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FINITELY QUASI-CONJUGATIVE RELATIONS

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ABSTRACT. In a former article, the first author introduces and analyzes quasiconjugative relations. In this paper, following Jiang Guanghao and Xu Luoshan's concepts of finitely conjugative and finitely dual normal relations on sets, the concept of finitely quasi-conjugative relations is introduced. A characterization of finitely quasi-conjugative relations is obtained. Particulary we show when the anti-order relation \leq is finitely quasi-conjugative.

1. Introduction

The concept of conjugative relations was introduced by Guanghao Jiang and Luoshan Xu ([1]), and the concept of dually normal relations was introduced and analyzed by Jiang Guanghao and Xu Luoshan in [2]. In former article [4], the first author introduced and analyzed a new class of relations in set - class of quasiconjugative relations on sets. In this article, as a continuation of article [4], following Jiang Guanghao and Xu Luoshan's concepts of finitely conjugative and finitely dual normal relations, introduced in articles [1] and [2], we introduce and analyze the notion of finitely quasi-conjugative relations.

For a set X, we call ρ a relation on X, if $\rho \subseteq X \times X$. Let $\mathcal{B}(X)$ be denote the set of all binary relations on X. For $\alpha, \beta \in \mathcal{B}(X)$, define

 $\beta \circ \alpha = \{ (x, z) \in X \times X : (\exists y \in X) ((x, y) \in \alpha \land (y, z) \in \beta) \}.$

The relation $\beta \circ \alpha$ is called the composition of α and β . It is well known that $(\mathcal{B}(X), \circ)$ is a semigroup. The latter family, with the composition, is not only a semigroup, but also a monoid. Namely, $Id_X = \{(x, x) : x \in X\}$ is its identity element. For a relation α on a set X, define $\alpha^{-1} = \{(x, y) \in X \times X : (y, x) \in \alpha\}$ and $\alpha^C = (X \times X) \setminus \alpha$.

Let A and B be subsets of X. For $\alpha \in \mathcal{B}(X)$, set

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 $A\alpha = \{y \in X : (\exists a \in A)((a, y) \in \alpha)\}, \ \alpha B = \{x \in X : (\exists b \in B)((x, b) \in \alpha)\}.$ It is easy to see that $A\alpha = \alpha^{-1}A$ holds. Specially, we put $a\alpha$ if $\{a\}\alpha$ and αb if $\alpha\{b\}$.

2. Quasi-conjugative relations

The following classes of elements in the semigroup $\mathcal{B}(X)$ have been investigated.

DEFINITION 2.1. For a relation $\alpha \in \mathcal{B}(X)$ we say that it is:

- regular if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha.$$

- normal ([3]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ (\alpha^{\rm C})^{-1}.$$

- dually normal ([2]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that $\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha.$

- conjugative ([1]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha$$

- dually conjugative ([1]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that $\alpha = \alpha \circ \beta \circ \alpha^{-1}$.

The notion of *quasi-conjugative relation* is introduced in paper [4] by the following definition.

DEFINITION 2.2. For a relation $\alpha \in \mathcal{B}(X)$ we say that it is a *quasi-conjugative* relation on X if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha^C.$$

The latter term is not convenient since thus a conjugative relation need not be quasi-conjugative. The same remark can be applied to the terms normal and regular too.

In the following propositions we give characterization of quasi-conjugative relations. Here, for a relation α , we use the notation α^* to sign the maximal elements of all family of relation β such that $\alpha^{-1} \circ \beta \circ \alpha^C \subseteq \alpha$.

PROPOSITION 2.1 ([4], Lemma 2.1). For a binary relation $\alpha \in \mathcal{B}(X)$, relation $\alpha^* = ((\alpha \circ \alpha^C \circ (\alpha^C)^{-1})^C)$

is the maximal element in family of all relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha^{-1} \circ \beta \circ \alpha^C \subseteq \alpha.$$

PROPOSITION 2.2 ([4], Theorem 2.1). For a binary relation α on a set X, the following conditions are equivalent:

- (1) α is a quasi-conjugative relation.
- (2) For all $x, z \in X$, if $(x, y) \in \alpha$, there exists $u, v \in X$ such that: (a) $(x, u) \in \alpha^C \land (y, v) \in \alpha$,
 - (b) $(\forall s, t \in X)((s, u) \in \alpha^C \land (t, v) \in \alpha \Longrightarrow (s, t) \in \alpha).$

(3)
$$\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^C$$
.

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PROPOSITION 2.3 ([4], Corollary 2.1). Let (X, \leq) be a poset. Then \leq is a quasi-conjugative relation on X if and only if for all $x, y \in X$ such that $x \leq y$ there exist elements $u, v \in X$ such that :

(a')
$$x \leq u \land y \leq v$$
, and
(b) $(\forall u \in V)(u \neq v)$

(b')
$$(\forall z \in X)(z \leq u \lor z \leq v).$$

3. Finitely quasi-conjugative relations

In this section we introduce the concept of finitely quasi-conjugative relations and give a characterization of this relations. For that we need the concept of *finite* extension of a relation. That notion and belonging notation we borrow from articles [1] and [2]. For any set X, let $X^{(<\omega)} = \{F \subseteq X : F \text{ is finite and nonempty }\}.$

DEFINITION 3.1. ([1], Definition 3.3; [2], Definition 3.4) Let α be a binary relation on a set X. Define a binary relation $\alpha^{(<\omega)}$ on $X^{(<\omega)}$, called the *finite* extension of α , such that

$$(\forall F, G \in X^{(<\omega)})((F,G) \in \alpha^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha).$$

From Definition 3.1, we immediately obtain that

$$(\forall F, G \in X^{(<\omega)})((F,G) \in (\alpha^C)^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha^C).$$

and

$$(\forall F, G \in X^{(<\omega)})((F,G) \in (\alpha^{-1})^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha^{-1} = \alpha F)$$

Now, we can introduce concept of *finitely quasi-conjugative relation*.

DEFINITION 3.2. A relation α on a set X is called *finitely quasi-conjugative* if there exists a relation β on X such that

$$\alpha^{(<\omega)} = (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}.$$

Although it seems, in accordance with Definition 2.2, it would be better to coll a relation α on X to be finitely quasi-conjugative if its finite extension to $X^{(<\omega)}$ is a quasi-conjugative relation, we will not use that option. That concept is different from our concept given by Definition 3.2.

Now we give an essential characterization of finitely quasi-conjugative relations.

THEOREM 3.1. A relation α on a set X if a finitely quasi-conjugative relation if and only if for all $F, G \in X^{(<\omega)}$, if $G \subseteq F\alpha$, then there are $U, V \in X^{(<\omega)}$, such that (i) $U \subseteq F\alpha^C$, $G \subseteq \alpha V$, and

(ii) for all $S, T \in X^{(\langle \omega \rangle)}$, if $U \subseteq S\alpha^C$ and $T \subseteq \alpha V$ then $T \subseteq S\alpha$.

PROOF. (\Longrightarrow) Let α be a finitely quasi-conjugative relation on set X. Then there is a relation $\beta^{(<\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$ such that $(\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} = \alpha^{(<\omega)}$. For all $(F,G) \in (X^{(<\omega)})^2$, if $G \subseteq F\alpha$, i.e., $(F,G) \in \alpha^{(<\omega)}$, thus $(F,G) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$. Whence there is $(U,V) \in (X^{(<\omega)})^2$ such that $(F,U) \in (\alpha^C)^{(<\omega)}$, $(U,V) \in \beta^{(<\omega)}$ and $(V,G) \in (\alpha^{-1})^{(<\omega)}$, i.e., $U \subseteq F\alpha^C$, $G \subseteq V\alpha^{-1} = \alpha V$. Hence we get the condition (i). Now we check the condition (ii). For all $(S,T) \in (X^{(<\omega)})^2$, if $U \subseteq S\alpha^C$ and $T \subseteq \alpha V$, i.e., $(S,U) \in (\alpha^C)^{(<\omega)}$ and $(V,T) \in (\alpha^{-1})^{(<\omega)}$, then by $(U,V) \in \beta^{(<\omega)}$, we have $(S,T) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$, i.e., $(S,T) \in \alpha^{(<\omega)}$. Hence $T \subseteq S\alpha$.

(\Leftarrow) Let α be a relation a set X such that for $F, G \in X^{(<\omega)}$ with $G \subseteq F\alpha$ there are $U, V \in X^{(<\omega)}$ such that conditions (i) and (ii) hold. Define a binary relation $\beta^{(<\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$ by

$$(F,G) \in \beta \iff (\forall S,T \in X^{(<\omega)})((F \subseteq S\alpha^C \land T \cap \alpha G \neq \emptyset) \Longrightarrow T \cap S\alpha \neq \emptyset).$$

First, check that (a) $(\alpha^{-1})^{(\langle\omega\rangle)} \circ \beta^{(\langle\omega\rangle)} \circ (\alpha^C)^{(\langle\omega\rangle)} \subseteq \alpha^{(\langle\omega)}$ holds. For all $H, W \in X^{(\langle\omega\rangle)}$, if $(H, W) \in (\alpha^{-1})^{(\langle\omega\rangle)} \circ \beta^{(\langle\omega\rangle)} \circ (\alpha^C)^{(\langle\omega)}$, then there are $F, G \in X^{(\langle\omega\rangle)}$ with $(H, F) \in (\alpha^C)^{(\langle\omega\rangle)}$, $(F, G) \in \beta^{(\langle\omega\rangle)}$ and $(G, W) \in (\alpha^{-1})^{(\langle\omega\rangle)}$. Then $F \subseteq H\alpha^C$ and $W \subseteq G\alpha^{-1} = \alpha G$. For all $w \in W$, let $S = H, T = \{w\}$. Then $F \subseteq S\alpha^C$ and $\alpha G \cap T \neq \emptyset$ because $w \in T$ and $w \in \alpha G$. Since $(F, G) \in \beta^{(\langle\omega\rangle)}$, we have that $F \subseteq S\alpha^C \land \alpha G \cap T \neq \emptyset$ implies $T \cap S\alpha \neq \emptyset$. Hence, $w \in S\alpha$, i.e. $W \subseteq S\alpha$. So, we have $(H, W) = (S, W) \in \alpha^{(\langle\omega\rangle)}$. Therefore, we have $(\alpha^{-1})^{(\langle\omega\rangle} \circ \beta^{(\langle\omega\rangle)} \circ (\alpha^C)^{(\langle\omega\rangle} \subseteq \alpha^{(\langle\omega\rangle)}$.

The second, check that (b) $\alpha^{(<\omega)} \subseteq (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^{C})^{(<\omega)}$ holds. For all $H, W \in X^{(<\omega)}$, if $(H, W) \in \alpha^{(<\omega)}$ (i.e., $W \subseteq H\alpha$), there are $A, B \in X^{(<\omega)}$ such that:

(i') $A \subseteq H\alpha^C$, $W \subseteq \alpha B$, and

(ii') for all $S, T \in X^{(<\omega)}$, if $A \subseteq S\alpha^C$ and $T \subseteq \alpha B$, then $T \subseteq S\alpha$.

Now, we have to show that $(A, B) \in \beta^{(<\omega)}$. Let be for all $(C, D) \in (X^{(<\omega)})^2$ the following $A \subseteq D\alpha^C$ and $D \cap \alpha B \neq \emptyset$ hold. From $D \cap \alpha B \neq \emptyset$ follows that here exists an element $d \in D \cap \alpha B (\neq \emptyset)$. So, $d \in D$ and $d \in \alpha B$. Put S = C and $T = \{d\}$. Then, by (ii'), we have

$$(A \subseteq S\alpha^C \land T = \{d\} \subseteq \alpha B) \Longrightarrow \{d\} = T \subseteq S\alpha,$$

i.e. $\emptyset \neq D \cap S\alpha = T \cap S\alpha$. Therefore, $(A, B) \in \beta^{(<\omega)}$ by definition of $\beta^{(<\omega)}$. Finally, for $(H, A) \in (\alpha)^{(<\omega)}$, $(A, B) \in \beta^{(<\omega)}$ and $(B, W) \in (\alpha^{-1})^{(<\omega)}$ follows that $(H, W) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$.

By assertion (a) and (b), finally we have
$$\alpha^{(<\omega)} = (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$$

COROLLARY 3.1. Let α be a relation on a set X. Then α is a finitely quasiconjugative on X if and only if for all elements $x, y \in X$ such that $(x, y) \in \alpha$ there are finite subsets $U, V \in X^{(<\omega)}$ such that

 $(1^0) \ (\forall u \in U)((x, u) \in \alpha^C) \ \land \ (\exists v \in V)((y, v) \in \alpha), \ and$

(2⁰) for all $S \in X^{(<\omega)}$ and $t \in X$ holds

$$(U \subseteq S\alpha^C \land (\exists v \in V)((y,v) \in \alpha)) \Longrightarrow (\exists s \in S)((s,t) \in \alpha) .$$

PROOF. Let α be a finitely quasi-conjugative relation on X and let x, y be elements of X such that $(x, y) \in \alpha$. If we put $F = \{x\}$ and $G = \{y\}$ in Theorem

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3.1 then there exist finite U and V of $X^{(<\omega)}$ such that conditions (1⁰) and (2⁰) hold.

Opposite, let for all elements $x, y \in X$ such that $(x, y) \in \alpha$ be there are Uand V of $X^{(<\omega)}$ such that conditions (1^0) and (2^0) hold. Define binary relation $\beta^{<\omega} \subseteq X^{<\omega} \times X^{<\omega}$ by

$$(A,B) \in \beta^{(<\omega)} \iff (\forall S \in X^{<\omega})(\forall t \in X)((A \subseteq S\alpha^C \land t \in \alpha B) \Longrightarrow t \in S\alpha).$$

The proof that the equality $(\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} = \alpha^{(<\omega)}$ holds is some as in the Theorem 3.1. So, the relation α is a finitely quasi-conjugative.

THEOREM 3.2. Let (L, \leq) be a poset. Then the relation \leq on L is a finitely quasi-conjugative relation if and only if for all $x, y \in L$ such that $x \leq y$, there exist finite subsets U and V of L such that

(a) $(\forall u \in U)(x \leq u)$ and $(\exists v \in V)(y \leq v)$ and

(b) $(\forall z \in L)((\exists u \in U)(z \leq u) \lor (\forall v \in V)(z \leq v)).$

PROOF. Let $x, y \in L$ such that $x \notin y$. Then by the finitely quasi-conjugativity of \notin , there exist finite subsets U and V of L such that

(1) $(\forall u \in U)(x \leq u) \land (\exists v \in V)(v \leq y)$, and

(2) for all $S \in L^{(<\omega)}$ and $t \in L$ the following holds

$$((\forall u \in U)(\exists s \in S)(s \leqslant u) \land (\exists v \in V)(t \notin v)) \Longrightarrow (\exists s' \in S)(s' \notin t).$$

For $z \in L$, let $S = \{z\} = \{t\}$. Then by (2), from

$$(\forall u \in U)(z \leqslant u) \land (\exists v \in V)(z \leqslant v)$$

implies $z \notin z$. It is a contradiction. Hence, we have

$$\neg((\forall u \in U)(z \leq u) \land (\exists v \in V)(z \leq v)).$$

So, finally, we have

$$(\exists u \in U)(z \leq u) \lor (\forall v \in V)(z \leq v).$$

Let for $(x, y) \in L^2$ be $x \notin y$ holds and let there exist finite subsets U and V of L satisfying conditions (a) and (b). So, the condition (a) is the condition (1^0) in Corollary 3.1.

Let $S \in L^{(<\omega)}$ and $t \in L$ with $(\forall u \in U)(\exists s \in S)(s \leq u)$ and $(\exists v \in V)(t \leq v)$ holds. Suppose that $(\forall s \in S)(s \leq t)$ holds. Then, by (b), for $S = \{s\}$ and z = s, we have

$$(\exists u \in U) (s \leq u) \lor (\forall v \in V) (s \leq v).$$

The first option is impossible because $s \leq u$. Then from $(\exists v \in V)(t \leq v)$ and $s \leq t$ follows $(\exists v \in V)(s \leq v)$. It is in contradiction with $(\forall v \in V)(s \leq v)$. So, must to be $\neg(\forall s \in S)(s \leq t)$. Thus $(\exists s \in S)(s \leq t)$. Hence \leq satisfies also condition (2^0) in Corollary 3.1. Finally, the relation \leq is a finitely quasi-conjugative relation on L.

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