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CONTRACTIVE OPERATORS ON TOPOLOGICAL VECTOR SPACES

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ABSTRACT. In this paper we define contractive bounded linear operators on partially ordered Haussdorff topological vector space and study theirs basic properties.

1. Introduction

I. Aranđelović and V. Mišić [**3**] (see also [**4**]) introduced the notion of a contractive linear operator on metric linear spaces and present some fixed point results with operator contractive condition which generalize some results from [**5**] and [**8**]. In [**2**] authors consider contractive linear operators on locally convex topological vector spaces.

In this paper we define contractive bounded linear operators on partially ordered (non-necessarily locally convex) Haussdorff topological vector spaces and study theirs basic properties.

2. Preliminaries

Let E be a linear topological space. Let E be a linear topological space. A subset P of E is called a cone if:

1) P is closed, nonempty and $P \neq \{0\}$;

2) $a, b \in \mathbf{R}$, $a, b > 0$, and $x, y \in P$ imply $ax + by \in P$;

3) $P \cap (-P) = \{0\}$.

Given a cone, $P \subseteq E$ we define partially ordering $\leqslant$ on E with respect to P by $x \leqslant y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leqslant y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$ (interior of P).

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Let E be a linear topological space and let $P \subseteq E$ be a cone. We say that P is a solid cone if and only if int$P \neq \emptyset$. Then c is an interior point of P if and only if $[-c, c]$ is a neighborhood of Θ in E.

Ordered topological vector space (E, P) is order-convex if its base of neighborhoods of zero consists of order-convex subsets. In this case the cone P is said to be normal, or P-saturated.

Let E be a Banach space and let P be a solid cone in E such that $\leqslant$ is a partially ordering on E with respect to P. P is a normal cone if and only if there exists a real number $K > 0$ such that $x \leqslant y$ implies

$$\|x\| \leqslant K\|y\| \tag{2.1}$$

for each $x, y \in P$. The least positive K satisfying (2.1) is called the normal constant of P. In [**8**] Sh. Rezapour and R. Hamlbarani proved that $K \geqslant 1$, when E is a Banach space.

Let $(E, |.\|)$ be a topological vector space and $P \subseteq E$ be a cone. P is a solid cone if and only if int$P \neq \emptyset$.

There exists solid cone which is non-normal.

Theorem 2.1. *(J. S. Vandergraft [**10**]) If the cone P of an ordered topological vector space (E, P) is normal and solid, then (E, P) is an ordered normed space.*

Theorem 2.2. *(Arandelović, Keckić [**2**]) There exists non-normable locally convex topological vector space which has solid cone.*

Theorem 2.3. *There exists non-locally convex and non-metrizable topological vector space which has solid cone.*

Proof. Let $X = [0, 1]$, $E = C_{\mathbf{R}}[0, 1]$ equipped with the strongest Hausdorff topology and $P = \{h \in E : h(t) \geqslant 0, t \in [0, 1]\}$. Then E is a Hausdorff non-locally convex, because its base in uncountable [**1**], and P is a non-normal solid cone, by Theorem 2.3. from [**6**]. $\square$

In the following we always suppose that E is a (non-necessarily locally convex) Haussdorff topological vector space, P is a solid cone in E such that $\leqslant$ is partially ordering on E with respect to P. By I we denote identity operator on E i.e. $I(x) = x$ for each $x \in E$. $\square$

3. Main Results

We start with the following definition.

Definition 3.1. If $A : E \to E$ is an one to one function such that $A(P) = P$, $I - A$ is one to one and $(I - A)(P) = P$ then A is contractive operator.

Example 3.1. Let n be a positive integer, $E = \mathbf{R}^n$, $P = \{(x_1, \ldots, x_n) \in E : x_i \geqslant 0 \; i = 1, \ldots, n\}$, $\lambda_1, \ldots, \lambda_n \in (0, 1)$ and $\Lambda = [a_{ij}]_{1 \leqslant i,j \leqslant n}$ be square matrix such that

$$a_{ij} = \begin{cases} 0 \quad 0, & i \neq j; \\ \lambda_j, & i = j \end{cases}, 1 \leqslant i, j \leqslant n.$$

Then $A : E \to E$ defined by $A(x) = \Lambda[x]$ is contractive bounded linear operator.

EXAMPLE 3.2. Let $E = \mathcal{C}^2([0,1])$ with the norm

$$\|f\| = \|f\|_\infty + \|f'\|_\infty,$$

and consider the cone

$$P = \{f \in E : f \geqslant 0\}.$$

Then P is non-normal solid cone in E (see [**8**]).

Then $A : E \to E$ defined by

$$A(f)|_x = \exp(-x)f(x)$$

is contractive bounded linear operator. We can see that $\|A\| = 3$, and so A is not contractive operator in sense of Banach.

Now we need the following Lemma.

LEMMA 3.1. *If $A : E \to E$ is the contractive bounded linear operator then*

1) there exists A^{-1} and it is bounded linear operator;

2) there exists $(I - A)^{-1}$ and it is bounded linear operator;

3) $A(x) \ll x$ for any $x \in \mathrm{int}P$;

4) $x \leqslant y$ implies $A(x) \leqslant A(y)$ for any $x, y \in P$;

5) $x \ll y$ implies $A(x) \ll A(y)$ for any $x, y \in P$;

6) $(I - A)(x) \ll x$ for any $x \in \mathrm{int}P$;

7) $I + A + \cdots + A^n = (I - A)^{-1} \circ (I - A^{n+1})$.

PROOF. 1) A^{-1} exists because A is one to one. For any $a, b \in E$ there exists $c, d \in E$ such that $a = A(c)$ and $b = A(d)$. From

$$A^{-1}(\alpha a + \beta b) = A^{-1}(\alpha A(c) + \beta A(d)) = A^{-1}(A(\alpha c + \beta d)) = \alpha A^{-1}(a) + \beta A^{-1}(b),$$

it follows that A^{-1} is linear. A is continuous because it is bounded, which implies that A^{-1} is continuous. So A^{-1} is bounded because it is linear.

2) $I - A$ is one to one bounded linear operator because I and A are one to one bounded linear operators. Now proof follows from 1).

3) $I - A$ is one to one bounded linear operator because I and A are one to one bounded linear operators. $(I - A)(P) = P$ and $(I - (I - A))(P) = A(P) = P$ because A is contractive.

From $(I - A)(P) = P$ by Open mapping theorem (see [**11**]) it follows that $(I - A)(\mathrm{int}P) \subseteq \mathrm{int}P$, which implies that $x - A(x) \in \mathrm{int}P$ for each $x \in \mathrm{int}P$.

4) From $x \leqslant y$ it follows $y - x \in P$ which implies $A(y - x) \in P$ because $A(P) = P$.

5) From $A(P) = P$ by Open mapping theorem (see [**11**]) it follows that $A(\mathrm{int}P) \subseteq \mathrm{int}P$. $x \ll y$ implies $y - x \in \mathrm{int}P$ which implies $A(y - x) \in \mathrm{int}P$ because $A(\mathrm{int}P) \subseteq \mathrm{int}P$.

6) It follows from $A(\mathrm{int}P) \subseteq \mathrm{int}P$ and $A(x) = x - (x - A(x))$.

7) It follows from $(I - A) \circ (I + A + \cdots + A^n) = I - A^{n+1}$. $\square$

In this section our main result is the following theorem.

THEOREM 3.1. *If $A : E \to E$ is the contractive bounded linear operator then for each $x \in P$ and any $c \in \mathrm{int}P$ there exists a positive integer n_0 such that*

$$A^n(x) \ll c$$

for all $n > n_0$.

PROOF. By Lemma 3.1 we get that

$$\begin{aligned}(I - A) \circ (n+1)A^n(x) &\leqslant (I - A) \circ (I + A + \cdots + A^n)(x) = \\ &= (I - A^{n+1})(x) = x - A^{n+1}(x) \leqslant x\end{aligned}$$

for any $x \in P$, because $A^n(x) \leqslant A^k(x)$ for any $k = 0, \dots, n$. So

$$(I - A) \circ (n+1)A^n(x) \leqslant x.$$

Hence

$$A^n(x) \leqslant \frac{1}{n+1}(I - A)^{-1}(x).$$

For any $0 \ll c$ we get that there exists a positive integer n_0 such that $n > n_0$ implies

$$\frac{1}{n+1}(I - A)^{-1}(x) \ll c,$$

because

$$\frac{1}{n+1}(I - A)^{-1}(x)$$

is a convergent sequence. So $n > n_0$ implies

$$A^n(x) \ll c.$$

$\square$

From Lemma 3.1 and Theorem 3.1 we obtain:

COROLLARY 3.1. *If $A : E \to E$ is a contractive bounded linear operator then*

$$\lim_{n\to\infty}(I + A + \cdots + A^n) = (I - A)^{-1}.$$

Next Corollary extends famous Volterra's fixed point theorem (see [**7**]).

COROLLARY 3.2. *If $A : E \to E$ is a contractive bounded linear operator then for any $z \in P$ equation*

$$x = z + A(x)$$

has a unique solution $y \in P$ and

$$y = \lim_{n\to\infty}(I + A + \cdots + A^n)(x)$$

for any $x \in \mathrm{int}P$.

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