'} \in V_1$. Therefore Then $|N(u_2) \cap V_1| = 2$. Therefore, V_1 is a dominating set of C_n^+. Therefore $u_3^{'}, u_5^{'}, u_6^{'} \ldots, u_{n-1}^{'}, u_n^{'} \in V_1$ and hence $|V_1| = n$. $\square$

Proposition 2.5. *Let $n \cong 0(mod\ 3)$. Suppose $\pi = \{V_1, V_2, V_3\}$ be a χ_{ee}- partition of C_n^+. Then $|V_i \cap V(C_n)| = \frac{n}{3}$, for all $i = 1, 2$ and 3.*

Proof. Clearly $|V_i \cap V(C_n)| \leqslant \frac{n}{3}$, for all $i = 1, 2$ and 3. Suppose $|V_i \cap V(C_n)| < \frac{n}{3}$. Therefore for some $j \neq i, 1 \leqslant j \leqslant 3$, $|V_i \cap V(C_n)| > \frac{n}{3}$ (since $\sum\limits_{i=1}^{i=3} |V_i \cap V(C_n)| = n$) a contradiction. Therefore, $|V_i \cap V(C_n)| = \frac{n}{3}$, for all $i = 1, 2$ *and* 3. $\square$

Proposition 2.6. *Let $n \cong 0(mod\ 3)$. Then $\chi_{ee}(C_n^+) > 3$.*

Proof. Suppose $\chi_{ee}(C_n^+) = 3$. Let $\pi = \{V_1, V_2, V_3\}$ be a χ_{ee}- partition of C_n^+.Then $|V_i \cap V(C_n)| = \frac{n}{3}$, for all $i = 1, 2$ *and* 3.Since $V_1 \cup V_2 \cup V_3 = V(C_n^+)$ for any $j, 1 \leqslant j \leqslant n$, $u_j^{'} \in V_i$, for some i, $1 \leqslant i \leqslant 3$. Suppose $u_j^{'} \in V_1$. Then by Lemma 2.32, $|V_1| = n$. Also, there exists $u_r^{'} \notin V_i$, for some r, $1 \leqslant r \leqslant n$. Therefore $u_r^{'} \in V_2$ *or* V_3. Therefore by lemma 2.2.32, $|V_2| = n$ or $|V_3| = n$. Since $V(C_n^+) = 2n$, one of V_2, V_3 is empty a contradiction. Therefore $\chi_{ee}(C_n^+) \geqslant 4$. $\square$

Proposition 2.7. $\chi_{ee}(C_n^+) \leqslant 4$.

Proof. **Case(i):** Let $n = 3k$. Then $\{\{u_1, u_4, u_2^{'}, u_3^{'}, u_5^{'}, u_6^{'}, \ldots, u_{3k}^{'}\}$, $\{u_2, u_5, u_8, \ldots, u_{3k-1}\}, \{u_3, u_7, u_{10}, \ldots, u_{3k-2}, u_1^{'}\}, \{u_6, u_9, u_{12}, \ldots, u_{3k}, u_4^{'}\}\}$ is an externally equitable independent partition of C_n^+. Therefore, $\chi_{ee}(C_n^+) \leqslant 4$.

Case(ii): Let $n = 3k + 1$. Then $\{\{u_1, u_4, u_2^{'}, u_3^{'}, u_5^{'}, u_6^{'}, \ldots, u_{3k+1}^{'}\}$, $\{u_2, u_6, u_9, \ldots, u_{3k}, u_4^{'}\}, \{u_3, u_7, u_{10}, \ldots, u_{3k+1}\}, \{u_5, u_8, u_{11}, \ldots, u_{3k-1}, u_1^{'}\}\}$ is an externally equitable independent partition of C_n^+.
Therefore $\chi_{ee}(C_n^+) \leqslant 4$.

Case(iii): Let $n = 3k + 2$. Then $\{u_1, u_4, u_2^{'}, u_3^{'}, u_5^{'}, u_6^{'}, \ldots, u_{3k+2}^{'}\}$, $\{u_2^{'}, u_6, u_9, \ldots, u_{3k}, u_4^{'}\}$, $\{u_3, u_7, u_{10}, \ldots, u_{3k+1}, u_1^{'}\}$, $\{u_5, u_8, u_{11}, \ldots, u_{3k+2}\}$ is an externally equitable independent partition of C_n^+.
Therefore $\chi_{ee}(C_n^+) \leqslant 4$. $\square$

THEOREM 2.6. $\chi_{ee}(C_n^+) = 4$.

PROOF. Follows from proposition 2.3 to 2.7.

□

REMARK 2.3. Let $G = (V, E)$ be a simple graph. Let S be an externally equitable independent set of G and for any $u \in V - S$, $|N(u) \cap S| \geqslant 2$. Then S is a dominating set of G. In particular, if $|N(u) \cap S| = r \geqslant 2$ then S is a $(r-1)-$ dominating set of G.

REMARK 2.4. If S is an externally equitable independent set of G and for any $u \in V - S$ $|N(u) \cup S| \geqslant 3$ then S does not contain any pendant vertex. $((ie)$ $deg_G(v) \geqslant 2$, for any $v \in V - S)$

OBSERVATION 2.2. Let G and H be two vertex disjoint graphs. Any externally equitable independent set S of $G + H$ contains either vertices from $V(G)$ or $V(H)$ and not from both. Also, if $S \subseteq V(G)$, then for any $u \in V(H)$, $|N(u) \cap S| = |S|$. Hence, $|N(v) \cap S| = |S| - 1$ or $|S|$ for any $v \in V - S$. Therefore S is a $(|S| - 1)$-dominating set of G. If for any $x \in V(G) - S$, $deg_G(x) \leqslant |S| - 2$ then S is not an externally equitable independent set of $G + H$.

OBSERVATION 2.3. Let G and H be two vertex disjoint graphs. Let S be an externally equitable independent set of G. Then G is an externally equitable independent set of $G \cup H$ if and only if for every $x \in V - S$, $|N(x) \cup S| \leqslant 1$. Similar results holds for H also. Also, if S is an externally equitable independent set of $G \cup H$ and S is a dominating set of G, then $|N(x) \cap S| = 1$, for every $x \in V - S$.

OBSERVATION 2.4. Let G and H be two vertex disjoint graphs. Let S be an externally equitable independent set of $G + H$. Then S is an externally equitable independent set of G if $S \subseteq V(G)$ or an externally equitable independent set of H if $S \subseteq V(H)$.

OBSERVATION 2.5. Let $\pi = \{S_1, S_2, \ldots, S_r\}$ be a χ_{ee} partition of G. $S_1, S_2, \ldots, S_r$ are externally equitable independent partition of $G + H$ if and only if every $u \in S_i$, $1 \leqslant i \leqslant r$, $n - |S_i| \geqslant deg_G(u) \geqslant n - |S_i| - (r-1)$.

OBSERVATION 2.6. Let $\pi = \{S_1, S_2, \ldots, S_t\}$ be externally equitable independent partition of $G + H$. Let $S_1, S_2, \ldots, S_r$ be an externally equitable independent sets of G and $S_{r+1}, S_{r+2}, \ldots, S_t$ be externally equitable independent sets of H. Then $\pi_1 = \{S_1, S_2, \ldots, S_r\}$ is an externally equitable independent partition of G and $\pi_2 = \{S_{r+1}, S_{r+2}, \ldots, S_t\}$ be an externally equitable independent partition of H . Also for every $u \in S_i$, $1 \leqslant i \leqslant r$, $n - |S_i| \geqslant deg_G(u) \geqslant n - |S_i| - (r-1)$ and $n - |S_j| \geqslant deg_H(u) \geqslant n - |S_j| - (t - r - 1)$ for every $u \in S_j$, $r + 1 \leqslant j \leqslant t$.

OBSERVATION 2.7. Let $\pi_1 = \{S_1, S_2, \ldots, S_r\}$ be an externally equitable independent partition of G and $\pi_2 = \{S_{r+1}, S_{r+2}, \ldots, S_t\}$ is an externally equitable independent partition of H . Then $\pi_1 \cup \pi_2$ is an externally equitable independent partition of $G + H$ if and only if for every i, $1 \leqslant i \leqslant r$ $n - |S_i| \geqslant deg_G(u) \geqslant n - |S_i| - (r-1)$ for every $u \in S_i$ and $n - |S_j| \geqslant deg_H(u) \geqslant n - |S_j| - (t - r - 1)$ for every $u \in S_j$, $r + 1 \leqslant j \leqslant t$. Let $k = max\{|S_i|, 1 \leqslant i \leqslant r\}$ and $l = min\{|S_i|, 1 \leqslant i \leqslant r\}$.

$n-k-(r-1) \leqslant deg_G(u) \leqslant n-l$, for every $u \in V(G)$. Therefore $\delta(G) \geqslant n-k-(r-1)$ and $\Delta(G) \leqslant n-l$.

ILLUSTRATION 2.2. Consider $G = K_3$ and $H = C_8$. Let $V(K_3) = \{u_1, u_2, u_3\}$ and $V(C_8) = \{v_1, v_2, \ldots v_8\}$. $\pi_1 = \{\{u_1\}, \{u_2\}, \{u_3\}\}$ and $\pi_2 = \{\{v_1, v_3, v_5, v_7\}, \{v_2, v_4, v_6, v_8\}\}$ are χ_{ee} partitions of G and H respectively. Let $S_1 = \{v_1, v_3, v_5, v_7\}$ and $S_2 = \{v_2, v_4, v_6, v_8\}$. If $\pi_1 \cup \pi_2$ is an externally equitable independent partition of $G + H$, then S_1 and S_2 are 3 dominating sets of H which is not possible since $\Delta(H) = 2$. A similar argument shows that a χ_{ee} partition of H cannot contain an element of cardinality three since any externally equitable independent set of cardinality three in C_8 is not a independent 2-dominating set of C_8. If π_2 contains an element of cardinality two then it is not a dominating set of C_8 since $\gamma(C_8) = 3$. Therefore, every element of π_2 is a singleton and hence $\chi_{ee}(K_3 + C_8) = 11$.

OBSERVATION 2.8. $\chi_{ee}(G + H) \leqslant |V(G)| + |V(H)|$ and the upper bound is sharp as seen in the above example.

OBSERVATION 2.9. Given any positive integer k, there exist graph G and H such that $\chi_{ee}(G + H) - (\chi_{ee}(G) + \chi_{ee}(H)) = k$.

PROOF. **Case 1:** k is even. Let $G = K_3$ and $H = C_{2k+2}$. $\chi_{ee}(G+H) = 2k+5$. $\chi_{ee}(G) = 3$ and $\chi_{ee}(H) = 2$. Hence the observation.

Case 2: k is odd. Consider $G = K_4$, $H = D_{k+2,k}$. $\chi_{ee}(G + H) = 2k + 4 + 4 = 2k + 8$, $\chi_{ee}(G) = 4$, $\chi_{ee}(H) = k + 4$. Therefore, $\chi_{ee}(G + H) - (\chi_{ee}(G) + \chi_{ee}(H)) = k$. □

OBSERVATION 2.10. Let G and H be two vertex disjoint graphs. Let $\pi_1 = \{V_1, V_2, \ldots . V_k\}$ and $\pi_2 = \{W_1, W_2, \ldots . W_r\}$ be χ_{ee}- partitions of G and H respectively. Let $r_i = \min\limits_{x \in V(G)-V_i} \{N(x) \cap V_i\}$, $1 \leqslant i \leqslant k$. Let $s_j = \min\limits_{x \in V(H)-W_j} \{N(y) \cap W_j\}$, $1 \leqslant j \leqslant r$. If $r = k$ and $r_i = s_i$ for every $i = 1$ to k, then $\{V_1 \cup W_1, V_2 \cup W_2, \ldots, V_k \cup W_k\}$ is an externally equitable independent partition of $G \cup H$. Hence $\chi_{ee}(G \cup H) \leqslant k$.

PROPOSITION 2.8. *Let G and H be two vertex disjoint graphs. Then $max\{\chi_{ee}(G), \chi_{ee}(H)\} \leqslant \chi_{ee}(G \cup H)$.*

PROOF. Let $\pi = \{V_1, V_2, \ldots . V_k\}$ be a $\chi_{ee}-$ partition of $G \cup H$. Let $\pi_1 = \{V_1 \cap V(G), V_2 \cap V(G), \ldots . V_k \cap V(G)\}$ and $\pi_2 = \{V_1 \cap V(H), V_2 \cap V(H), \ldots . V_k \cap V(H)\}$. Clearly π_1 and π_2 are independent partitions of G and H respectively. Let $x, y \in V(G) - (V_i \cap V(G))$, $1 \leqslant i \leqslant k$. Therefore $x, y \in V(G \cup H) - V_i$. Therefore $|(N(x) \cap V_i) - (N(y) \cap V_i)| \leqslant 1$. Since x and y are not adjacent with any vertex of H, $N(x) \cap V_i = N(x) \cap (V_i \cap V(G))$ and $N(y) \cap V_i = N(y) \cap (V_i \cap V(G))$. Therefore $|N(x) \cap (V_i \cap V(G)) - N(y) \cap (V_i \cap V(G))| \leqslant 1$ for every i, $1 \leqslant i \leqslant k$. Therefore π_1 is an externally equitable independent partition of G. Similarly π_2 is an externally equitable independent partition of H. Therefore $\chi_{ee}(G) \leqslant k$ and $\chi_{ee}(H) \leqslant k$. Therefore $\max\{\chi_{ee}(G), \chi_{ee}(H)\} \leqslant \chi_{ee}(G \cup H)$. □

REMARK 2.5. The bound is sharp as seen from the following example: Let $G = K_3$ and $H = C_8$. $V(G) = \{u_1, u_2, u_3\}$ and $V(H) = \{v_1, v_2, \ldots v_8\}$. $\chi_{ee}(G) = 3$ and

$\chi_{ee}(H) = 2$. $\{\{u_1, v_1, v_3, v_5, v_7\}, \{u_2, v_2, v_4, v_6, v_8\}, \{u_3\}\}$ is an externally equitable independent partition of $G \cup H$. Therefore $\chi_{ee}(G \cup H) \leqslant 3$. But $\chi_{ee}(G \cup H) \geqslant 3$, $\chi_{ee}(G \cup H) = 3 = \max\ \{\chi_{ee}(G), \chi_{ee}(H)\}$.

THEOREM 2.7. *Let G and H be two vertex disjoint graphs. Let π_1 and π_2 be two partitions of G and H respectively satisfying the following. Let $\pi_1 = \{V_1, V_2, \ldots V_k\}$, $\pi_2 = \{W_1, W_2, \ldots W_r\}$. Any vertex in $V(G) - V_i$ is adjacent with either a or $a + 1$ vertices of V_i $1 \leqslant i \leqslant k$ and any vertex in $V(H) - W_j$ is adjacent with either a or $a + 1$ vertices of W_j $1 \leqslant i \leqslant r$. Then $\chi_{ee}(G \cup H) \leqslant max\{\chi_{ee}(G), \chi_{ee}(H)\}$.*

PROOF. Let with out loss of generality $k \geqslant r$. Consider $\pi = \{V_1 \cup W_1, V_2 \cup W_2, \ldots, V_r \cup W_r, \ldots, V_k\}$. Clearly π is an externally equitable independent partition of $G \cup H$. Therefore $\chi_{ee}(G \cup H) \leqslant k = \max\{\chi_{ee}(G), \chi_{ee}(H)\}$. □

REMARK 2.6. Let G and H be two vertex disjoint graphs. Let π_1 and π_2 be two partitions of G and H respectively satisfying the following. Let $\pi_1 = \{V_1, V_2, \ldots V_k\}$, $\pi_2 = \{W_1, W_2, \ldots W_r\}$. Any vertex in $V(G) - V_i$ is adjacent with either a or $a + 1$ vertices of V_i $1 \leqslant i \leqslant k$ and any vertex in $V(H) - W_j$ is adjacent with either a or $a + 1$ vertices of W_j $1 \leqslant i \leqslant r$. Then $\chi_{ee}(G \cup H) \leqslant \max\{k, r\}$.

References

[1] R. B. Allan and R. C. Laskar, *On domination and independent domination numbers of a graph*, Discrete Math. 23 (1978), 73-76.

[2] A. Anitha, S. Arumugam and E. Sampathkumar, *Degree Equitable Sets in a Graph*, International J.Math. Combin., 3(2009), 32-47.

[3] A. Anitha, S. Arumugam, S. B. Rao and E. Sampathkumar, *Degree Equitable Chromatic number of a a Graph*, JCMCC, 75(2010), 187-199.

[4] C. Berge, *Theory of graphs and its applications*, Dunod, Paris, 1958.

[5] K. M. Dharmalingam, *Studies in Graph Theory - Equitable domiantion and Bottleneck domination*, Ph.D Thesis, 2006.

[6] K. M. Dharmalingam and V.Swaminathan, *Degree Equitable domiantion in Graphs*, Kragujevac J. Math., 35(1)(2011),191-197.

[7] G. H. Fricke, Teresa W. Haynes, S. T. Hetniemi, S. M. Hedetniemi and R. C. Laskar, *Excellent trees*, Bull. Inst. Combin. Appl., 34 (2002), 27-38.

[8] F. Harary, *Graph Theory*, ddison Wesley, Reading Mass, 1972.

[9] S. T. Hedetniemi and R. C. Laskar, *Bibliography on domination in graphs and some basic definitions of domination parameters*, Discrete Math. 86(1990) 257-277.

[10] Peter Dankelmann Durban Gayla S. Domke, Wayne Goddard, Paul Grobler, Johannes H. Hattingh and Henda C. Swart, *Maximum Sizes of Graphs with Given Domination*, ParametersPreprint submitted to Elsevier Science 15 October 2004.

[11] Terasa W. Haynes, Stephen T. Hedetneimi, Peter J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., 1998.

[12] W.Meyer, *Equitable coloring*, Amer.Math. Monthly, 80 (1973), 920-922.

Received by editor 05.01.2012; available on internet 03.09.2012

Associate Professor, Department of Mathematics, Sethu Institute of Technology, Virudhunagar District, Madurai, India

Ramanujan Research Center in Mathematics, Saraswathi Narayanan College, Madurai, India
E-mail address: sulanesri@yahoo.com